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ΚΥΠΡΙΑΚΗ ΜΑΘΗΜΑΤΙΚΗ ΕΤΑΙΡΕΙΑ
Nicosia - Cyprus

Editor
Athanasios Gagatsis

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Learning Mathematics for Using its Language in a Universal Way

BRUNO D'AMORE: Universidad Distrital Francisco José de Caldas, Bogotà, Colombia

ABSTRACT: *Mathematics is the only discipline whose contents are more or less the same in all the countries of the world in which it is taught, depending on the age of the students. Recently Unesco has published a long document outlining the mathematical knowledge that is necessary for future citizenship. We all tend to emphasize that Mathematics does not merely have practical applications, but that its extraordinary importance lies in the language that it is able to develop and that this is one of the principal objectives of its complex process of teaching / learning. We must enable future citizens to use mathematical language to interpret all natural phenomena and the disciplines that humanity is able to develop. Among these are the Arts and in particular Music and the Plastic Arts. For decades now, many art critics use mathematical language to interpret the phenomenon of artistic creation and to describe the work of artists who often are not even aware of the mathematics they are using. The descriptive and rational power of mathematical language here reveals all its extraordinary effectiveness. In this sense, it is ever more important to study better and in more depth the Mathematics Education in order to understand the dynamics of "learning situations". Mathematics Education is an autonomous science that has assumed enormous importance in recent decades; the research continues to enrich its contents, thanks also to the contribution of other domains of human knowledge.*

Key words: *Mathematics education, Mathematical language, Art and mathematics.*

SOME BASES OF MATHEMATICS EDUCATION

A certain amount of confusion between Pedagogy and Didactics still exists today. The term Pedagogy can be interpreted in various ways, and it is often seen as a specific aspect of Philosophy that examines fundamental terms such as ethics, education, the relationship between learner and teacher, the role of the school and so on. Didactics, on the other hand, places emphasis on concepts like learning, general features of cognitive construction, the individual and society in education, the relationship between the learner and Knowledge or the teacher and Knowledge, and so on.

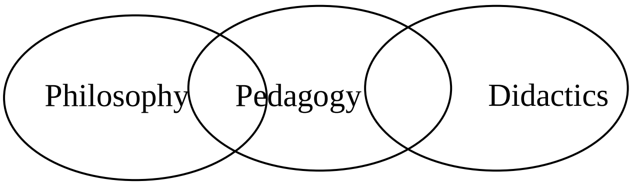


Figure 1. Philosophy – Pedagogy – Didactis in relations.

Today we can identify at least three different specific areas of Didactics: General Didactics, concerned with the broadest aspects of this discipline; Special Didactics, concerned with non-normal aspects of teaching and learning, with individual needs and particular teaching and learning situations; and Disciplinary Didactics, concerned with the specificity of the teaching and learning of given disciplines.

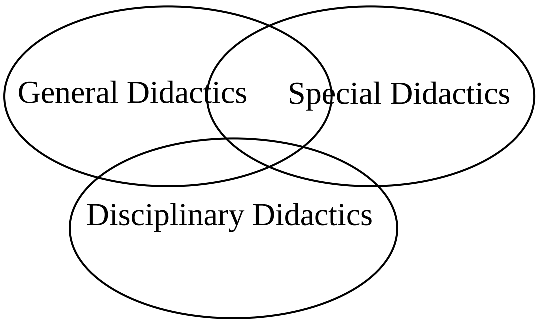


Figure 2. Relations between General, Special, and Disciplinary Didactics.

The pedagogues of the 18th and 19th centuries already focused on the difficulties of learning without distinguishing among the specificities of the disciplines. Today, it is clear that learning Mathematics is different from learning to swim or the History of Art. A clear demonstration of this is the fact that there are learners who have difficulties only in learning Mathematics. During the 1980s, the specificity of different forms of learning, together with our increasing understanding of given subjects, led to the idea of *episteme* and consequently to an Epistemology of specific learning, for example of the Mathematics learning, in order to highlight the characteristics that distinguish it from that of other subjects.

Today, we have a specific theory that within a plurality of disciplinary Didactics it enables us to focus on a given learning. In our case, there is a Didactics of Mathematics (or Mathematics Education), which may have certain characteristics in common with the Didactics of Modern Languages within the field of general Didactics, but which is a part of the specific areas of interest of Mathematics rather than of Pedagogy. In general, those who concern themselves with the Mathematics Education are Mathematicians.

FUTURE TRENDS IN MATHEMATICS

Mathematics is the only discipline whose contents are more or less the same in all the countries of the world in which it is taught, depending on the age of the students rather than the different geographical areas or social conditions. Recently, Unesco has published a long document outlining the mathematical knowledge that is necessary for future citizenship (Artigue, 2011). A particularly interesting aspect of this document is the distinction between basic mathematical knowledge for every citizen and advanced knowledge necessary for a significant use of Mathematics in a critical and analytic way, both from a personal and professional perspective.

As proposed in this document, we all tend to emphasize that Mathematics does not merely have practical applications, but that its extraordinary importance lies in the language that it is able to develop and that this is one of the principal objectives of its complex process of teaching-learning. We must enable future citizens to use mathematical language to interpret all natural phenomena and the disciplines that humanity is able to develop. Among these are the Arts and in particular Music and the Plastic Arts. A researcher in Mathematics Education may decide to devote all his work to this social duty.

PERSONAL COMMITMENTS IN MATHEMATICS EDUCATION

I began my research in Mathematics Education as a young Mathematician and I would never have imagined that I would have to abandon pure mathematical research to dedicate myself entirely to that applied to teaching and learning. In 1971, two years after my degree in Mathematics, I answered to a call on the part of a journal for an analysis of some didactic proposals by a group of researchers, but only five years later I decided with the conviction of a personal decision to dedicate my work to Mathematics Education. In this new field, I began the study of the interference between different types of language at school, for example the conflict between formal language and everyday language in the practice in school.

My research concentrated on the error, widespread in the 1980s, of trying to transform the resolution of a problem into an algorithm, a tendency based on an ingenuous interpretation of the books of George Polya. Moreover, I demonstrated the impossibility of this kind of reductionism and began investigating the linguistic causes of difficulties encountered in dealing with problems by students at all school levels. I examined at length the work environment “laboratory” as a learning environment in which learning by doing transforms the problem into a concrete need, interpreting student behaviour in terms of the theory of situations and providing tools for positive and negative analysis of the phenomenon.

With Martha Isabel Fandiño Pinilla, I studied certain specific aspects of learning such as the number zero, the relations between area and perimeter of bi-dimensional figures, the idea of mathematical infinity, which led me to become Chief Organizer of the Topic Group 14: Infinite processes throughout the curriculum, at the 8th ICME, Seville, July

14-21 in 1996. On this occasion, a member of the advisory panel was Raymond Duval. I have worked on the theme of the convictions of teachers on mathematical infinity in a doctoral thesis conducted in Italy and discussed it in Slovakia.

I have dedicated much energy in promoting the need of the study of the learning of Mathematics in young learners, even at nursery school, since this kind of situation affords insights into “ingenuous” learning applicable at other levels of schooling. Together with Francesco Speranza, I have dedicated years of research and experimentation to classroom practice with regard to the history of Mathematics and this has led to an ongoing production of articles on the epistemology of Mathematics, in progress today.

In 1986, I founded an annual national (soon to become international) conference which every year in November attracts thousands of participants and to which some of the most famous international researchers have contributed and which reaches its 27th edition this year. At the same time, I founded and directed for over 20 years a journal on Mathematics Education which was closed when I moved from Italy to Colombia and which has published research by the major international scholars and reached a B international classification.

My research has also contained elements of Ethnomathematics and analysis of key terms such as “competence”, together with Martha Isabel Fandiño Pinilla and Juan Godino. A particularly surprising result came from a long-term study of spontaneous demonstrations produced by 9th and 10th grade students from which emerged that some demonstrations considered unacceptable by teachers were so far the simple reason that their logic was not Aristotelian but rather that of the Indian *nyaya*. This logic was much more concrete and closer to the basic needs of the students who tried to anchor their reasoning to example and thesis, considered as point of departure (elements common to *nyaya*) rather than to logical deduction (in turn, based on material implication).

I have also worked at length on the difficulties encountered by students in the cognitive construction of mathematical objects, identifying tools for interpreting, describing, and evaluating errors (e.g., work on TEP together with Hermann Maier). I have participated with seminars and papers at international conferences in Europe, America, and Asia. I have particular pleasure in recalling numerous occasions working with my colleague and friend Athanasios Gagatsis, first in Thesaloniki and then in Nicosia, with whom I share a deep interest in the field of representations and semiotics.

I have always been against the aberrant notion that the emergence of a new theory announces the death of previous theories. In my opinion, although significant and profound theories have more recently developed (to which I have also contributed, including the EOS of Juan Godino), the foundational theories remain the basis of the Mathematics Education. First and foremost is the theory of situations by Guy Rousseau, a historical foundation of our discipline. In this respect, I have always sought to give new value to the foundational disciplines through a demonstration of their reciprocal coherence and moreover, necessity, as regards specific problems of interpretation of classroom situations, the concept that has been central to my 42 years of research.

I have constantly studied the convictions that students have about Mathematics and about their way of operating in Mathematics, immediately understanding the need to place at the centre of the interpretation of classroom situations the convictions of teachers, an area to which I have dedicated many years of research, together with Martha Isabel Fandiño Pinilla, and for which I have developed tools of analysis. The convictions of teachers determine the nature of mathematical work in the classroom and exert massive influence on the convictions of students.

Finally, fascinated by the studies first by Raymond Duval and then by Luis Radford, I began to dedicate my work to the multiform presence of semiotics within the process of teaching and learning in Mathematics. I searched the history to study the evolution of Mathematics for examples and stimuli regarding the definition of the various mathematical objects in order to analyse them first from an epistemological and then from a didactic point of view, trying to identify opportune definitions of "mathematical object". I have long studied, together with Martha Isabel Fandiño Pinilla, the variations in meaning that teachers and learners attribute to different representations obtained through transformations of treatment carried out. In this field, we have published various articles, participated in international conferences and supervised two doctoral theses, in Italy and in Colombia, while still retaining that the question remains partially open.

At this point, I was ready to examine my deepest philosophical convictions, moving from a modern but ingenuous realism to a mature pragmatism in which today I firmly believe and which is embedded in my having placed anthropological theories at the base of my description of didactic phenomena. I have always devoted time to the diffusion and popularizing of Mathematics, trying to reach students and adults who do not appreciate Mathematics simply because they do not know it. In this field, in recent years together with Martha Isabel Fandiño Pinilla and other colleagues, I have written numerous books and also received prizes.

I have also always been fascinated by the language of Mathematics and how it may be understood as a basis, driven not just by erudite motives, such as the unity of human culture and the refusal of the attempt, to divide learning into "two cultures", but also by the fascination that poetry and figurative arts have had for me from an early age.

AN EXAMPLE: DANTE AND THE MATHEMATICS

Thus, I have devoted quite a few of my studies to the presence of Mathematics in the works of Dante Alighieri and in particular in his monumental jewel of universal poetry, the "Comedìa" (the Divine Comedy). This has led to the publication of numerous articles and books as well as to the presentation of papers at international conferences. Seven hundred years after the writing of this monument that enhances human knowledge, I provided guidance on the interpretation of certain parts that had remained hidden, due to the mathematical ignorance on behalf of critics and historians of literature. Today, many of my interpretations have been accepted by experts on Dante and I have been able to use these results to reiterate the groundlessness of the division of cultures. In the Middle Ages, Dante was able to use the (elementary) Mathematics of the time in describing nature and human feelings, theology and logic, to use metaphors and to narrate in a way far more profound than the mathematical pseudo-culture of certain writers allows today. Thus, this is another reason to understand and to use Mathematics especially on the part of the Humanists.

Figure 3. The cover of an old edition of the Divine Comedy (Alighieri, 1907).

Figure 4. Dante e it re Regni, (Di Michelino, 1465).

Let us take a couple of examples of how Mathematics can help understand certain lines of the “Divine Comedy”, previously obscure to literary critics who were not fond of this discipline. As a first example, we can consider an arithmetic reference in Paradiso XXVIII, lines 91 to 93:

> ...And every spark behind its fire did speed;
> Thousands there were beyond the numbering
> To which the doubled chessboard squares will lead...

The great number referred to is that of the angels who are born in succession, instant by instant, witnesses to the glory of God, counted not by doubling the number, but thousand by thousand. How great is this number of angels? Dante states that their growth by thousands is beyond the doubling of the chessboard squares. This is clearly a reference to the famous legend of Sissa Nassir, the inventor of chess. As a reward from his enthusiastic Sovereign, he asked for something apparently modest: on a chessboard eight squares by eight, he asked for a grain of rice on the first square, doubled on the second square to make two, doubled on the third square to make four, doubled on the fourth square to make eight, and so on until the last and sixty-fourth square. The calculation today is relatively easy, especially with the use of a calculator, but with the Roman system becomes, to say the least, arduous, and the grains due to Sissa Nassir are 18 446 744 073 709 551 615, a number almost unreadable. Today, with a more compact method of writing, we use the so-called scientific notation: $1.8447 \cdot 10^{19}$.

In order to have an idea of the enormity of this number we can imagine to distribute the grains over the whole surface of the Earth, the measure of which, expressed in current terms (and not in those of Dante’s age) including seas, oceans, deserts, mountains, glaciers, etc. is around $5.0995 \cdot 10^{18}$ cm². As we distribute them, we find 3,62 grains (let us say, approximately three and a half) for every cm² of the Earth. (Thus it is clearly the reason why the Sovereign felt himself teased and rather than award Sissa Nassir his prize he cut off his head instead thereby obtaining a considerable saving by withholding the gift).

But the number of angels, rather than doubling, increased by thousands. If we proceed with this calculation (1 grain on the first square, 1000 on the second, 1000000 on the third, 1000000000 on the fourth, and so on), we have an immense, but still finite, number: 10^{189} (in other words, $2 \cdot 10^{170}$ angels for each cm^2 of the Earth. We may indeed be thankful to the angels for being immaterial!). Yet, beyond this jovial aspect of the story, there are two elements of great interest.

The first is that Dante could have said that the angels born in succession, instant by instant, witnesses to the glory of God, are infinite. By comparison to the infinite, even an immense number like 10^{189} is a drop in the ocean. The choice of a very great number has more impact than the adjective “infinite”. Paradoxically, the number provokes thought more than the infinite.

The second is that many authors assert that Dante was not aware of the Arab-Indian numbers that had begun to circulate in Europe but were known only among a few scholars. But to get an idea of the immense value of those angels, it is necessary to use a positional system, not like the Roman system, which lacked that characteristic. A simple research in the libraries of Florence shows that the second son of Dante, Jacopo, was a student at one of the three Florentine public schools, Santa Trinita, where he studied Mathematics under the guidance of the prestigious scholar Paolo dell'Abbaco, who undoubtedly taught the obligatory Roman system, but also offered to his brighter students the basics of the new Arab system that was circulating precisely in Tuscany. Today we can't by no means be so sure that Dante was unaware of the positional arithmetic system!

The second example is certainly one of the most famous mathematical references in Dante (Paradiso XXXIII, lines 133 to 138):

> ...As the geometer in thought will strain
> To measure out the circle, nor can tell
> The principle he lacks, so toils in vain,
>
> Such was I at this new seen miracle;
> I longed to see how image and circle blend
> And how the image comes therein to dwell...

The "new seen" referred to here is the direct contact between Dante and God, through sight. The "Divine Comedy" is almost over and we are at the final lines. The Poet has travelled through "Inferno, Purgatorio and Paradiso". Shortly his journey will be over and he will return to Earth. The final part is the great fortune for him to have had visual contact with God.

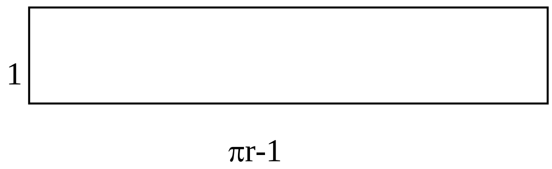
He must find a metaphor that enables him to explain the greatness of what is happening and to correlate the "new seen" with something capable of rendering the idea. And so he turns to geometry, to that "to measure out the circle".

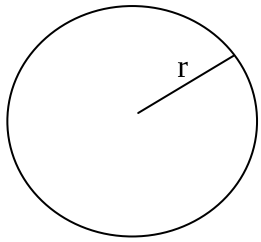
The metaphor is by no means simple, but has been erroneously interpreted for centuries. One of the most common critical texts offers the following exegesis: "like the geometer who concentrates all his mental faculties on the *insoluble problem* of squaring the circle" (emphasis added), "such was I before that extraordinary vision, that in vain" (Par, 33, 133-138).

What exactly is the problem of the squaring of the circle? The question can be posed in two complementary ways: (1) given a circumference, find a square or a rectangle whose perimeter has the same length as the circumference; and (2) given a circle, find a square or a rectangle that has the same area as the circle.

This problem was brilliantly solved in Ancient Greece, for example, by Dinostratus in the fifth century B.C. (but not only by him). It was something well known among educated people, not only Mathematicians as it was also explained by Plato.

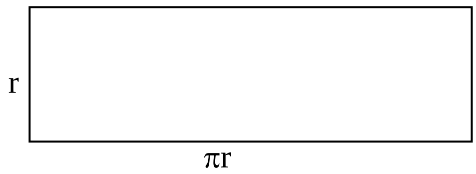
From a more modest scholastic point of view, everyone, towards the end of the primary school, can remember having learned that a circumference with radius r measures $2\pi r$. Therefore, if we take a rectangle with sides 1 and $\pi r-1$ (with $r > 1/\pi$), the lengths of the circumference and the perimeter are the same. In this way, as any ten-year old child knows, the area of a circle with radius r is πr^2 and so a rectangle with sides πr and r will have an area equal to that of the circle.

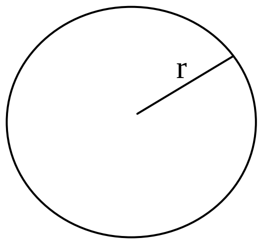




perimeter of the rectangle: $2\pi r$

measure of the circumference: $2\pi r$





area of the rectangle: πr^2

area of the circle: πr^2

Where, then, lies the impossibility of the problem? In Dante, something is implicit. It has always been well known that the Greek Mathematicians preferred ruler-and-compass solutions (a way of expressing something that goes beyond the mere reference to the two instruments, but which we can take here as a meaning working indeed with an (unscaled) ruler and a compass). The solution offered by Dinostratus and those of other Greek scholars are correct but they were not obtained by using ruler and compass.

For many centuries, unsuccessfully, the Greeks and subsequently all other Mathematicians tried to square the circle with these instruments. Today, we know that this is impossible, as demonstrated by Carl Louis Ferdinand von Lindemann only in 1882. The Greeks must have supposed it, even though implicitly. It is by no accident that the three most loved and studied problems (the “three classical problems of Greek geometry”, quoted by Plato) were perennially examined: the quadrature of the circle, the duplication of the cube, and the trisection of an arbitrary angle.

The question is, since Dante does not explicitly say “with ruler and compass”, did he fall into the same error of modern critics, or did he know the question well and imagined his readers also knew of it likewise so as to consider further erudition superfluous? We will never have an answer to this question. The geometrical knowledge widely disclosed by Dante in many parts of his works leads me to venture that we are in the presence of another example of the current defeat of the unity of culture. In Dante, the "two cultures" co-existed, something, alas, that is not like that for many readers today, not only non-Mathematicians but also anti-Mathematicians.

ANOTHER EXAMPLE

In the same way, I have devoted a part of my study to the relationships between Mathematics and figurative arts. I have studied this relationship between their languages by not only by seeking out authors and works that obviously lend themselves to mathematical interpretations, such as the decorative Arab tiles at the Alhambra in Granada but also the works of the Italian and German Renaissance, the work of Maurits Escher or Oscar Reutersvärd to name a few. My idea was to study the entire history of art and to go beyond the typical literary, philosophical or psychological critical interpretations and to build new ones which are rational, mathematical and formal.

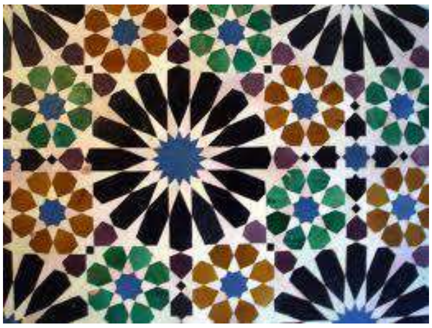


Figure 5. A tile Arabic geometric medieval (D'Amore, 2015, p. 158).

Figure 6. A typical geometric design of Maurits Escher (D'Amore, 2015, p. 428).

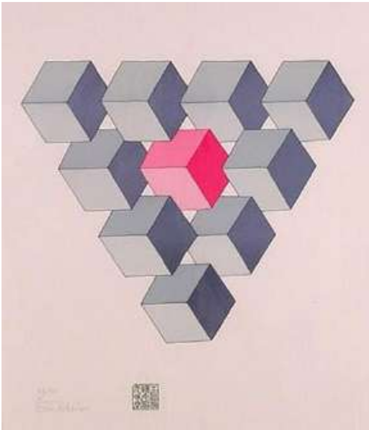


Figure 7. A typical geometric design of Oscar Reutersvärd (D'Amore, 2015, p. 454).

In this way, I was able to build a dialogue with critics and artists in order to influence a new way of practicing and writing art criticism, organizing international exhibitions in the field of conceptual art, offering a perspective known as "exact art" which was subsequently taken up by well-known artists. The name "exact art" evokes, within art, the fact that mathematics, within science, is known as "the exact science":

exact art : art = exact science : science

On this theme, I wrote hundreds of books and articles and my latest work is about to be published, even though its size (1000 pages and 1000 full-colour images) poses evident editorial problems.

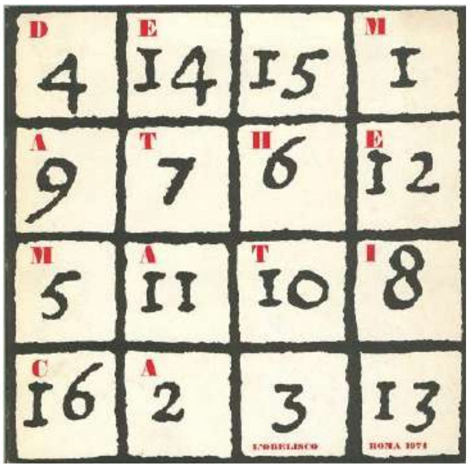


Figure 8. Cover of D'Amore and Menna (1974).

Even though many believe that the study of impossible perspectives has its origin in the middle of the nineteenth century this is quite incorrect. Human intelligence manifests in numerous ways one of which being the love of contradiction. After thousands of years of having searched for absolute, mathematical, formal, perfect rules of perspective representation and having found them it was then decided to contradict them for the pure cultural and intellectual love of a challenge. Thus, begins another story, in reverse, the attempt to represent in the plane, and so in the picture, impossible perspectives that surprise those who discover them and amuse those who analyze them.

In 1754, a book was published by the English scholar of architectonic drawing John Joshua Kirby (1716 – 1774); its almost endless title (as was common at the time) was "Dr. Brook Taylor's Method of Perspective Made Easy both in Theory and Practice, Being an attempt to make the art of perspective easy and familiar to adapt it entirely to the arts of design; and to make it an entertaining study to any gentleman who shall choose so polite an amusement". The book was printed by W. Craighton at Ipswich, near London, and the publishers were J. Swan, F. Noble, and J. Noble. The book is itself odd in that, for example, the numbering of the pages is not always progressive. But what makes it most noteworthy is the fact that the illustrations are by the great painter, drawer and engraver, William Hogarth (1697 – 1764), author of many irreverent satirical prints that made a considerable impression at that time.

Particularly famous and often quoted is the figure found in the front, the piece entitled "Prospective Absurdities".

Figure 9. Prospective Absurdities (Hogarth, 1754).

Let us consider some of its features.

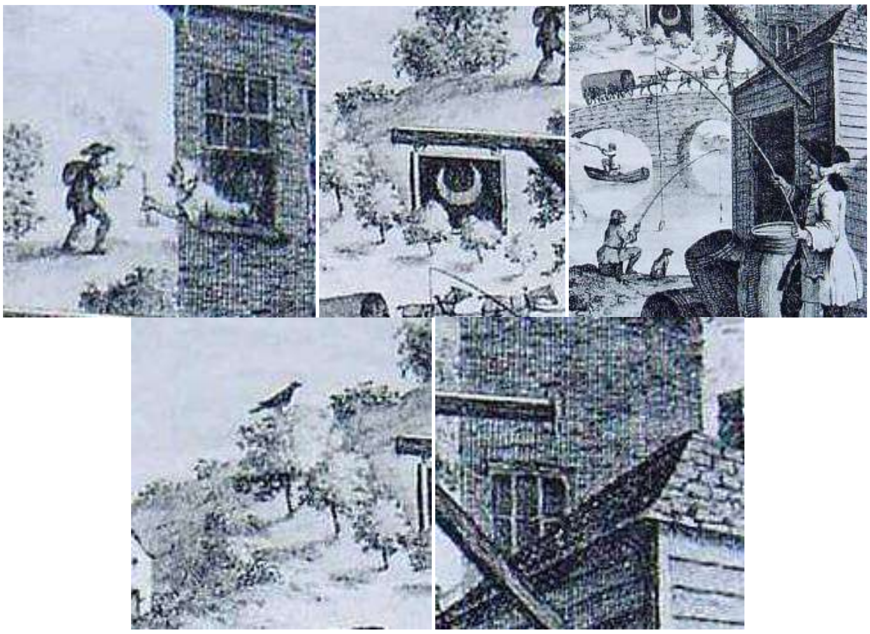


Figure 10. Some details of Prospective Absurdities.

Here the play is clear: near-far and in front-behind are inverted, also by means of a subtle play with proportions and measures. The artists most quoted in terms of impossible perspectives are certainly the Dutchman Maurits Cornelis Escher (1898 – 1972) and the Sweden Oscar Reutersvärd (1915 – 2002).

As regards perspectives and impossible drawings, the studies of the Penroses are always quoted: father (Lionel Sharples, Psychologist, 1898 – 1972) and son (Roger, born in 1931, Mathematician and Physicist, famous scholar of space-time and black holes, as well as notable narrator), in particular with reference to an article published in the British Journal of Psychology in 1958 (Penrose & Penrose, 1958) in which appears a celebrated impossible “triangle”.

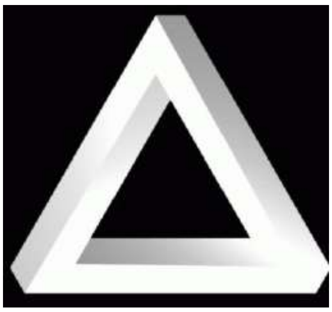


Figure 11. Tribar (Penrose & Penrose, 1958).

But the first impossible drawing by Reutersvärd dates from 1934, much before the triangle of the Penroses (1958). Among the famous optical illusions, one of the first (1832) was the Necker's "Cube", from the name of the Swiss crystallographer Louis Albert Necker (1786 – 1861), that appears in the Escher's "Belvedere".

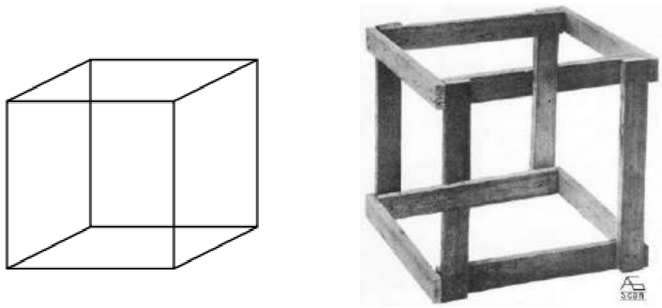


Figure 12. Albert Necker's cube (Necker, 1832).

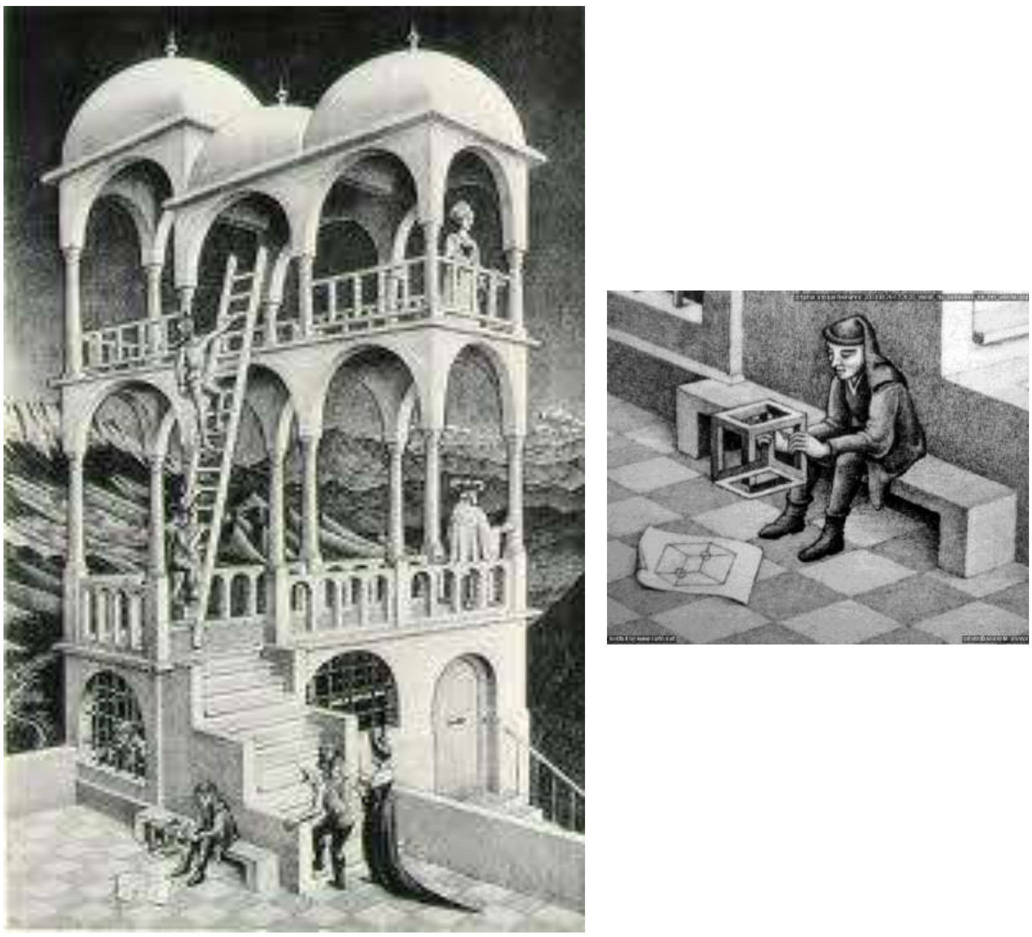


Figure 13. Belvedere, lithograph and a detail (Escher, 1956).

In the interesting book by Jan Gullberg (1936 – 1998), “Mathematics, from the birth of numbers”, published in 1997, in the chapter dedicated to Geometry, there is a reference (p. 374) to “Phantasmagorical Geometries”. Apart from a brief mention of the work of the Penroses, the discussion is based on the work of Oscar Reutersvärd.

I have also been closely bound by friendship to his family and by a work collaboration with Oscar throughout my life. My wife, Martha, and I possess a collection of several hundred original works of his.

CONCLUSION

For many decades art critics have used mathematical language to interpret the phenomenon of artistic creation and to describe the work of artists, many of whom are quite unaware of the Mathematics they use in their work. Here emerges all the descriptive and rational power of mathematical language. Beyond general motivations and concrete applications it becomes ever more important to study thoroughly and more deeply the Mathematics Education in order to understand the development of classroom situations, mathematical learning, and its multiple languages, the semiotic nuances that enable representation of objects hidden to the senses and that therefore can be communicated and made accessible only through semiotic representations in appropriate registers, via the two different semiotic transformations: treatment and conversion. The Mathematics Education is an autonomous science that in recent decades has assumed great importance. Scientific research produces ever richer results, also thanks to the contributions of other domains of human knowledge.

NOTE

This paper is the text of the Lectio Magistralis given from the author on 15 October 2013 in the Aula Magna of the University of Cyprus, Nicosia, on the occasion of the granting of the title of PhD Honoris Causa in Social Science and Education.

ACKNOWLEDGMENT

I want to dedicate this prestigious award to my wife Martha, companion, partner and accomplice, indefatigable collaborator in every adventure, both cultural and existential, someone of unmatchable human depth and extraordinary critical ability. Without her support and belief in me, all this would not have been possible. Today, as ever, any award given to me is in fact given to both of us.

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Mathematics Word Problems Illustrated: An Analysis of Flemish Mathematics Textbooks

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ABSTRACT*: In the mathematics class, elementary school pupils are confronted a lot with mathematical word problems. In their mathematical textbooks, these word problems are often accompanied with illustrations. In the present study, we systematically investigated, for three different elementary grades, the occurrence of illustrations next to mathematical word problems in four commonly used mathematics textbooks in Flanders. Apart from their occurrence, we also investigated their function and nature. The study first of all showed that the majority of the word problems in the textbooks were accompanied with illustrations, and that there was a significant difference between textbooks, and an interaction between grade and textbook. Second, half of all the illustrations were essential to solve the word problems, but there were considerable differences between grades as well as textbooks. Third, half of the illustrations were representations of a single element mentioned in the problem or the whole problem situation, whereas the other half represented the problem's mathematical structure. These percentages also differed between grades and textbooks. In the conclusion of this article, we discuss these findings and raise several questions for future research concerning the use of illustrations in mathematics textbooks by textbook makers and by the teachers in the mathematics class.*

Keywords: *Elementary school mathematics, Textbook research, Illustrations, Word problems.*

INTRODUCTION

Mathematical word problem solving is an important aspect of elementary mathematics education. It allows pupils to build new mathematical knowledge and apply concepts that they have learned in class. Furthermore, it allows pupils to apply and adapt a variety of appropriate strategies to solve problems that arise in the mathematics class and in other contexts (NCTM, 2000; Verschaffel, Greer & De Corte, 2000). Research however shows that pupils tend to have difficulties with solving word problems and that many errors are due to pupils' reliance on superficial coping strategies (Verschaffel & De Corte, 1997; Verschaffel et al., 2000). Pupils for instance tend to only look for the key words and the given numbers in the problem, instead of reading the whole problem and trying to build a coherent internal representation of it.

These mathematical word problems are most of the time presented to pupils in their textbooks for elementary mathematics. Moreover, as one opens a typical mathematics textbook in Flanders, one can see that they are often accompanied with illustrations. The inclusion of illustrations in mathematics textbooks or in any other textbook can serve different functions (Berends & Van Lieshout, 2009; Elia & Philippou, 2004; Levie & Lentz, 1982; Levin, 1981). Sometimes, they have nothing to do with the information or problem given in the text; they merely decorate the page with a view to make the textbook visually more attractive for the pupils. In other cases, they depict the information that is provided in the text or problem, with a view to help the pupils to imagine the problem situation. Still, in other cases, they provide extra information that is absolutely necessary to understand the textual fragment or to solve the problem (Berends & Van Lieshout, 2009; Elia & Philippou, 2004; Levie & Lentz, 1982; Levin, 1981).

Because of the importance of word problems in the elementary mathematics curriculum and pupils' difficulties with them, it seems worthwhile to investigate the potential impact of illustrations on pupils' word problem solving processes. So in the present study, we analysed: (a) how often elementary school children are confronted with illustrations next to word problems in their mathematics textbooks, (b) what functions these illustrations fulfil, (c) what the nature of these illustrations is, and (d) whether the occurrence, function and nature of these illustrations differs between grades and textbooks.

To make the study feasible, we restricted ourselves to three different grades and four different mathematics textbooks. Before presenting the research objectives and theoretical framework, we first give a short overview of the literature about the inclusion of word problems in the mathematics class and about the use of illustrations in mathematical textbooks.

The Inclusion of Word Problems in the Mathematics Class

Word problems are defined by Verschaffel et al., (2000, p. ix) as:

> Verbal descriptions of problem situations wherein one or more questions are raised the answer to which can be obtained by the application of mathematical operations to numerical data available in the problem statement. In their most typical form, word problems take the form of brief texts describing the essentials of some situation wherein some quantities are explicitly given and others are not, and wherein the solver - typically a student who is confronted with the problem in the context of a mathematics lesson or a mathematics test - is required to give a numerical answer to a specific question by making explicit and exclusive use of the quantities given in the text and mathematical relationships between those quantities inferred from the text.

An example of a classical word problem is 'Pete wins 3 marbles in a game and now has 8 marbles. How many marbles did he have before the game?' (Verschaffel et al., 2000, p. ix).

Verschaffel et al. (2000) describe several reasons why word problems are included in the elementary school curriculum. First of all, by means of solving word problems, pupils can practice the mathematics that they may need in real life situations (application function). Second, word problems show pupils that the mathematics that they learn at school, is applicable in situations in everyday life (motivation function). Third, they also help to think creatively and develop their problem-solving abilities (thought-provoking function) or to assess these skills (selection function). And, lastly, when used at the start of a teaching/learning process they may help pupils to develop new mathematical concepts (concept-formation function). Many authors consider the first function, namely the application function, as the historically most important one (Verschaffel et al., 2000). So, word problems are essentially used in the mathematics class as a sort of substitute for problem situations with which pupils will be confronted with in everyday life.

The ideal process of solving word problems involves a number of phases (Verschaffel et al., 2000). As shown in Figure 1, first, learners have to understand the described situation and come to a situation model. Second, based on the situational model, they have to construct a mathematical model. Third, learners have to work through the mathematical model using disciplinary methods to derive some mathematical result(s). In the fourth phase, they have to interpret the mathematical outcome(s), and lastly they have to communicate the interpreted result(s) in the form of an answer to the initial problem. This modelling process is not a strictly linear process; it rather has to be considered cyclic (Blum & Niss, 1991; Burkhardt, 1994; Verschaffel et al., 2000). So, ideally, when learners solve word problems, they should go through all the phases of the modelling cycle.

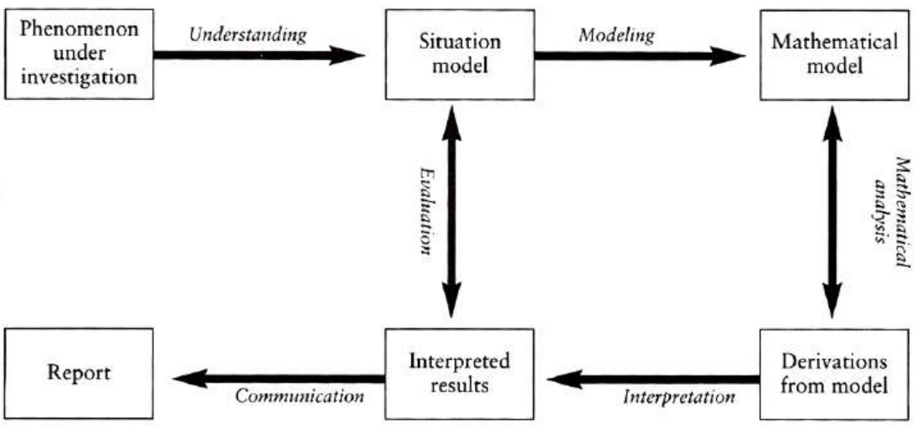


Figure 1. Schematic diagram of the mathematical modelling process (adopted from Verschaffel et al. (2000, p. xii)).

Learners tend to have difficulties with solving mathematical word problems and often make many errors. There is quite some research evidence showing that many of these errors are due to superficial coping strategies, in which learners tend to immediately do some calculations with the numbers in the problems, without trying to construct a meaningful representation of the problem situation and without checking afterwards if the outcome of these calculations make any sense in relation to the problem statement (Verschaffel & De Corte, 1997). Well-documented examples of such superficial coping strategies are performing the arithmetic operation or mathematical formula with the numbers in the problem that was taught most recently, or looking for certain key words in the text of the word problem that are associated with a certain arithmetic operation (e.g., the words ‘more’ or ‘altogether’ being associated with addition versus the words ‘less’ or ‘lose’ that are associated with subtraction).

Closely related to pupils’ reliance on superficial coping strategies is their tendency to exclude real-world knowledge and considerations when trying to make sense of a word problem and looking for its mathematical solution. This tendency was experimentally shown by Greer (1993) and Verschaffel, De Corte, and Lasure (1994), who presented elementary school pupils (age 10-11) with a set of problematic word problems in which the reality of the context should be taken in to account to come to a situationally meaningful mathematical model and ultimate answer. An example of such a problematic item is: ‘A man wants to have a rope long enough to stretch between two poles 12 metres apart, but he has only pieces of rope 1.5 metres long. How many of these pieces would he need to tie together to stretch between the poles?’ This problem cannot be solved by dividing 12 by 1.5. One has to take into account that some extra rope is needed to tie the knots and tie the rope around the pole, so the answer more than eight ropes would be considered as a realistic reaction. Verschaffel et al. (1994) found only 17.0% realistic reactions to these problematic items. These findings were afterwards replicated by different authors and in various countries (for an overview see Verschaffel, Greer, Van Dooren, & Mukhopadhyay, 2009). In sum, research has amply documented that many pupils approach word problems without the inclination to construct a meaningful representation of the problem situation described in the problem text and to interpret the outcome of their computational work in terms of the original problem statement.

The Use of Illustrations in Textbooks

Illustrations are omnipresent in educational textbooks. They cover a large percentage of the pages. Evans, Watson and Willows (1987) calculated the percentage of pages with a non-text configuration (i.e., illustrated pages) of the total number of pages from Canadian reading textbooks, mathematics textbooks and science textbooks from grade 1 to grade 12. They found that primary grades textbooks are highly illustrated: 90% to 100% of the pages had at least one illustration. This percentage of illustrations gradually declined as the grade level increased. For the intermediate level (grade 7-10) the percentage was 50 to 70%, and for the senior level (grade 11-12) 30 to 60% (Evans et al., 1987). Furthermore, over the generations there seems to be an increase in the use of illustrations in textbooks. Berglund (as cited in Pettersson, 2002) observed that 'for each generation the pictures of the textbooks have become more in number, larger, more elegant, and more colourful' (p. 246).

The inclusion of pictures in textbooks, or books in general, may have various reasons. Many categorisations of these functions have been proposed (Berends & Van Lieshout, 2009; Elia & Philippou, 2004; Levie & Lentz, 1982; Levin, 1981). For instance, based on a literature review, Levin (1981) proposed eight functions of illustrations accompanying prose. Pictures can have a *decoration function* (i.e., enhance a book's attractiveness), a *remuneration function* (i.e., increase publisher's sales), a *motivation function* (i.e., increase children's interest in the text), *reiteration function* (i.e., provide additional exposure to the text), a *representation function* (i.e., make the text information more concrete), an *organization function* (i.e., make the text information more integrated), an *interpretation function* (i.e., make the text information more comprehensible), and a *transformation function* (i.e., make the text information more memorable).

Another categorisation is the one of Levie and Lentz (1982), who made a distinction between the *attentional*, *affective*, *cognitive* and *compensatory function*. They indicate that pictures first of all may attract attention to the material or within the material. Second, they may enhance enjoyment of the reader and affect emotions and attitudes. Third, including pictures can have a cognitive function. For example, they can facilitate learning from a text by improving comprehension and retention of the information or they can provide additional information. A last function of pictures may be compensatory in the sense that they can help poor readers more than good readers.

Apart from these more general categorisations, Berends and van Lieshout (2009) and Elia and Philippou (2004) focus specifically on the functions of illustrations in the process of word problem solving. Berends and van Lieshout (2009) differentiate between *bare*, *useless*, *helpful* and *essential illustrations*. They define a *bare illustration* as an abstract symbolic representation of a word problem, for example 18 - 4 = _. *Useless illustrations* do not contain information to solve the word problem. *Helpful illustrations* contain numerical information that is also presented in the word problem. Lastly, *essential illustrations* contain information that is missing in the word problem but that is crucial for solving it. Berends and van Lieshout (2009) investigated the influence of these different types of illustrations on the speed and accuracy of performance of primary school children's word problem solving. They found that, across the four different types of illustrations, speed of performance dropped: Problems presented together with bare illustrations were solved the fastest whereas problems presented with essential illustrations the slowest. In addition to speed of performance, the accuracy of performance only dropped when problems were presented together with essential illustrations for which children had to look at the illustration to find the necessary information.

In the categorisation of Elia and Philippou (2004), four functions of illustrations are distinguished. They defined *decorative illustrations* as illustrations that depict something that has no link with the word problem, *representational illustrations* as illustrations that represent a part or the whole content of the problem, *organizational illustrations* as illustrations that provide directions to support the solution procedure and *informational illustrations* as illustrations that contain essential information to solve the problem. They investigated the effect of these different kind of illustrations in word problem solving and found that decorative illustrations were ineffective, whereas representational, informational and organizational illustrations had a significant positive effect on mathematical problem solving. Elia and Philippou concluded: 'The use of the pictures leaded frequently students to internal conflicts, which in turn enabled them to find correct solutions for the problems.' (p. 332).

Both the categorisations of Berends and van Lieshout (2009) and of Elia and Philippou (2004) are relevant for the present research in the sense that they focus on illustrations next the mathematical word problems and highlight some important aspects of illustrations. For instance, except for the first category of bare illustrations, Berends and van Lieshout (2009) seem to differentiate between three potential functions that illustrations may have for finding the solution of a word problem: They may be either useless for solving the problem, helpful or essential. In addition, the categorisation of Elia and Philippou (2004) seems to focus more on what is depicted in the illustrations (e.g., the whole content of the problem or only a part of it). In both categorisations, these two dimensions (i.e., the function of the illustrations and what they depict) seem to get blurred somehow, while we think that this distinction is important. Indeed, two illustrations can both depict the whole problem situation, but can differ in whether they are essential to solve the problem or not. Likewise, two illustrations that differ in nature (one representing the whole problem situation and another one representing only one element) may both be essential. For the present study, we will, as explained in a following section, make a categorisation that is strongly based on the ones of Berends and van Lieshout (2009) and Elia and Philippou (2004), but that makes a clear distinction between the function of the illustrations (i.e., their purpose) and the nature of the illustrations (i.e., what they depict).

So, as shown in the categories of Berends and van Lieshout (2009) and Elia and Philippou (2004), but also the ones of Levin (1981) and Levie and Lentz (1982), the numerous illustrations in textbooks and books in general can serve different functions: they can increase children's motivation, attract their attention, merely decorate the page. In textbooks in general, one would expect that illustrations would be particularly inserted and designed with the aim to help pupils to understand and learn the subject matter or to help them to solve a given problem. This is however not the case. Lingons (as cited in Pettersson, 2002) investigated the function of illustrations in textbooks and found that the majority of the illustrations were considered by the teachers as unnecessary; only 43% of colour photos and 43% of drawn illustrations in textbooks were interpreted as relevant to the text by the teachers. Other researchers (Levin & Mayer, 1993; Mayer, 1993; Woodward, 1993) came to the same conclusion. Mayer

(1993) for example did a survey of 6th grade science textbooks and found that of all illustrations 23% were merely decorative ones that did not enhance the message of the text while 62% were representational ones that portrayed a single element and just served to direct the reader's attention. Accordingly, only 15% of the illustrations were considered as important for learning: 5% depicted relations among elements and 10% explained how systems work. These illustrations that are important for learning can also be called organizational illustrations following the categorisation of Elia and Philippou (2004).

Research Objectives

In order to obtain a clear view on the illustrations pupils encounter in their mathematics textbooks, and specifically on illustrations next to mathematical word problems, we systematically investigated the occurrence of illustrations next to word problems in their textbooks. We tried to answer three main research questions. First, we investigated how often elementary school children are confronted in their mathematics textbooks with illustrations next to word problems, and whether there is a difference between the different grades (*Research question 1*). Second, we looked at the function of these illustrations next to word problems. More specifically, we analysed whether these illustrations are essential, non-essential or irrelevant to solve the problems, and if the distribution of essential and non-essential illustrations differed between the grades (*Research question 2*). Third, we investigated the nature of the illustrations. More specifically, we analysed what element(s) or aspect(s) of the word problems are depicted by these illustrations (i.e., a single element of the problem situation, the global problem situation, the mathematical structure hidden in the problem). Again, we also looked for differences between the grades (*Research question 3*).

These three research questions were investigated by analysing four different textbooks. Apart from investigating these research questions, we also checked whether there were differences between these four textbooks concerning the occurrence, the function, and the nature of the illustrations.

METHOD

Analytic Framework

To empirically investigate the occurrence, function and nature of illustrations next the word problems, the concepts 'word problem' and 'illustration' need to be operationalized. First, for the operationalization of the word problems, we started from Verschaffel et al.'s (2000) above-mentioned definition. The three important aspects of this definition are that: 1) there has to be a verbal description of a problem situation, 2) there must be a question, and 3) in order to find the answer one has to perform at least one arithmetic operation with (some of) the numerical data in the problem statement. The further operationalization of this definition led to the following list of more specific criteria to be used for deciding whether or not a certain task in a textbook was to be considered as a word problem or not.

- There has to be a description of the problem situation. This description can be verbal or pictorial and can be part of the problem text or the heading accompanying the problem.
- As part of the description of the problem situation, some numerical data should be given. This information can again be given in a verbal or pictorial way, but also in the form of a table and a figure. It is also possible that the information is measurable in the classroom.
- There must be a question. This question can be explicitly asked or be implicitly apparent from the description of the problem.
- In order to find the answer one has to perform at least one arithmetic operation with (some of) the numerical data in the problem statement. This arithmetic operation can be given, however only as an empty formula or with only one number filled in, not with all the numbers given.

Second, we created a categorisation for the concept 'illustration'. We considered all the visualizations or depictions such as drawings, sketches, paintings, photographs (i.e., graphical representations) that were presented next to the word problems as illustrations. Also, tables and figures were considered as illustrations. For reasons that were explained above, rather than adopting one of the existent categorisation schemes for categorising illustrations, we developed our own system wherein we made a distinction between the function and nature of the illustrations. This categorisation was mainly based on the one of Berends and van Lieshout (2009) and Elia and Philippou (2004).

Concerning the function of the illustrations, we distinguished three categories: irrelevant, non-essential, and essential. Illustrations were categorised as *irrelevant* when they had no connection whatsoever with the situation described in a word problem. When there was a connection with the situation described in the word problem, but the illustration was not necessary to solve it in the sense that it did not contain information that was not given in the word problem itself, but was needed to find the solution, it was categorised as *non-essential.* Lastly, when the illustration next to the word problem had a connection to the situation described in the problem, and when it was absolutely essential to solve the problem in the sense that it contained some numerical information needed to find the solution to the problem, that was not present in the word problem itself, the illustration was categorised as *essential*. For an example of each of these three categories see Table 1.

Table 1

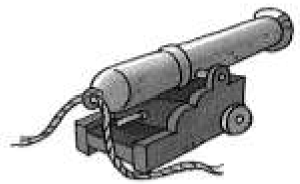
Examples of the Different Functions of Illustrations

Function	Word Problem	Illustration
Irrelevant	In every box, there are 10 bags of cookies. You filled three boxes and have 4 bags of cookies left. How many more bags do you need to fill in the box?	
Non-essential	Today there were 453 supporters for the competition. Last week there were 472 supporters. How many supporters were there in total for both competitions?	
Essential	There is a discount on the wine. Father buys 36 bottles. How much does he have to pay?	 * Buy 4 bottles, get 2 bottles for free

Concerning the nature of the illustrations, we only categorized the illustrations that were non-essential or essential. We used the following categorisations: one element, global situation, and structure (chunking, table, and graph). The categorisation *one element* was given to the illustrations (either non-essential or essential) that depicted one object from the word problem. When an illustration provided a depiction of the overall problem scene, or at least more than a single element, it was categorised as *global situation*. Lastly, when an illustration was designed in such a way that (part of) the mathematical structure underlying the word problem was shown, it was categorised as *structure*. A distinction was made between chunking, table, and graph. Illustrations were categorised as chunking when (some) objects in the illustration that were part of the question, were organised in different groups and consequently made the mathematical structure more clear. When the illustration was in the form of a table (i.e., an organised set of data, typically displayed in columns and rows) it was categorised as table. Lastly, the categorisation graph was given when the word problem was accompanied with ‘a diagram that represents a system of connections or interrelations among two or more things by a number of distinctive dots, lines, bars, etc' ('graph', n.d.). See Table 2 for examples of the different categories of the nature of the illustrations.

Table 2

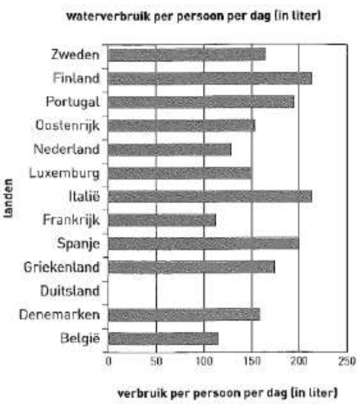
Examples of the Different Natures of Illustrations

Nature	Word problem	Illustration
One element	Captain One-eye paid 1680 gold pieces for 5 cannons. How much does each cannon cost?	
Global situation	Tom went to the store. In his shopping bag, he has: 1kg 250g leek, half a kg meat, 500 g grapes, and 800g bread. The shopping bag itself weights 700 g. How much weight does Tom has to carry?	
Structure		
Chunking	Lisa has 24 lollipops. She divided them in bags of three. How many friends can she give a bag with lollipops?	
Table	Next Saturday Yasmin goes with her mother, father and sister to the zoo in 'Antwerpen'. They live in the centre of 'Korbeek-Lo' and want to go to the station of 'Leuven' by bus. Then, they will take the train to 'Antwerpen'. How long does it take to go from 'Korbeek-Lo' to the station in 'Leuven'?	

8 BIERBEEK, CENTRUM – LEUVEN – BERTEM, ROND PUNT
9 KORBEEK-LO, PIMBERG – LEUVEN – BERTEM, ROND PUNT

zaterdag		
	8	9
Bierbeek De Bone		
Korbeek-Lo Pimberg		8:48
Bierbeek Sint-Camillus		|
Korbeek-Lo Station		|
Korbeek-Lo Centrum		8:50
Bierbeek Brent	8:23	|
Lovenjoel Bruul	8:27	|
Lovenjoel Gemeentehuis	8:28	|
Lovenjoel Salve Mater	8:29	|
Lovenjoel Zoet. Centrum	8:30	|
Korbeek-Lo Osselaan	8:37	8:52
Leuven Spaanse Kroon	8:40	8:55
Leuven Elisabethlaan	8:42	8:57
Leuven Station	8:50	9:05
Leuven Fochplein	8:54	9:09
Lijn(en) 3301, 3303, 3313	Info: 016 313737	

vertrekuren van treinen uit Leuven		
vertrek	bestemming	aankomst
9:19	Hasselt	10:41
9:29	Luik	10:05
9:33	Brussel-Centraal	10:03
9:40	Antwerpen-Centraal	10:27

Graph	How much is the difference between the highest and lowest water consumption in litters per person and per day?	

Textbooks

To find an answer to our research questions, we browsed through four commonly used mathematics textbooks in Flanders; Textbook A, B, C and D. This selection was based on the study of Avau and Thomassen (2011), which revealed that these were among the most commonly used textbooks for elementary school mathematics in Flanders (together representing 70% of the textbooks used in Flanders).

Between these four textbooks there is expected to be a big similarity in the nature and sequence of the learning contents, because they are in accordance with the attainment targets. These targets are minimum objectives that pupils should achieve at the end of primary school ('Ontwikkelingsdoelen en eindtermen', 2010). From these four textbooks we selected the booklets for the 2^{nd} (age 7-8), 4^{th} (age 9-10) and 6^{th} (age 11-12) grade.

Data collection

In each textbook, we first looked for every item whether it was, according to our definition, a word problem or not. If an item was considered as a word problem, we coded if it was accompanied by an illustration or not, based on our definition of illustrations. Furthermore, for all the coded illustrations, we first coded their function (i.e., irrelevant, non-essential or essential), and, second, for the non-essential and the essential illustrations, also their nature (one element, global situation or structure-chunking, structure-table or structure-graph).

Furthermore, textbooks sometimes present series of word problems accompanied by one single illustration next to this set of (related) word problems. We then coded the illustrations in relation to the word problem to which they belonged, so each problem was coded separately using each time the same common illustration. In some of these problem chains, the next problem builds on the outcome(s) of the previous one. Stated differently, the unknown of the previous problem acts as a given in the next one. In such cases, we based our coding of an illustration for a given problem on the assumption that the pupil had solved the previous problem correctly, and, consequently, that the pupil had access to all required numerical information to correctly solve the problem at hand. It is therefore possible that the function and nature of the same illustration accompanying a chain of word problems changes depending on the problem. For instance, an illustration can be coded as 'essential' for the first problem in a chain if a number that is needed to solve the problem was actually given in that illustration, but as 'non-essential' for the next problem when that problem could be solved by means of the given numbers in the text and the number that acted as the solution of the previous problem, without the necessity to look in the illustration for a missing number.

It is also possible that a single word problem or a chain of related word problems is accompanied by several illustrations. In these cases, only one code was given to the word problem and the accompanying illustrations. When a single word problem was accompanied by more than one illustration, it was first of all possible that the illustrations have the same function and nature. In those cases, there was, of course, no coding problem. A second possibility was that the illustrations have a different function. When one of the illustrations was essential to solve the problem, we always took the code of this illustration (its function and nature). Take for instance a word problem that is accompanied by a table (or graph) that is essential to solve the problem and by a non-essential one-element illustration or an illustration that depicts the global situation – a combination that was quite common in the textbooks we explored. In that case we scored it as essential and scored the nature of that essential illustration (for the example above the code would be essential, structure table (or graph)). When a non-essential illustration was combined with an irrelevant one, we choose to code the non-essential one, and thus also coded the nature of that non-essential illustration. A third possibility, which occurred extremely rare, was that the function of the different illustrations accompanying a given word problem was the same but their nature was different. For instance, an essential one-element illustration and an essential global situation illustration. In those rare cases, the researchers made a joint decision as to what code was most appropriate for its nature. When a chain of word problems was accompanied by multiple illustrations, we again coded the illustrations in relation to the word problem to which they belonged. Each problem was coded separately and we looked which specific illustration was related to solve that problem. This illustration then was coded for that word problem. So for example when there were several word problems accompanied with two essential structured illustrations, one a graph and one a table, for the word problems that were related to the graph the illustration was coded as essential, structure-graph, and for the problems that were related to the table the illustration was coded as essential, structure-table.

Furthermore, some textbooks made intensive and systematic use of specific instructional pictograms, such as a pictogram of a clock (meaning that the problem should be solved under time pressure), a pictogram representing a pair of children (meaning that the problem should be solved in pairs) or a pictogram of a calculator (meaning pupils are allowed to use their calculator). Likewise, some textbooks contain illustrated characters (a fictitious boy and/or girl, who show up from time to time to give the pupils a warning or a concrete hint). These pictograms and illustrated characters were excluded from our analysis because of their instructional function and differences between textbooks.

Analyses

Before analysing the data, the reliability of the coding was checked. For the coding of the word problems it was not possible to compute the inter-rater reliability. During the coding, whenever there was doubt about an item being a word problem or not, this item was discussed between the researchers until agreement was reached. For the coding of the illustrations, the inter-rater reliability was calculated for the number, the function and the nature of the illustrations. It was calculated for 20% of all the booklets (i.e., three or four booklets per grade for each textbook). Concerning the question if a word problem was accompanied by an illustration or not, there was an almost perfect agreement between the researchers, kappa=.976. But also concerning the function and nature of the illustrations, the agreement was very high, respectively kappa= .918 and kappa =.943.

The actual data analysis consisted of three parts. First, we investigated the number of illustrations next to the word problems. We did a descriptive analysis to see how often word problems were accompanied with illustrations and to compare the number of illustrations in the different grades and textbooks. By means of a logistic regression analysis (Generalized Linear Models procedure or GLM) we examined whether the differences between the grades and textbooks were significant.

Second, we investigated the function of the illustrations next to the word problems. A descriptive analysis was performed to reveal the functions of the illustrations, first overall, and afterwards for the different grades and textbooks. Again a Generalized Linear Models (GLM) analysis was performed to examine for significant differences in the function of the illustrations in the different grades and textbooks.

Third, we analysed the nature of the illustrations. We did again first a descriptive analysis, and afterwards a GLM to see if there were significant differences between the grades and textbooks.

Lastly, we also did an additional analysis in which we calculated chi-squares to investigate whether there was a connection between the function and nature of the illustrations.

RESULTS

The Number of Illustrations

First, altogether there were 2746 word problems and 2070 illustrations. Thus, the total percentage of illustrated word problems was 75.4%. So, in the textbooks, the majority of the word problems were presented together with an illustration. Second, the percentages decreased slightly between 2nd and 6th grade: 78.1% illustrated word problems in the 2nd grade ($N_{word\ problems}$=730 and $N_{illustrations}$= 570), 76.2% in the 4th grade ($N_{word\ problems}$=858 and $N_{illustrations}$= 654), and 73.1% in the 6th grade ($N_{word\ problems}$=1158 and $N_{illustrations}$= 846). To determine if these differences were significant, a GLM procedure was performed in SPSS. The analysis revealed no significant effect, *Wald* $X^2(2,2746)=0,11$, $p=.944$, leading to the conclusion that in each grade the number of illustrated word problems was about the same.

When looking at the percentage for each textbook, we noticed considerable differences between textbooks: Textbook A had the highest percentage of illustrated word problems, 90.4% ($N_{word\ problems}$=730 and $N_{illustrations}$= 660), whereas Textbook D had the lowest, 60.9% ($N_{word\ problems}$=932 and $N_{illustrations}$= 568). The percentages for Textbook C and Textbook B were in between, respectively 78.6% ($N_{word\ problems}$=702 and $N_{illustrations}$= 552) and 75.9% ($N_{word\ problems}$=382 and $N_{illustrations}$= 290). The GLM analysis revealed that these differences between textbooks were significant, *Wald* $X^2(3,2746)=159.83$, $p<.001$. As the pairwise comparisons showed, they all differed significantly from each other in the number of illustrations next to word problems. So these data show that some textbooks use a lot of illustrations next to word problems, whereas others use considerably fewer illustrations.

Lastly, we also analysed whether there was an interaction between grade and textbook. The GLM analysis revealed that there was an interaction between grade and textbook, *Wald* $X^2(6,2746)=35.72$, $p<.001$ (see Figure 2). The percentage of illustrations increased per grade for some textbooks, whereas it decreased for others. In Textbook A and Textbook C, the percentage illustrations tended to increase per grade, however, the pairwise comparisons did not show any significant difference (the 2nd and the 6th grade from Textbook A differed with a p-value of .050). For Textbook B, the percentage of illustrations declined significantly from the 2nd grade to the 4th grade, $p=.004$, and the 6th grade, $p=.001$. Lastly, for Textbook D, there was a significant increase from the 2nd to the 4th grade $p=.008$, and then again a decrease from the 4th to the 6th grade, $p=.004$.

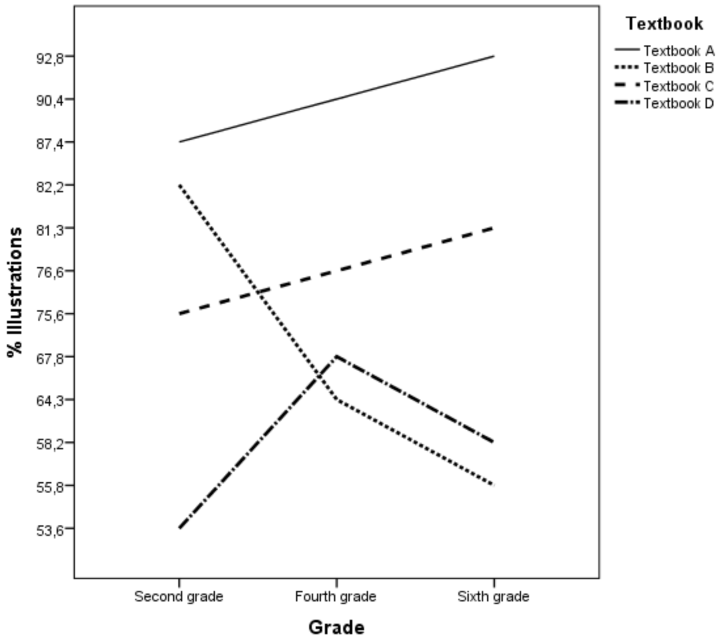


Figure 2. Interaction between grade and textbook for the percentage of illustrations next to word problems.

The Function of the Illustrations

As for the illustrations' function, we analysed the percentage of illustrations that were essential, non-essential or irrelevant to solve the problems. In general, 53.6% of all the illustrations that were presented next to the word problems were essential, 44.0% were non-essential, and 2.4% irrelevant. We furthermore looked at these percentages per grade and per textbook. Because the percentage of irrelevant illustrations is negligible, we only analysed the proportion of essential versus non-essential illustrations (In what follows we only report the data for the essential illustrations, since those for the non-essential ones are their complement).

Concerning the percentage of essential illustrations per grade, a slight increase from the 2^{nd} to the 4^{th} grade, and then a stagnation to the 6^{th} grade was visible. The percentages were respectively 40.7%, 60.4% and 60.3%. The GLM analysis revealed a main effect of grade, *Wald* $X^2(2,2021)=61.95$, $p<.001$. There was a significant difference between the 2^{nd} and the 4^{th} grade, $p<.001$ and the 2^{nd} and the 6^{th} grade, $p<.001$.

Furthermore, there were some differences between the textbooks. The textbook with the least essential illustrations was Textbook B (37.1%). Textbook A and Textbook D had somewhat more, respectively 44.1% and 61.9%, while the textbook with the highest percentage of essential illustrations was Textbook C (70.0%). A GLM analysis revealed a main effect of textbook, *Wald* $X^2(3,2021)= 59.70$, $p<.001$. There was a significant difference between Textbook A and Textbook C, $p<.001$, between Textbook A and Textbook B, $p=.001$, between Textbook C and Textbook D, $p<.001$, and between

Textbook B and Textbook D, p=.010. So the four textbooks differed substantially with respect to the frequency with which they inserted illustrated word problems wherein the illustration contained numerical information that was essential to solve the problem.

Lastly, we also found an interaction between grade and textbook, *Wald* $X^2(6,2070)=123.49$, $p<.001$. As shown in Figure 3, some textbooks tended to have more essential illustrations with increasing grade whereas in one other textbook this percentage decreased as the grade increased. For Textbook C, there was an increase in the percentage of essential illustrations from the 2nd to the 6th grade. This increase was significant from the 2nd to the 6th grade, $p<.001$, and the 4th and 6th grade, p=.007. For Textbook B and Textbook D, there was an increase from the 2nd to the 4th grade and then a decrease to the 6th grade. For Textbook B, this difference between grades was significant between the 2nd and the 4th grade, $p<.001$, the 2nd and the 6th grade, p=.003, and the 4th and the 6th grade, $p<.001$. For Textbook D, this difference was significant between the 2nd and the 4th grade, $p<.001$, and the 2nd and the 6th grade, $p<.001$. Lastly, there was a decrease from the 2nd to the 6th grade for Textbook A. This decrease was significant from the 2nd to the 4th grade, $p<.001$, and also from the 2nd to the 6th grade, $p<.001$.

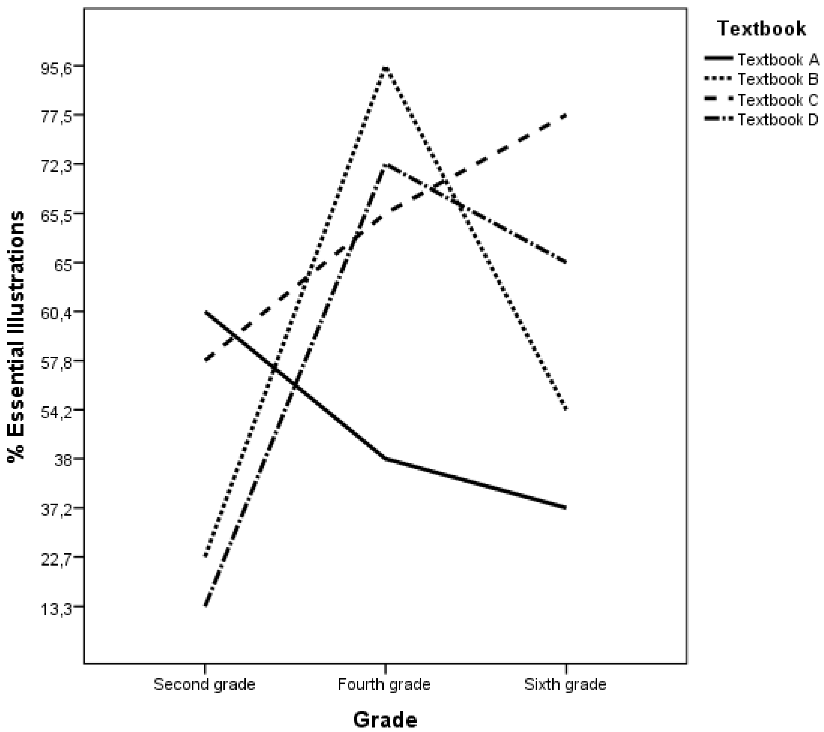


Figure 3. Interaction between textbook and grade for the percentage of essential illustrations next to word problems.

The Nature of the Illustrations

Lastly, for the illustrations that were relevant (i.e., the non-essential and essential illustrations), we analysed their nature. In general, 49.1% of the illustrations were designed in such a way that they revealed the underlying mathematical structure, 29.5% were illustrations that depicted one element, and 21.4% depicted the global situation. Of the 49.1% illustrations that were coded as structure, 26.6% were tables, 17.9% were chunking, and 4.6% were graphs. When looking at the percentages per grade (see Table 3), some differences between the three grades can be noticed. There are more one element and global situation illustrations and fewer structural illustrations in the 2nd grade than in the two other grades, whereas the percentages are quite similar between the 4th and 6th grade.

Table 3

Percentage of Illustrations per Nature and per Grade

Nature	Grade		
	Second grade	Fourth grade	Sixth grade
One element	37.9	26.6	26.0
Global situation	35.2	17.1	15.3
Structure	27.0	56.3	58.6
Chunking	*10.5*	*17.9*	*22.9*
Graph	*3.6*	*6.2*	*4.1*
Table	*12.9*	*32.2*	*31.7*

We again performed a GLM analysis to see whether the differences between the grades were significant. Concerning the illustrations consisting of one element, there was a main effect of grade, *Wald* $X^2(2,2021)=17.22$, $p<.001$. There was a significant difference between the 2nd and the 4th grade, $p<.001$, and the 2nd and the 6th grade, $p=.003$. For the illustrations categorised as structure, there was again a main effect of grade, *Wald* $X^2(2,2021)=134.25$, $p<.001$. There was a significant difference between the 2nd and the 4th grade, $p<.001$, and between the 2nd and the 6th grade, $p<.001$. For the illustrations that depict the global situation the analysis could not be performed due to too few observations in a particular category.

We also analysed the percentages of the nature of the illustrations per textbook. These percentages show that, compared to the other textbooks, Textbook C had high number of structured illustrations and a low number of one element illustrations. See Table 4 for an overview of the percentages of illustrations per nature and per textbook.

Table 4

Percentage of Illustrations per Nature and per Textbook

Nature	Textbook			
	Textbook A	Textbook B	Textbook C	Textbook D
One element	38.0	39.6	15.7	27.9
Global situation	19.9	28.6	22.7	18.1
Structure	42.1	31.8	61.6	54.0
Chunking	*11.9*	*7.5*	*30.7*	*17.6*
Graph	*3.8*	*6.8*	*1.6*	*7.4*
Table	*26.3*	*17.5*	*29.3*	*29.0*

The GLM analysis showed that, for the one element illustrations, there was a main effect of textbook, *Wald* $X^2(3,2021)=59.38$, $p<.001$. There was a significant difference between Textbook A and Textbook C, $p<.001$, between Textbook A and Textbook B, $p<.001$ and Textbook A and Textbook D, $p=.001$. For the illustrations categorised as structure, there was a main effect of textbook, *Wald* $X^2(3,2021)=30.38$, $p<.001$. There was a significant difference between Textbook A and Textbook C, $p<.001$, between Textbook A and Textbook B, $p<.001$ and between Textbook C and Textbook D, $p=.001$. For the illustrations that depict the global situation the analysis could not be performed because of too little observations for one textbook.

We also analysed the interaction effect between grade and textbook for the nature of the illustrations. For the one element illustrations and the illustrations that depicted the mathematical structure there was an interaction between textbook and grade, respectively *Wald* $X^2(6,2021)=23.00$, $p<.001$, and *Wald* $X^2(6,2021)=104.01$, $p<.001$. For the global situation illustrations, the interaction analysis could not be performed because of too little observations. Figure 4a shows the interaction between grade and textbook for the one element illustrations, Figure 4b the illustrations that depict the mathematical structure, and Figure 4c for the global situation illustrations for which no analysis could be performed.

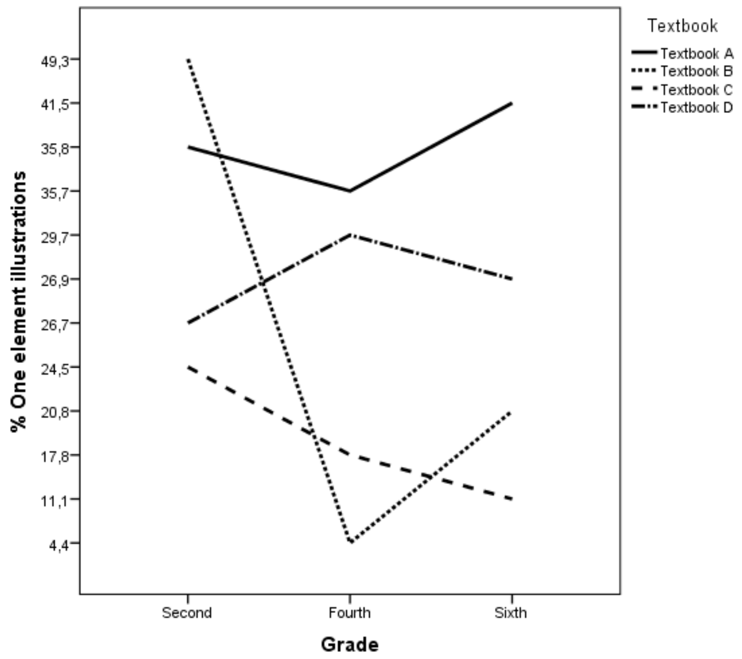


Figure 4a. The interaction between grade and textbook for the one element illustrations.

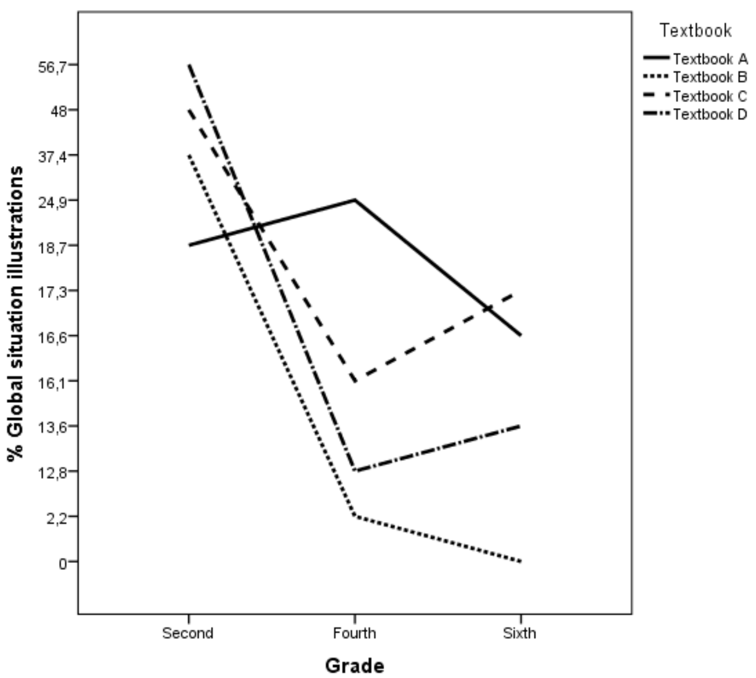


Figure 4b. The interaction between grade and textbook for the global situation illustrations.

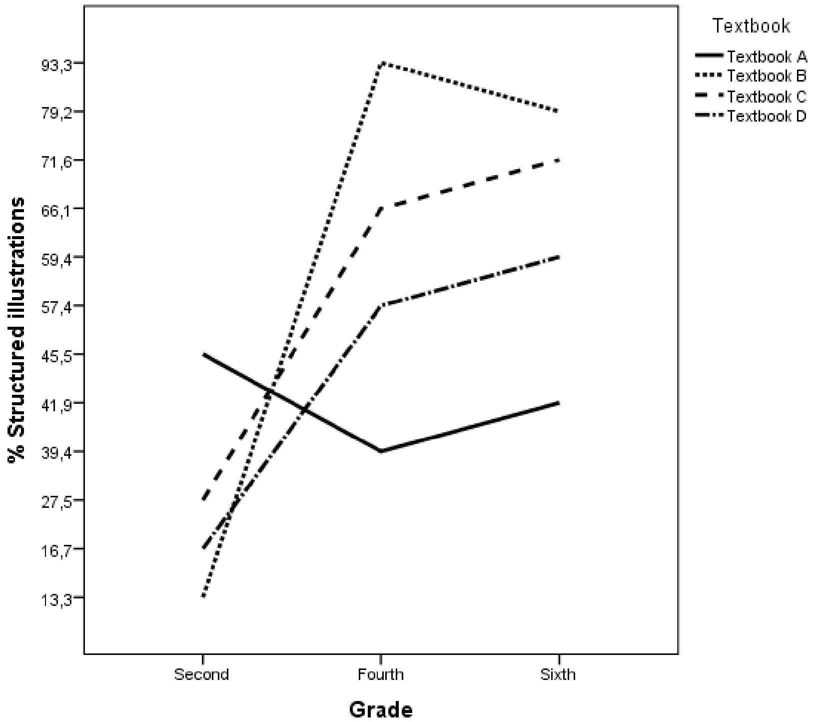


Figure 4c. The interaction between grade and textbook for the illustrations that depict the mathematical structure.

Lastly, in an additional analysis, we looked at the nature of the illustrations for the essential and the non-essential illustrations. As shown in Table 5, the majority of the essential illustrations were illustrations that showed the mathematical structure. For the non-essential illustrations, there were less structure illustrations and more one element and global situation illustrations. The chi-square test between the function and nature of the illustrations was calculated and showed a significant effect, *Wald* $X^2(2,2021)=786.97$, $p<.001$. So, this shows that the illustrations that contain information that is essential to solve the word problems depict most of the time the mathematical structure, whereas the non-essential illustrations depict most of the time one element or the global situation.

Table 5

Percentage (and Values) of the Nature of the Illustrations for the Essential and Non-Essential Illustrations

Function	Nature			
	One element	Global situation	Structure	**Total**
Essential	6.9 (140)	6.1 (123)	41.9 (847)	**54.9 (1110)**
Non-essential	22.6 (456)	15.3 (309)	7.2 (146)	**45.1 (911)**
Total	**29.5 (596)**	**21.4 (432)**	**49.1 (993)**	**100.0 (2021)**

When looking more closely at the percentages of the structured illustrations for the essential and non-essential illustrations, one can see that most of the essential illustrations are tables or chunking, whereas only few illustrations are graphs (see Table 6). For the non-essential illustrations, there are almost no tables and graphs, and so almost only chunking illustrations. The chi-square test between the function and the different structure illustrations again showed a significant effect, *Wald* $X^2(2,2021)=145.757$, $p<.001$.

Table 6

Percentage (and Values) of the Structured Illustrations for the Essential and Non-Essential Illustrations

Function	Nature			
	Chunking	Graph	Table	**Total**
Essential	24.67 (245)	5.36 (83)	52.27 (519)	**85.30 (847)**
Non-essential	11.78 (117)	1.01 (10)	1.91 (19)	**14.71 (146)**
Total	**36.46 (362)**	**9.37 (93)**	**54.18 (538)**	**100 (993)**

CONCLUSION AND DISCUSSION

In the present study, we tried to uncover how often elementary school children are confronted in their mathematics textbooks with illustrations next to word problems, whether these illustrations are essential or not to solve the problems (i.e., their function), and what these illustrations depict (i.e., their nature). To investigate this, we browsed through four frequently used mathematics textbooks (Textbook A, B, C, and D), for three different grades (i.e., the second, fourth, and sixth grade) and collected data about the occurrence of illustrations next to word problems, their function and their nature.

Our first research question was about the number of illustrations next to word problems and whether there was a difference between grades and textbooks (*Research question 1*). Our findings show that about 3/4 of the word problems were accompanied with an illustration. There were no differences between grades, but there were differences between textbooks, and there was also an interaction between textbook and grade. In some textbooks there was an increase in number of illustrations per grade whereas in others there was a decrease or no clear pattern.

Second, we wanted to get a better view on the function of the illustrations next to the word problems, and again whether there were differences between the different grades and textbooks (*Research question 2*). Our findings show that there were almost no irrelevant illustrations. Slightly more than half of the illustrations next to the word problems were essential, in the sense that they were necessary to solve the word problems, whereas a bit less than half were non-essential, meaning that they did not contain essential information to solve the word problems. Furthermore, concerning the number of essential illustrations, there were differences between the grades and the textbooks, and there was an interaction between grade and textbook which was however again rather obscure. In one textbook there was an increase of essential illustrations per grade, in another textbook there was a decrease, and in the two other ones there was no clear pattern.

Lastly, we investigated the nature of the illustrations and whether this differed between grades and textbooks (*Research question 3*). The analysis showed that half of the illustrations depicted either one element or either the global situation of the context of the word problem, whereas the other half of the illustrations depicted the mathematical structure. Furthermore, for the one-element illustrations and the structured illustrations, there were significant differences between grades and textbooks as well as an interaction effect, which, however, again did not reveal a clear pattern.

Finally, there tended to be a relation between the function and nature of the illustrations: whereas non-essential illustrations most of the time depicted one element or the global situation, essential ones mostly depicted the mathematical structure.

So, our findings clearly show that in their mathematics textbooks, pupils are very often confronted with illustrations next to word problems, and that half of these illustrations are essential to solve the problems, whereas the other half is non-essential. Concerning the nature of the illustrations, half of the illustrations show the underlying mathematical structures, whereas the other half depicts one element or the global situation. Furthermore, there are big differences in the occurrence, function and nature between the different grades and textbooks.

These findings are important in the sense that they may help to explain how pupils think about and behave towards word problems, and especially about the illustrations that accompany them, in an instructional, assessment or research setting. Furthermore, they also uncover several questions that could be asked concerning illustrations in mathematics textbooks and in the mathematics class that could be addressed in future research.

A first series of questions relates to how textbook writers decide where to put an illustration next to a word problem and where not, and how they choose what exactly they depict. Are they aware of the fact that illustrations can have different functions next to the word problems? Did they consciously decide to include a certain overall percentage of essential, non-essential, and irrelevant illustrations? And did they follow a certain strategy with respect to the distribution of these various kinds of illustrations over the various grades? Or was their only intention with the illustrations to enhance the attractiveness of the textbooks and did they give their illustrator(s) only some vague instructions as to where to put and how to make the illustrations?

A second list of questions relates to teachers' beliefs about illustrations next to word problems and how they actually deal with them in their mathematics class. Do they draw their pupils' attention to the illustrations, do they teach their pupils how to look at the illustrations, and do they tell them how they have to use these illustrations? Also do they make a distinction between the different kinds of illustrations, and do they discuss with their pupils the status of the textual and pictorial information in a word problem and what to do in case there is a tension between the two kinds of information? And, lastly, do the teacher manuals of the textbooks contain any background information for teachers about these illustrations and how to use them in their classroom practice?

This brings us to the last series of questions. What are pupils' beliefs concerning illustrations next to word problems and how they handle them? Do they make a distinction between different kinds of illustrations (based on their function and nature) and do they adjust their looking behaviour and the place they give to the pictorial information in their situational model based on the function and nature of the illustrations? It is a possibility that pupils just do not know how to use these illustrations. They are confronted with numerous illustrations in their textbooks, some of them depict the global situation and are non-essential, others depict one element of the word problem but are essential because they contain a certain number that is not given in the problem. Also, in some cases, they do not need the illustrations, whereas in others they do need them. So maybe they just do not know how to approach all these different illustrations. It could therefore be argued that there is a need for more instruction about how they should use the illustrations in their textbooks. As Mayer (1993, p. 258) remarked:

> Children receive hundreds of hours of instruction in how to read words, [..] but a minuscule amount of time is devoted to teaching children how to read illustrations. Similarly, they receive hundreds of hours of instruction on how to write words but almost none on how to use illustrations effectively.

Also, are there any differences between pupils in how they use and interpret these illustrations in their textbooks next to the word problems? Do high achievers in mathematics use illustrations differently than low achievers? For example are low achievers helped by illustrations or are these illustrations distracting? Also, are there any differences between visualisers (i.e., 'individuals who prefer to use visual methods when attempting mathematical problems which may be solved by both visual and nonvisual methods' (Presmeg, 1986, p. 42)) and non-visualisers in the use of illustrations next to word problems? And lastly, is there any influence of pupils' linguistic proficiency, for example, when they solve word problems in another language than in their mother tongue? It could be that pupils with a lower linguistic proficiency make more use of the illustrations and are more benefitted by illustrations than other pupils.

Although all these questions need to be investigated in future research, based on the present findings and the previous studies about how pupils solve mathematical word problems (Verschaffel et al., 2000; Verschaffel & De Corte, 1997), we would argue that textbook writers but also teachers should pay sufficient attention to the use of illustrations next to word problems, their function, and their nature. Also, further research is needed concerning pupils' beliefs about illustrations and individual differences in the use of illustrations in mathematical textbooks.

To end this article, we realize that apart from the analysis of illustrations next to word problems, in terms of their frequency, function and nature, one could also analyse illustrations from other perspectives. One could, for example, analyse illustrations from a psychological or pedagogical perspective such as analysing the authenticity of the illustrations, or in other words their proximity to real life (i.e., black and white line drawing, an illustration in colour, a photograph) or from a sociological perspective (i.e., a role-reinforcing or role-breaking function from a gender or ethnic or disability perspective). These perspectives can also be included in future research.

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Relationships between Student Performance on Arithmetic Word Problems Eye-Fixation Duration Variables and Number Notation: Number Words vs. Arabic Numerals

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JÁNOS STEKLÁCS: Kecskemét College, Teacher Training Faculty, Kecskemét, Hungary

ABSTRACT: *This study investigated 4th grade students' eye-movements during arithmetic word problem solving. The sample consisted of 24 students (13 boys and 11 girls, mean age 10 years and 5 months). The students solved a 2x2 system of four tasks: (1) addition–numerals, (2) subtraction–number words, (3) subtraction–numerals, and (4) addition–number words. Besides performance and response time, fixation duration variables were computed: fixation duration on the text (FDT) and fixation duration on the number areas (FDN), and fixation duration on the keyword (FDK). Significant correlations were found between FDN and FDK, both for Task 1 and Task 3, but not for Task 2 and Task 4, suggesting that the number format played a significant role in the problem solving process. Our research may yield new results about the practical educational use of word problems with different number notations.*

Key words: *Eye-movement, Arithmetic, Number notation.*

INTRODUCTION

To what extent mathematics education fulfills the role of fostering useful mathematical knowledge is often measured by means of word problems. Csíkos, Kelemen, and Verschaffel (2011) highlight the importance of word problems in mathematics classroom practice by drawing attention to their significance in skill application, i.e., word problems may (or should) be a means of applying mathematical knowledge and skills in real-world like situations. Some mathematical word problems engage higher-order thinking skills by requiring students to build genuine mathematical models of an everyday situation (Verschaffel, Greer, & De Corte, 2000). Other types of word problems can be solved by means of only using a superficial solution strategy (see Verschaffel & De Corte, 1997), in other words, by searching for figures in the text, and connecting them with an arithmetic operation. In between these two sides of the coin, an interesting type of word problems requires students to search for figures in the text, but mechanically executing an arithmetic operation closely associated with a keyword in the text will lead to wrong results. In this research we used word problems of the compare type (a comparison between two quantities is required) using a so-called “inconsistent” keyword due to which the superficial solution strategy would fail.

Inconsistent Word Problems of the Compare Type

During the elementary school years, compare word problems are of special importance. Compare word problems have their steady place as a distinct category beside other types of simple arithmetic word problems. Early classifications identified four clusters of simple arithmetic word problems: change, combine, compare and equalize types (see Radatz, 1983; Stigler, Fuson, Ham, & Sook Kim, 1986); in recent publications combine, compare and change types are distinguished and defined (see Jitendra, Griffin, Deatline-Buchman, & Sczesniak, 2007; Riley & Greeno, 1998). In compare word problems, two values of a variable are given, and there is a relational statement connecting those two values. There is an extensive body of research about the difficulties compare word problems cause for elementary school children (see for example Mwangi & Sweller, 1998).

Inconsistent word problems are a special type of arithmetic word problem, where students are required to execute one arithmetic operation (e.g., addition), and there is a keyword presented in the text, which is a relational term inconsistent with the required operation (e.g., “less” when addition is the required operation (van der Schoot, Bakker Arkema, Horsley, & van Lieshout, 2009)). Here the term “required” refers to the strategy of connecting the two values of the text with one basic operation. A simple example of an inconsistent compare word problem is the following:

The oldest man in the village is 112 years old. He is 8 years older than his wife. How old is his wife?

In the above word problem the basic operation that connects the two numbers found in the text is subtraction, whereas the relational term “older” would have been consistent with addition (while the relational term consistent with subtraction would be “younger”).

According to Hegarty, Mayer, and Green’s (1992) results among undergraduate students, inconsistent word problems take more time to solve than problems with a consistent keyword, and research provided evidence for the longer fixations needed for inconsistent word problems (Verschaffel, De Corte, & Pauwels, 1992).

Van der Shoot et al. (2009) revealed among 5th and 6th grade children that besides inconsistency, another factor called “markedness” makes the solution process more difficult. Markedness refers to the phenomenon that words expressing the ‘negative’ quality of antonymous pairs of words are semantically more complex, therefore their presence may lengthen the solution process. For instance, the appearance of “less than” or “smaller than” may make the solution process more difficult.

Verschaffel, De Corte, and Pauwels (1992) showed that this consistency-inconsistency effect could be detected in data gathered among 3rd grade students, but not among university students. These seemingly contradictory results could be reconciled when university students faced real challenges when solving compare problems, i.e. when they had to find the solution to the tasks rather than just state what operations had to be computed.

Stimulus Modality in Arithmetic Comparisons

"Number reading is … architecturally similar to word reading" (Cohen, Dehaene, & Verstichel, 1994, p. 279). According to the triple code theory developed by Dehaene and summarized by Dehaene, Molko, Cohen, and Wilson (2004), dedicated brain circuits are engaged in recognizing the number of objects in a set. This model suggests that when solving simple compare word problems, different left and right segments of the brain (intraparietal sulcus) are activated. Different malfunctions in arithmetic computations can be associated with different neural correlates (Dehaene, Piazza, Pinel & Cohen, 2003), and this suggests that there are different neural coding systems for number words and Arabic numerals, and proper functioning is associated with appropriate number representation in case of both types of coding (Dehaene et al., 2004).

Number comparison is a prerequisite to solving compare word problems. Research on number comparison using event-related potential (ERP) suggests that the time needed for determining whether a quantity is above or below five does not interact with number notation (see Temple & Posner, 1998).

Another prerequisite component of solving compare word problems is the execution of simple arithmetic operations such as addition or subtraction. In a research with young Hungarian adults, ERP brain activity results suggested that simple one-digit addition required more time when presented in word number format (Szűcs & Csépe, 2004). Similarly, Rayner (1998) summarizes results from an eye-movement study: when reading numbers, fixation times vary with the number of syllables and with the frequency and magnitude of the numbers.

In previous eye-movement research on inconsistent word problems, stimulus materials contained numerical values in the Arabic numeral format (De Corte et al., 1990; Hegarty et al., 1992; van der Schoot et al., 2009). Since the effort needed for solving compare word problems can be measured by means of eye-fixation durations, De Corte et al. (1990) suggested several different time-related measures attributable to different difficulties such word problems may cause. One of these possible measures is the duration of fixations on textual components and on the numbers of the word problem. Within the textual components, the keywords of the inconsistent word problems are of special importance.

The Current Practice in the Textbooks

In Hungary, the currently used textbooks and task booklets overwhelmingly use numerals to represent numbers in word problems. Arithmetic word problems containing number words instead of numerals may find their place in further research and in educational practice, too. One of the authors of one of the most widely used 4th grade Hungarian mathematics textbooks (I. Libor, personal communication, March 11, 2013) proposed three factors that may explain this phenomenon: (1) According to the rules of Hungarian spelling, numbers can be written either in Arabic numeral format or in words, but in *common practice* (italicized by us) numbers that can be written in a short phrase are written in words (e.g., ten million). (2) Teaching practice suggests that using number words makes a word problem more difficult and therefore cardinal numbers are usually written in Arabic numeral format whereas ordinal numbers and simple fractions (where the enumerator is 1) are written as number words in word problems. A brief analysis based on a sample series of word problems published on the website of the largest textbook publisher (Szöveges feladatok, 2015) shows that out of 25 sample tasks, only one contained numbers words for cardinal numbers and another one mixed the Arabic cardinal numbers and the number words. Ordinal numbers were written as number words. (3) The textbook review process has a bias towards the "common practice", i.e. towards the use of Arabic numerals in word problem texts.

As part of this section on the current textbook practice, for the sake of international comparison, two key elements of the transparency of Hungarian number words will be given. A recent summary from Cankaya, LeFevre, and Sowinski (2012) contains examples for naming two- and three-digit numbers in German, Dutch, English, French, Czech, Basque, and Chinese. Compared to them, Hungarian composition of number words between 10 and 1000 is rather similar to the Chinese way of naming two- and three-digit numbers, and the authors call this system "regular". A second characteristic is that numbers below 2000 are written as one single number word in Hungarian.

Aims of the Current Research

Combining the ideas from these two previous lines of research, we investigated the role of keywords and the role of the notation of numbers in the text in solving inconsistent word problems. The aim of the investigation is to reveal the relative (if any) effects of the following factors on performance and on different eye-fixation measures: the arithmetic operation to be computed and number notation (Arabic vs. number words).

We hypothesize that (1) the duration of fixations on different parts of the word problems, i.e. text area (excluding keyword area) and number area, and (2) the notation of the number presented in the text (number words vs. Arabic numbers) will have an effect on students' performance. Furthermore, it is hypothesized that (3) number notation and the operation to be computed in a task will have an effect on reaction time and on fixation duration.

METHODS

Sample

The sample consisted of 24 students (13 boys and 11 girls, mean age 10 years and 5 months). They all attended the same school in a large county seat town and had diverse social-economic status backgrounds. The heterogeneity of the sample in terms of their family background may strengthen the generalizability of our results. The sample – although being so-called convenient sample – has an appropriate size for applying various statistical methods with the aim of documenting the phenomenon to be investigated. Also, this sample size allowed for providing uniform experimental settings throughout the experiment. None of the participants suffered from eye disease or from dyscalculia.

Tasks

The students solved four experimental tasks and two buffer tasks. The four experimental tasks comprised a 2x2 system with the arithmetic operation needed for an effective solution as one variable, and the number format as another variable: Task 1: addition – numerals; Task 2: subtraction – number words; Task 3: subtraction – numerals; Task 4: addition – number words. All tasks contained an inconsistent keyword (e.g., shorter, when addition was required). The four tasks were presented on the screen of the eye tracking system in three lines. The two statements about the quantities formed two lines, and the third line contained the question. The three lines were center-aligned. The English translation of the tasks (based on the translation – re-translation method in developing the translated text) is shown in Table 1.

Table 1

Task System of the Four Word Problems Used in the Experiment

Task	Text	Operation	Modality
1	John has 115 books. He has 8 books less than Grete has. How many books does Grete have?	addition	numerals
2	The oldest man in the village is one hundred and twelve years old. He is eight years older than his wife. How old is his wife?	subtraction	number words
3	The highest pyramid in Egypt is 137 m high. It is 20 meters higher than the Great Lighthouse. How many meters high is the Great Lighthouse?	subtraction	numerals
4	The running track is one hundred and twenty-five meters long. It is seventeen meters shorter than our street. How long is our street?	addition	number words

The first buffer task was a very easy warm-up task of the consistent type that contained numbers in the Arabic numeral notation and required students to simply add two numbers. The next four tasks formed the core part of the research and the last task (the second buffer task) was a puzzle-like, 'tricky' word problem, which is not discussed in the current analysis but seems to be worth being analyzed from a linguistic, social and socio-psychological viewpoint.

Procedure and Measures

Data collection was done in the school. Before the experiment started, children received information about the general aim of obtaining information about their mathematical task solving abilities. The students were then individually tested in a quiet room. They were given a few minutes to look around and examine computer-like screen of the eye tracking system and then the investigation began with the usual calibration process. Eye movements were registered with a Tobii T120 eye tracker.

Having received the answer to each task, the research assistant switched to the next task by pressing the space button on her computer, and then noted down the student's answer without giving any feedback as to whether the answer was right or wrong. Having completed all tasks, the student had to leave the room and go to another room without the chance of informing his or her classmates about either the tasks or the procedure.

Five dependent variables were computed for each task: (1) performance, (2) response time (RT), total fixation duration time (TFD), and fixation time on (3) the text components (excluding the keyword) (FDT), on (4) number areas (FDN), and on (5) the keyword (FDK). Since RT and TFD are very closely correlated (Pearson-correlations between .83 and .99), they could be used interchangeably.

RESULTS

The results will be presented according to the three research questions we proposed. Before doing so, some statistical analyses are required to ensure the reliability and validity of our analyses.

Uniformity of the length of texts

In order to eliminate a factor of text length that might possibly cause difficulties in interpreting the results, we compared the number of words in each task. The numbers of words in the tasks were: 16, 15, 21, and 17, respectively. Uniform distribution can be assumed; Chi-square = 1.20, $p = .75$.

Normal distribution assumption

Before showing the results of various quantitative statistical methods, the assumption of normal distribution required for several analyses has been investigated. Although t-tests and ANOVA are robust to the violation of the normal distribution assumption (Schmider, Ziegler, Danay, Beyer, & Bühner, 2010), due to the sample size we took care special attention to provide reliable analyses. According to the one-sample Kolmogorov-Smirnov-tests, the assumption of the normal distribution of the quantitative variables being involved in the next analyses can be hold in all but two cases. It should be noted that for four tasks and for five eye-fixation measures it meant in eighteen cases the assumption of the normal distribution could be held. The two exceptional cases were: fixation duration on the text (FDT) and fixation duration on the numbers (FDN) in Task 3 ($p = .03$, and $p = .01$, respectively). For the performance measures the dichotomous nature of that variable prevented us from having normal distribution variables, therefore McNemar-tests will be used.

Descriptive statistics of the basic quantitative measures

The results show that one of the two tasks with number words (Task 4) proved to be more difficult than the other three tasks. The percentages of correct solutions for the four tasks were 46%, 46%, 50%, and 21%, respectively. Mauchly's W statistics showed that the variances of the differences in the six possible pair-wise comparisons can be considered equal ($W = .73$, $p = .23$). Consequently, the univariate repeated measures statistics can be used in the analyses without corrections. Table 2 shows the basic descriptive statistics of the task solution process and Table 3 summarizes the results of paired-samples t-tests for each pair of tasks.

Table 2

Descriptive Statistics of the Task Solution Process – Mean Values (SD in Parentheses) of the Main Variables

Task	Perf.	TFD	TFC	FDK	FDT	FDN
1	.46 (.51)	25.01 (13.78)	68.67 (31.83)	1.33 (1.11)	10.56 (4.24)	12.25 (8.79)
2	.46 (.51)	29.72 (18.90)	84.29 (41.22)	1.82 (1.24)	14.91 (6.61)	14.15 (14.60)
3	.50 (.51)	26.83 (19.68)	79.83 (43.62)	1.71 (1.78)	18.89 (13.02)	7.44 (6.83)
4	.21 (.41)	30.70 (14.21)	91.83 (38.77)	2.29 (2.08)	14.40 (7.44)	15.07 (8.60)

Note: (1) Perf. = mean solution rate (i.e., percentage of correct solutions); TFD = total fixation duration; TFC = total fixation count; FDK = fixation duration on the keyword; FDT = fixation duration on the text (other than the keyword); FDN = fixation duration on the numbers. (2) Fixation duration values are given in seconds.

Table 3

Results of the Paired-Samples t-tests (p in Parentheses) for Pair-Wise Task Measure Comparisons

Task pair	TFD	TFC	FDK	FDT	FDN
1 – 2	1.48 (.15)	*2.08 (0.05)*	1.26 (.22)	*3.70 (<.01)*	.81 (.43)
1 – 3	.53 (.60)	1.31 (.20)	.77 (.45)	*3.17 (<.01)*	*3.23 (<.01)*
1 – 4	*2.53 (.02)*	*3.26 (<.01)*	*2.12 (.05)*	*2.90 (<.01)*	*2.26 (.03)*
2 – 3	1.27 (.22)	.60 (.55)	.31 (.76)	1.70 (.10)	*3.79 (<.01)*
2 – 4	.37 (.71)	1.25 (.22)	1.17 (.26)	.57 (.58)	.37 (.71)
3 – 4	1.05 (.31)	1.27 (.22)	1.15 (.26)	1.80 (.09)	*4.43 (<.01)*

Note: (1) Mean = mean solution rate (i.e., percentage of correct solutions); TFD = total fixation duration; TFC = total fixation count; FDK = fixation duration on the keyword; FDT = fixation duration on the text (other than the keyword); FDN = fixation duration on the numbers. (2)Fixation duration values are given in seconds. (3) Significant values are italicized.

Pair-wise comparisons of the performance rates required us to use the Wilcoxon-tests. These comparisons showed that the solution rate on Task 4 is significantly lower than on Task 2 and Task 3 ($Z = 2.12$, $p = .03$; $Z = 2.11$, $p = .04$, respectively).

Hypothesis 1: Connection between performance and fixation duration measures

In hypothesis 1, we proposed that the fixation duration measures (TFD, FDT, FDK, FDN) have an effect on students' performance on the task. Correlations between the five different task measures were computed. There were no significant correlations between performance and FDN or FDK, except for Task 3, where we found a significant negative correlation between performance and FDT on the one hand ($r = -.44$, $p = .03$), and between performance and RT on the other ($r = -.40$, $p =.05$).

Both tasks with number words (Task 2 and 4) required longer fixation duration than tasks with numerals (Task 1 and 3). The difference was significant between Task 1 and Task 4. As for total fixation counts, Task 1 required significantly fewer fixations than Tasks 2 and 4. As for the fixation duration measures, the differences in favor of Task 1 (in the sense of shorter fixation compared to the other tasks) can be in part attributed to the slightly shorter text of the word problem. However, differences in fixation duration on the number components of the word problem indicate that word numbers may require longer time to read than numerals while reading the text and solving the word problem.

Hypotheses 2 and 3: The role of number notation

To test the effects of number notation (Hypotheses 2 and 3), 2x2 repeated measures ANOVAs were computed. The first factor is number notation and the second is the operation (addition vs. subtraction). Table 4 shows the eta-squared effect sizes of the two factors for each dependent variable.

Table 4

Results of Two-Way Repeated Measures ANOVA and Effect Sizes of the Number Notation (Numeral vs. Number Word) and of the Operation (Addition vs. Subtraction) on the Dependent Variables

Dependent variable	Factor in ANOVA repeated measures	F(1,23)	p	Eta-squared (%)
Response time	Number notation	10.507	.004	31.4
	Arithmetic operation	1.943	.177	(7.8)
	Interaction	0.774	.388	(3.3)
FDT	Number notation	0.020	.888	(0.1)
	Arithmetic operation	9.015	.007	29.1
	Interaction	9.190	.006	29.5
FDN	Number notation	18.391	<.001	44.4
	Arithmetic operation	2.726	.112	(10.6)
	Interaction	3.425	.077	(10.3)
FDK	Number notation	3.126	.091	(12.4)
	Arithmetic operation	0.127	.725	(0.6)
	Interaction	2.463	.131	(10.1)

Note: FDT = fixation duration on the text (other than the keyword); FDN = fixation duration on the numbers; FDK = fixation duration on the keyword. Non-significant eta-squared values are shown in parentheses.

Table 4 shows the outstandingly relevant role of number notation in response time and this was due to the effect of notation on fixation duration on numbers. However, no such strong effect of number notation was found for performance, fixation duration on text components and on fixation duration on keyword. Interestingly, the arithmetic operation to be computed had a significant effect on fixation duration on text components, with higher FDT values for subtraction problems.

To further investigate the role of number notation, correlations between fixation duration measures within a task are shown in Table 5.

Table 5

Correlations Between FDT, FDN and FDK Variables Within a Given Task

		FDN (for the given task)	FDK (for the given task)
Task 1	FDT	.742	.800
	FDN	1	.690
Task 2	FDT	.505	.575
	FDN	1	(.068)
Task 3	FDT	.901	.935
	FDN	1	.830
Task 4	FDT	.469	.739
	FDN	1	(.218)

Note: Coefficients in parentheses are not significant at the p = .05 level. FDT = fixation duration on the text (other than the keyword); FDN = fixation duration on the numbers; FDK = fixation duration on the keyword.

The differences between the correlation coefficients are significant in any case when the smaller coefficient is either .218 or .068 and the bigger coefficient is greater than or equal to .690. Table 5 suggests that the connections between fixation duration length variables are of different strengths, revealing the possible role of number modality in fixation duration on different parts of the word problem.

In the cases of Task 2 and Task 4 (where the numbers were given in the number word format) correlations between fixation duration on numbers and fixation duration on keyword proved to be non-significant. Since correlations between other fixation duration variables were significant in all tasks, this peculiarity needs further interpretation. The significant correlations between fixation durations on text, numbers and keyword (FDT, FDN and FDK) indicate that the more time a student spends on a given part of the word problem text, the more time he or she spends on other types of text components. The exception is the correlations between fixation duration time on the number word and fixation duration on the keyword. This can be interpreted as follows: when the number word notation is used, the more time a student spends on the number word, the longer he or she will fixate on text components of the word problem except for on the keyword component. This may furthermore indicate that albeit longer fixation time is needed for the number word component, the keyword component will not require more fixation time. What is curious is the increased fixation duration on the text components when the numbers are given in number word format.

DISCUSSION AND CONCLUSION

This research provided evidence about the role that number notation may play in arithmetic word problem solving. The main novelty of this research was focusing on number notation. Surprisingly few studies have examined the role of number notation in an educational context. One possible reason of the scarcity of such research is that in real classroom settings there is a lot of variation in student characteristics that may interfere with task-related variables. For example, as De Corte et al. (1990) pointed out, research on arithmetic word problem solving should use tasks in which students' reading and computational skills play a negligible role. In our current study, one of these two challenges has been eliminated (i.e., the tasks were well within the computational skills of 10 year old students), and the other factor was an important component of the dependent variables (i.e., fixation duration measures on textual components).

Our results confirmed the hypothesis about the effects of number notation on different fixation duration measures. In accordance with our expectations based on the literature (Rayner, 1998), significantly longer response time and fixation duration on numbers is needed when using number word notation. However, number notation proved to have no important role in performance, fixation duration on text components, and in fixation duration on keyword. Since longer fixation duration on text components and longer reaction time are associated with a lower level of performance (albeit the correlation was significant only for Task 4), but number notation has no significant effect on performance, the longer FDT and RT in the tasks involving number words may point to a "compensation" effect. It means that the longer time needed for the completion of the problem is not accompanied by worse performance when the number element of the word problem is presented in a word format.

In the light of the importance of so-called common practice when choosing number notation in school word problem texts, research should focus on the possible advantages or disadvantages of different number notations. Namely, research should examine the advantages or disadvantages of using number words instead of Arabic numbers in the text of word problems. Of course, it is better to be able to solve arithmetic compare word problems with either Arabic numerals or number words than to be able to solve word problems with only one of these two notations. Since in this experiment the number notation did not play a crucial role in students' performance and in their previous classroom practice students encountered mainly Arabic numerals in word problems, two possible explanations can be considered.

First, the current classroom practice sufficiently develops students' word problem solving skills with either notation. Second, being unusual, the use of number words may alter the task solving process. According to this second explanation, the gains students possibly attain from the unusual number notation may raise awareness in a thinking process that would have otherwise been (over-) automatized. The possibility of modifying the thinking process is supported by the non-significant correlations between fixation duration on numbers and fixation duration on keyword in Task 2 and Task 4 (number word notation). Since there is still a significant correlation between fixation durations on the number word and on the text components, these results might be interpreted as follows. While solving arithmetic compare word problems students slow down their reading on the number words and on the text components in general but not on the keyword of the task. Albeit the keyword plays an important role in deciding which arithmetic operation the student should choose, it seems that these keywords can fulfill their role in tasks with number words in a relatively short time.

As a final conclusion about the practical implication of our results, we encourage textbook writers and teachers to construct and use arithmetic compare word problems with diverse notations of numbers. Further research with bigger samples may provide evidence about how different number notations change the task solving process. Additionally, cross-cultural (cross-linguistic) studies can reveal to what extent performance and number notation are independent of each other.

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What is Included in "At Most"?

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ABSTRACT: *The aim of this research is to examine mathematics teachers' and student teachers' understanding of the quantifying conditions "at most" and "at least". Twenty-eight teachers and 17 student teachers were given a sorting task including these conditions. Subsequently a discussion was conducted with the student teachers. The participants displayed no difficulty with the condition "at least" but a majority (16 teachers and 10 student teachers) did not understand that the condition "at most" can include "none".*

Key words: *Quantifying conditions, Mathematics teachers, Student teachers.*

INTRODUCTION

The aim of this research is to expose and discuss mathematics teachers' and student teachers' understanding of the quantifying conditions "at most" and "at least". This research was instigated after observing a recurring mistake among teachers attempting a sorting task whose instructions included these conditions.

It is accepted that children's understanding of numerically quantified expressions takes many years to develop. Even the simplest expression "three mistakes" can be interpreted as "exactly three", "at most three", or "at least three", depending on the context. Musolini (2004) combined developmental psychology and theoretical linguistics to study this understanding. He found that by the age of five children have implicit knowledge of the non-exact interpretations in certain contexts, while they do not yet know the meaning of corresponding expressions "at most" and "at least" which convey these senses explicitly. Teachers must be able to develop their students' understanding of these quantifying conditions and to do so must themselves have a well-based knowledge and must know meanings for terms and explanations for concepts (Ball, Thames, & Phelps, 2008).

It has been noticed, specifically in the solution of mathematical word problems, that there is a need to bridge between mathematical language that requires a knowledge of the mathematical components and natural everyday language. The focus on research in this field is on students' ability to translate from everyday language into mathematical language. Sometimes the same words bear different meanings. Ilany and Margolin (2010) give the example of the word "diagonal" which has a specific meaning in mathematics, but in everyday life it can indicate any line which is not perpendicular. Our research focuses on two terms which are well understood in natural language and bear the same meaning in mathematical language, yet the "translation" proves problematic.

OUTLINE OF THE RESEARCH

The following sorting activity was given to 28 5th and 6th grade mathematics teachers and to 17 students studying for teaching mathematics in special education. The teachers were participants in an online professional development course aimed at integrating the computer into the teaching of mathematics. They were required to upload their solutions onto the course website, where they received feedback and discussed the assignments and other theoretical issues in a discussion forum. The student teachers were given the task as part of a course on the basics of geometry, after learning about families of quadrilaterals and inclusion relations. The completed tasks were handed in to the lecturer who examined them and returned them to the students in the following lesson, when he led a discussion based on the results. This discussion was recorded and some parts of it are replicated here.

THE ACTIVITY

The participants were shown a picture (Figure 1) of 16 quadrilaterals (adapted and developed from Britton and Stump (2001)).

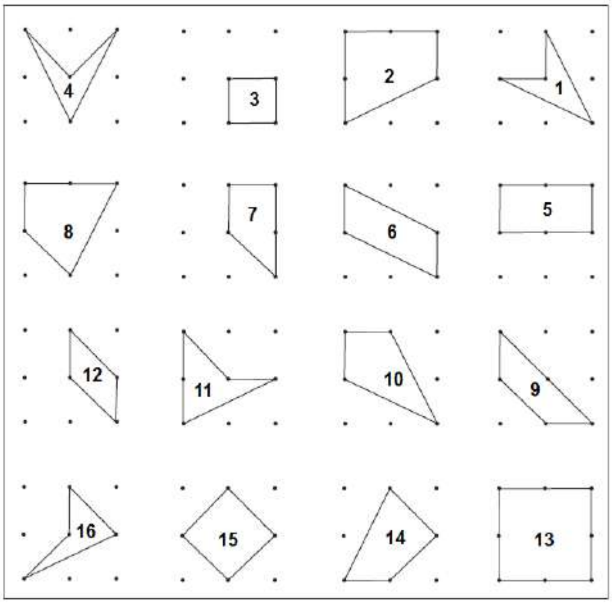

Figure 1. Collection of quadrilaterals.

Initially they were requested to sort the quadrilaterals into four categories as shown in Figure 2.

Angles / Sides	At least one right angle	No right angle
Two pairs of parallel sides		
At most one pair of parallel sides		

Figure 2. Sorting the quadrilaterals.

This activity was developed by the mathematics team at the Center for Educational Technology (CET) as part of a professional development course in Geometry for 5th and 6th grades teachers. Subsequently the participants were asked to provide a suitable name for the quadrilaterals in the first row of the table and a suitable name for the quadrilaterals in the uppermost left corner. Finally, they were given an identical table without headings and asked to create their own categories and to sort the quadrilaterals into the new table.

FINDINGS

We will report and analyse the results of the three parts of the task. The first part of the task dealt with sorting the quadrilaterals and compare between the teachers' answers and the students' answers. Table 1 shows how many correct and wrong responses were given in each of the two groups – teachers and student teachers.

Table 1

Distribution of Correct and Incorrect Answers

Answers / Population	Correct answer	Incorrect answer (at least one mistake)
Teachers (N=28)	6 (21%)	22 (79%)
Students (N=17)	0	17 (100%)

We will present the most frequently occurring mistakes and attempt to characterize them.

The Teachers' Mistakes

As seen in Table 1, 22 teachers sorted the quadrilaterals incorrectly**.** Of these, 12 completed the table as shown in Figure 3.

Angles / Sides	At least one right angle	No right angle
Two pairs of parallel sides	3, 5, 13, 15	6, 12
At most one pair of parallel sides	2, 7	9

Figure 3. Incorrect sorting of 12 teachers.

The teachers sorted all the quadrilaterals in the first row correctly, but in the second row, in the intersection of "No right angle" and " At most one pair of parallel sides" they placed only quadrilateral number 9. These teachers overlooked the four quadrilaterals (numbers 1, 4, 11, and 16) with no parallel sides. In the intersection of "At least one right angle" and "At most one pair of parallel sides" they placed only quadrilaterals number 2 and 7, omitting the three quadrilaterals (numbers 8, 10, and 14) with no parallel sides. This leads us to suspect that the teachers were unaware that the condition "At most one pair of parallel sides" includes two possibilities: one pair of parallel sides and no parallel sides.
Four teachers filled in the table as shown in Figure 4.

Angles / Sides	At least one right angle	No right angle
Two pairs of parallel sides	3, 5, 13, 15	6, 12
At most one pair of parallel sides	2, 7, 9	

Figure 4. Incorrect sorting of four teachers.

As before, these teachers completed the first row correctly but placed no quadrilaterals in the intersection of "No right angle" and "At most one pair of parallel sides". In the intersection of "At least one right angle" and "At most one pair of parallel sides", they placed quadrilaterals number 2, 7, and 9. It is clear that these teachers also did not consider the condition "At most one pair of parallel sides" to include quadrilaterals with no parallel sides. It is not clear why they did not recognise that quadrilateral 9 belonged in the second column. This needs to be investigated in further studies. The remaining six teachers each made different, non-recurring mistakes which did not appear to follow a pattern.

The Students' Mistakes

The students' mistakes were similar to those of the teachers. In the lesson following the completion of the sorting task the lecturer returned the students' papers and instigated and recorded a discussion on the different solutions.

Ten of the 17 students filled in the table exactly as shown in Figure 3. It appears as if the students like the teachers, did not understand the condition "At most one pair of parallel sides" and thus did not take into account those quadrilaterals with no parallel sides.

At the beginning of the discussion the lecturer checked if the students understood the structure of the table, for example:

Lecturer: Which quadrilaterals would belong in the lower right-hand box?
Student: Those that have no right angle and at most one pair of parallel sides.

Responses like this convinced the lecturer that the students understood the concept of intersection between properties. Subsequently he decided to inspect each quadrilateral and asked the students where they thought it belonged in the table.

In the following episode he points to quadrilateral number 1 and asks to which column it belongs. Many students answer together and agree that it belongs to the column "No right angle". One of them, Anna, speaks out on her own.

Lecturer: Why?
Students: Because it doesn't have any right angles.
Anna: What's going on here? It does have, it has one right angle.
Students: But that's not an angle belonging to the quadrilateral.
Anna: Oh, right.

The lecturer ensures that Anna understands that the right angle is not interior to the quadrilateral, and that the interior angle measures 270^o. Anna was not alone in her mistake – others also related to quadrilaterals numbers 1 and 4 as containing one right angle. In both cases the right angle is external to the quadrilateral.

The discussion on the first quadrilateral continues with the question of which row it belongs to. The students call out different answers and there is no agreement. The most frequent response is "it doesn't belong in either row". This corresponds to the way many of the participants completed the table as shown in Figure 3, in which quadrilateral number 1 does not appear. The lecturer addresses another of the students:

Lecturer: Which row do you think this quadrilateral belongs to?
Dina: Neither. It doesn't fulfill either of the definitions.
Lecturer: Let's check together. Why not the first row?
Dina: Because it has no parallel sides.

The lecturer, to encourage Dina, shows that this is correct by extending the sides to show that they intersect.

Lecturer: And what about the second row?
Dina: That's also not suitable.
Lecturer: Why?
Dina: Because it doesn't have anything at all.

Following this response the lecturer decided to check whether this non-understanding of "at most" was general or depended on the context. Therefore he inquired if she understood this condition in a real life context:

Lecturer: Let's say you're at the dentist who tells you to eat at most three pieces of chewing gum a day. What do you understand from that?
Dina: I'm allowed to eat up to three pieces of gum.
Lecturer: If you eat one, is that okay?
Dina: Yes.
Lecturer: If you don't eat any, is that okay?
Dina: Yes.

This episode suggests that the frequent occurring error stems not from a non-understanding of the condition "at most" but from an inability to connect it to a mathematical context. It appeared to help the other students understand their mistake, as evident in the continuing discussion. Five of the students filled in the table as shown in Figure 5.

Angles / Sides	At least one right angle	No right angle
Two pairs of parallel sides	3, 5, 13, 15	6, 12
At most one pair of parallel sides	2, 7, 8, 10	1, 4, 9, 11, 14, 16

Figure 5. Incorrect sorting of five students.

Here the mistake is placing quadrilateral number 14 wrongly. The students stated that they could not identify a right angle in that quadrilateral. This could stem from a lack of experience with geometric shapes.

In the second task in which the teachers and the students were asked to provide a suitable name for the quadrilaterals in the first row of the table and a suitable name for the quadrilaterals in the uppermost left corner, most of them answered correctly. All of them noticed correctly that in the first row are the parallelograms, and 94% of the students and 96% of the teachers answered correctly that in uppermost left corner are the rectangles. There were some participants who did not answer the question and some who answered that there are squares and rectangles. They did not know that squares are included in the rectangles although they got the task after they learned it.

In the third part of the task they were given an identical table without headings and asked to create their own categories and to sort the quadrilaterals into the new table. There were two kind of categories they provided: (a) using attributes of quadrilaterals like the different kind of angles: there is a right angle in the quadrilateral, there are not right angles in the quadrilateral, all the angles are acute (even though there are not such quadrilaterals). Other attributes they provided dealt with symmetry; equal or unequal opposite sides or angles; equal or an equal adjacent sides or angles; convex or concave quadrilateral; and (b) the participants provided headings similar to the given in the original task using "At most" and "At least" quantifying conditions.

Furthermore, in Table 2, we compare between the teachers' and the students' preferences by the headings they chose.

Table 2

Distribution of Categories Used by Participants

Categories / Participants	Using simple attributes	Using quantifying conditions
Teachers (N=28)	15 (54%)	13 (46%)
Students (N= 17)	4 (24%)	13 (76%)

However, when we checked their work we found that most of them, although they chose the headings by themselves they could not cope with the task. Only few of them answered correctly. Only two of the teachers answered correctly and they used simple attributes, and only two of the students answered correctly and they used quantifying conditions.

We can see that most of them failed in sorting the quadrilaterals. Although they chose by themselves how to sort they did not succeed. Analysing their mistakes we could find some kind of mistakes:

a. The most typical mistakes we found were like those in the first task: not understanding the quantifying conditions and therefore not including all the needed quadrilaterals, or adding quadrilaterals incorrectly.

However we noticed some more mistakes like:

b. Not recognizing right angle on the geoboard when it is not stereotyped, they missed the right angle on quadrilateral no. 14 or 15 (Figure 1).

c. Not finding equal sides on the geoboard, they did not see the equal sides in quadrilateral no. 1, and some of them missed the equal sides in quadrilateral no. 2 or no. 7.

d. Some of the participants gave the category: all the angles in the quadrilateral are acute. Not only did they not know that this is an impossible condition, they thought that quadrilaterals no.1 and no. 4 suit this condition.

There were much more mistakes, but they were idiosyncratic, and it was not possible to find other commonalities between them.

DISCUSSION AND CONCLUSION

The aim of this research was to expose common mistakes made by primary school teachers and student teachers with respect to the two quantifying conditions "at most" and "at least". These conditions appeared as part of a task involving sorting quadrilaterals according to specific properties. The participants understood the condition "at least" but not the condition "at most". They understood "at most one" to include "one" but not "none".

We found difficulties in the grasp of this condition "at most" in a mathematical context as opposed to a real-life situation. We strongly recommend that mathematics teachers and teacher educators be made aware of the difficulties involved in making the transition from real-life to mathematical situations. Since children's understanding develops slowly and since the learner will meet these two quantifying conditions frequently in his school career in different branches of mathematics such as calculus, probability, etc., we recommend developing tasks that aim to deepen the knowledge and understanding of these terms as early as primary school.

The activity described in this paper exposed other common mistakes which will be analysed at a later on. This research is a pilot study for a more extensive research with a wider participant base and in different topics in mathematics in order to find out whether it is context- based.

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The Teacher's Knowledge of Mathematics: Developing Student Teacher Knowledge and Competences

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ABSTRACT: *Several researchers on mathematics education seek to answer questions such as: Which is or what makes up the knowledge of the mathematics teacher? Which is the knowledge to be developed in student teachers? And, without a doubt: which are the competences of a mathematics teacher? The definitions of the terms used when answering the questions cited in the specialized bibliography on the topic are many. This article presents a revision that collects different definitions on the knowledge of the mathematics teacher, on the knowledge of the student teachers, on the competences for the mathematics teacher and on the management of the teaching-learning process. This revision is organized by presenting a chronological evolution of each of the terms. Each referring section presents a short description of the definition considered the most current, correct and complete.*

Key words: *Teacher´s knowledge, Student teacher´s knowledge, Competence, Management of teaching-learning process.*

INTRODUCTION

For a long time has been of great interest for many researchers on mathematics education to answer questions such as: Which is or what makes up the knowledge of a mathematics teacher? Which is the knowledge to be developed in student teachers? And, without a doubt: which are the competences of a mathematics teacher? There are many researchers who have undertook the task to respond to each one of these questions and of course, there are various documents and research papers in which researchers try to make explicit their responses to these questions. However, when you want to conduct a study involving the terms associated with the questions exposed initially, the information obtained is quite large, making hard to determine which are the current definitions and which are the ones more accurate in relation to each of these terms.

This article comes as part of the theoretical construction to raise the issue of research for the doctoral thesis entitled "Changing conceptions of a group of future teachers of mathematics on their management of the teaching-learning process in a learning environment based on the resolution of problems". It established a bibliographic review that collected various definitions on teachers' knowledge of mathematics, on the knowledge of the student teachers, on the term competence and mathematics teacher's competences, as well as on the management of the teaching-learning process. When structuring this article, I deemed appropriate to make a chronological evolution of each of the terms presented and finish each one of the paragraphs with the definition that I deem to be the most current, correct and complete.

In this review, I will mention at first place the various definitions that on teachers' knowledge have presented different authors, beginning with that established by Shulman (1986, 1987) and eventually including the proposal of Delaney, Ball, Hill, Schilling and Zopf (2008). This section lists the positions on teachers' knowledge from around nine renowned authors on this term. Secondly, I will present papers that reveal how the considerations on the teachers' knowledge impinge on the features and requirements made to the training programs for teachers of mathematics, in particular in the development of the competencies as a fundamental part of the student teachers' knowledge. Then I will include texts that show different meanings for the term competence, starting with the etymology of the word. Finally, I will present research papers and texts that define the practice of the mathematics teacher and the management of the teaching-learning process.

Regarding to Mathematics Teachers' Knowledge

Since a few years ago, there have appeared various responses to the question: what should be the knowledge and abilities of teachers and professors of mathematics? One of the answers referenced by many authors is the proposal by Shulman (1986, 1987), who considers that categories addressing a teacher's knowledge should include basically: knowledge of the content, general teaching knowledge, knowledge of the curriculum, teaching knowledge of content, knowledge of the educational contexts and the knowledge of the objectives, aims and the educational values, as well as its philosophical and historical foundations.

Shulman (1987) considers that the didactical knowledge of the content gains a particular interest because it identifies the bodies of knowledge distinctive to the teaching. According to this author, this knowledge represents the mixture between subject and didactics, which will take you to an understanding of how certain topics and problems are able to organize by themselves, and also represented and adapted to the various interests and capabilities of students and are exposed to its teaching.

Bromme (1998), based partially on that proposed by Shulman (1986) on the professional knowledge of the mathematics teacher, considers appropriate to differentiate between: (1) Knowledge of mathematics, which include inter alia mathematical principles, rules and ways of thinking and mathematical techniques; (2) Curricular knowledge, which appears described in the curriculum, and also codified in textbooks and other teaching tools; (3) Knowledge of the class, which appear through the establishment of relations and a special balance tailored to the specific circumstances of the class; (4) Knowledge of what students are learning, that is to say that teacher must be aware of the mathematics understanding of their students.

In relation to the above considerations, Bromme (1988) wonders about how it is allowed the maintenance of different types of knowledge such as those mentioned. That is to say, he asked about the understanding on the nature of the knowledge concerning the school and to the course, with regard to the purposes and objectives to be achieved. As a response to these questions, this author claims that these are precisely the meta-knowledge that allow you to set some coherence between the different skills, because they define the framework of guidance in which knowledge is valued, as well as its relationship with the profession itself. Bromme (1988) also defines the metaknowledge as the philosophy of the teacher in terms of mathematics and education, even if its philosophical nature has very specific effects on the didactic practice.

On the didactic practice of mathematics teacher, Bromme (1988) considers other knowledge that mathematics teacher must have, namely: (1) Knowledge on teaching of the subject, which have a special character since psychological and pedagogical information and teacher's own experiences are brought together with the mathematical knowledge; and (2) Pedagogical knowledge, all of which refer to the knowledge that is valid while maintaining a relative independence of the subject. The latter also includes knowing certain methodological aspects of the class, ways of managing difficult behavior, and the organization of the school center.

Classifications of Bromme (1988) and Shulman (1986) on the teachers' knowledge necessarily imply a separation (at least analytical) between the different teachers' knowledge. According to Gomez (2007) that separation may generates difficulties, for example it is complex to imagine a teacher who understands how to organize, represent and adapt topics, particular problems or issues to the diverse interests and abilities of students regardless of the knowledge he has of apprentices. In other words, the knowledge on the apprentices should be included within the pedagogical content knowledge, although Shulman (1987) presents them as independent. For Gomez (2007), Simon partially resolves these difficulties by identifying the abilities that are put into play when, based on the evaluation of the knowledge of the students, teacher formulates the hypothetical trajectory of learning.

Simon (1997) mentions the following knowledge for the mathematics teacher: (1) Knowledge of mathematics; (2) Knowledge of math activities and representations; (3) Hypotheses about the students' knowledge; (4) Theories from mathematics teachers, their teaching and learning; and (5) Knowledge of the students' learning about a specific topic. According to Gomez (2007), although Simon (1997) aims to define a structure of the relationship between these knowledge and the components of the hypothetical trajectory of learning, this is not achieved as he suggests that all the types of knowledge, except for math activities and their representations, affect all components of the hypothetical trajectory of learning.

Regardless of Simon (1997) failing to establish the relationship between knowledge and the components of the learning path, Gómez (2007) considers that the taxonomy of the teachers' knowledge exposed by this author differs from the taxonomies by Shulman (1986) and Bromme (1988) due to its functional nature, as it assumes a position with respect to the students learning. He proposes a scheme of instruction consistent with that position and he identifies the abilities that are considered necessary to fulfill this teaching. In addition, Gómez (2007) found that in most taxonomies of teachers' knowledge analyzed by him, including those associated to the teaching of science, it can be seen a common core: knowledge of the discipline, about how "represent it in the classroom", on students and on instructional strategies. According to this author, these explanations are looking to characterize, in some way, the integration between the knowledge of content and pedagogy. In other words, these are efforts aimed to define the notion of pedagogical content knowledge.

Passing through the work of Shulman (1987), Bromme (1988) and Simon (1997), and Ball and Cohen (1999) make a more explicit definition of the knowledge about the discipline of the mathematics teacher and point out that knowing the "mathematics that are to be taught" implies much more than the idea of "know the math of the curriculum". It implies for the student teacher to learn about the mathematical content from the perspective that such content is to be learnt by someone. This condition relies on recognizing that if you get to "know the mathematics that must be taught in order to someone to learn" it will imply a specific knowledge of mathematics linked to the professional task of teaching math (Ball & Cohen, 1999, p. 8).

In accordance with the aforementioned, for Llinares, Valls, & Roig (2008) the professional knowledge of mathematics teacher is seen as integrated by different domains (knowledge about the organization of the curriculum, the modes of representation and most appropriate examples in each time, the skills in mathematics management and communication in the classroom, knowledge in epistemology of mathematics etc.) (García, 1997; Llinares, 2000; Escudero & Sánchez, 2007; Gavilán, García & Llinares, 2007a, 2007b; D'Amore, 2007). However, for Llinares (2008) the trait that characterizes the teachers' knowledge does not only lies on what he knows (domains of knowledge) but in what he does with what he knows (use of knowledge) (Eraut, 1996 cited by Llinares, 2008). Delaney, Ball, Hill, Schilling, and Zopf (2008) propose four categories for the teachers' knowledge: (1) Common knowledge of the content, as the knowledge and the math skills that any educated adult is expected to have; (2) Specialized knowledge of the content, such as the knowledge that the teacher requires in his work and that goes beyond the knowledge of an experienced literate adult; (3) Knowledge of the content and the students; and (4) Knowledge of the content and teaching. This definition of the teachers' knowledge is in force, because the subsequent proposals, although envisaged to involve different aspects associated with the teacher's knowledge, are fully covered by the proposals set by Delaney et al. (2008). In addition, this definition and the proposed components thereof precise or complement in many cases that exposed by authors that preceded them, reason by which it may be considered as the most complete.

Regarding to the Development of Student Teachers' Knowledge

Based on that exposed by Ball and Cohen (1999), Llinares (2000, 2004, 2008) believes that the way in which the students teachers and mathematics teachers need to understand the mathematics differs from the way in which other professionals require learning about them. As a result, according to this author, in a training program for teachers the mathematical content must be "different" of mathematics seen in other different professional spheres (architects, professional mathematicians, engineers, economists and so on). Another aspect that sets out a radical difference in the training of teachers when compared to other professionals is that the future mathematics teachers should be familiar with epistemological knowledge of mathematics. In order to make decisions in the classroom, teachers exercise either explicit or implicit use of all kinds of knowledge, of methods, of beliefs and perceptions about how you look for knowledge, learn it and organize it. Usually you create this epistemological background, essentially, in an empirical way to respond to the educational needs (D'Amore, 2007).

Taking into account the considerations on the training of teachers, Llinares et al. (2008), state that it is generated a need among teachers and student teachers to investigate the potential of the "mathematical situations", by seeing them as instruments of mathematical learning. A previous task aimed to see the mathematical situations as tools for the learning of student teachers is to explore the mathematical possibilities of the problem, identify potential targets to achieve via the resolution of this task in an educative context and try to anticipate possible strategies of the students (Llinares et al., 2008).

Llinares et al. (2008) consider that in order to perform an analysis of the teaching situation, the student teachers need to understand the task and the mathematics involved. In addition, according to these authors such situations do not only imply solving the problem by designing strategies, guessing relations that must be tested or generalizing through modifying the presentation of the problem, but also considering the problem as an instrument with which it is possible to generate mathematical learning. In this way, according to these authors, the introduction of that "didactical" in the analysis of the math activities, when you see them as a learning tool, becomes in itself in a didactic goal for the trainer of teachers.

Llinares (2008) stresses the importance of the use of knowledge in the resolution of the problematic situations generated in their professional activity, i.e. the practice of teaching mathematics understood as: (1) Conducting a few tasks (system of activities) to achieve an end; (2) Using some instruments; and (3) Justify its use (Llinares, 2008). When considering the teaching of mathematics as a practice to be understood and learned, Llinares (2004, 2008) identifies three systems of activities that articulate it and the components of the professional knowledge that enable you to perform it, namely: (1) Analyze, diagnose, and assign meaning to the mathematical output of their pupils and compare it with what he intended (objectives); (2) Plan and organize the mathematical content to teach it - to define action plans; and (3) Provide direction and manage the mathematics communication in the classroom.

In order to develop each of these "systems of activity", the student teachers must become competent in the different aspects that define them, and therefore "know" what is the foundation of such system, in this way the respective teaching competence is generated (Llinares, 2004, 2008). Steaming from this consideration emerges naturally a demand to talk about the competence as a fundamental part of the knowledge of mathematics teachers and that of mathematics student teachers.

Regarding to Competence

The concept of competence has a double meaning, according to the dictionary from the Spanish Royal Academy (2001); the first meaning comes from the Latin "competentĭa" associated with the verb "competĕre" that means compete. By using this notion is possible to find the following meanings: dispute or argument between two or more people about something; competence seen as opposition or rivalry between two or more who aspire to obtain the same thing; the third meaning refers to a situation involving companies that compete in a market by offering to or demanding the same product or service, and finally there is a meaning related to sports competition.

The second meaning has given rise to the word "competent", in which there are three meanings: competence seen as responsibility, skill, aptitude, suitability for making or intervene in a particular case and as a real power of attorney of a judge or other authority for the knowledge or resolution of an issue. This way, there are two verbs "compete" and "relate" that come from the same Latin verb "competĕre", but that differ substantially. However, these two verbs give origin to the noun "competition". That makes extremely difficult its understanding and may give rise to potential confusions.

Gómez (2007), Rico and Lupiáñez (2008), and Lupiáñez (2009) claim that has been commonly accepted that one of the first specific meanings assigned to the term competence comes from Chomsky's linguistics work (1957, 1965, 1980). According to these authors, under the Chomsky's linguistics perspective the competence in a language is defined as the mastery of the principles that govern the language and the performance as the manifestation of the rules that derive from the use of language. According to Puig (2008), the linguistic competence is presented in opposition to the performances, which are specific in said competence that is updated, "and the opposition is based on the fact that Chomsky argues forcefully that the Linguistic theory is concerned primarily with an ideal speaker-listener … who know its language perfectly … and is unaffected by such grammatically irrelevant conditions" (p. 3).

Hymes (1971) proposes going beyond the definitions of competence and performance set by Chomsky (1965), because he considers they are based on a personal performance idea with socio-cultural importance. This author defines competence as the capabilities of a person, but that do not depend exclusively on the knowledge, but also on the ability to use them. Hymes (1971) defines the knowledge, the ability in its use and role models as elements of competence. According to Bustamante (2003), these definitions have points in common with the definition provided by Chomsky, because competence is personal, the systematic possibility belongs to the structure, the system exceeds the individual, and the skill related to the use incorporates cognitive elements. Despite the similarities, Bustamante (2003) highlights aspects that make the definitions by Hymes different from those outlined by Chomsky, namely: the systematic possibility also has to do with the structure of other systems of signs; the knowledge varies from one individual to another; the skill in the use incorporates non-cognitive aspects; the performance is not a deficient execution of the competence; and the competences of individuals are over determined and the context is key for its interpretation.

The differences and similarities presented between the definitions by Chomsky and Hymes are an example of that explained by Weinert (2001). According to the former, there are many different theoretical approaches and the meanings vary depending on the point of view and the underlying objectives associated with the use of the term competence, both in the scientific debate as in political discourse. This review will include the various meanings given to the concept of competence.

Braslavsky (1993) defines that a competence is a know-how with knowledge and with consciousness concerning to the impact of such skill. For Coolahan (1996), the competence is the overall capacity based on the knowledge, experiences, values, and attitudes that a person has developed through commitment to educational practices. On the other hand, Perrenoud (1997) also speaks of the competence in terms of capacity, but understood as the skill to perform effectively in a number of situations, capacity which is based on the knowledge, but that is not restricted to them.

At the beginning of the twenty-first century, Roegiers (2000) defines the competence as the possibility of mobilizing an integrated set of resources in order to resolve an issue that belongs to a family of situations. Prieto (2002) picks up the idea of competence in terms of capacity and considers that the competences tend to convey the meaning of what the person is capable of or is competent to execute, the degree of preparation, sufficiency, or responsibility for certain tasks.

Beckers (2002) points out that competence mobilized various resources under the service of an action with a precise goal. According to this author, the competence is the skill that allows the subject to mobilize, in an integrated manner, your internal resources (knowledge, know-how, and attitudes) and external, in order to effectively resolve a family of complex tasks for him. Also, in 2002, the education network of the European Commission (EURYDICE) defined the competence as the skill or power to act effectively in a certain context (EURYDICE, 2002).

In view of the conceptual issues and discussions about the definitions, Rychen and Tiana (2004) propose a functional approach to the competencies, such as that taken for the initiative of Definition and Selection of Competencies: Theoretical and Conceptual Foundations (DESECO) and carried out jointly by the Swiss Federal Office of Statistics and the Organization for Economic Cooperation and Development (OECD-Switzerland). This conceptualization entails a holistic approach in the sense that it integrates and links external demands, individual features (including ethics and values) and the context as essential elements of competent performance. In this context, a competence is defined as "the skill to meet complex requirements in a satisfactory manner or to carry out a task or activity" (p. 21). From this perspective these authors define competence as: "combination of interrelated cognitive and practical skills, knowledge, motivation, values and ethics, attitudes, emotions, and other social and behavior components that together can be mobilized for effective action in a particular context" (Rychen & Tiana, 2004, p. 58).

In accordance with the abovementioned, it is found greater logic to what was said by Weinert (2004) who claimed that if we restrict our approach to the use of the competence term in philosophy, psychology, linguistics, sociology, political science and economics fields, there is still a wide variety of definitions. However, this author claimed that in all of these disciplines competence is interpreted as a fairly specialized system of abilities and capabilities necessary or sufficient to achieve a specific goal.

Rico and Lupiáñez (2008) collected and analyzed different approaches to the notion of competence. They established three central ideas (invariants) that, more or less explicitly, influence in all of these notions: cognitive components or other types that fall into the definition that each author provides about competence, purpose or purposes assigned to him, as well as the context in which competence is found or performed.

With regard to the meaning of competence in education, Rico and Lupiáñez (2008) summarized three important ideas, namely:

- the competence reveals itself, works through performance, and it expresses in various ways, either generic or specific as performing, interpreting, and solving problems, facing complex demands or applying knowledge to the practice;
- the competence is shown via the personal and social development of the competent subject, which is also expressed in various ways related to how to live, develop capacities, take decisions, continue learning, work, or improve the quality of life.

Competence always refers to an application context. There is a clear emphasis on action and development, which comes from the cognitive and attitudinal components and takes place in a specific context, and in a contextualized way. The references are wide and sometimes, imprecise, but leave no room for doubt. The last analysis mentioned certain or established situations, and references to contexts have greater diversity and convey a better accuracy: academic, professional or social contexts in a variety of areas, in educational practice or in society.

Rodriguez (2007) presents a conception of competence in which you can observe the three components stated by Rico and Lupiáñez (2008), because this author believes that the competence is a concept that integrates: "the “know-how” - theoretical or propositional knowledge.. derived from the empirical or logical assertions about the world -, knowing how to do - practical knowledge or development of the skills and abilities required to operate in the world - and to know how to be - experiential knowledge, also known as the "know". That is, the set of rules, values, attitudes and circumstances that allow you to interact successfully in the social environment" (p. 146).

Aspects of the competences related to such a knowledge mentioned by Rodriguez (2007) can be largely identified in the definition by Vasco (2011) of this term. Vasco defined that competence "can be described more precisely as a set of knowledge, skills, attitudes, understandings, as well as cognitive, metacognitive, affective and psychomotor attitudes properly interrelated to facilitate the efficient and flexible performance with a sense of an activity or a certain type of task in relatively new and challenging contexts" (p. 21).

In the definition by Vasco (2011) on competence, you can identify each of the features that others have also identified on this term. However, it is possible that when conducting an analysis that requires the identification of the competences that a teacher or teacher student requires, such definition turns out to be little operational. This is why it is practical to go to the definition established in 2008 by D'Amore, Godino, and Fandiño (2008, p.12), who state that competence is a complex and dynamic concept:

> complex: it is about a set of two components: use (exogenous) and domain (endogenous); even it involves a cognitive, creative and interpretative processing of knowledge that relates different contents; dynamic: the use and domain are not the only expressions of the competence; the competence seen as an object in itself encompasses not only knowledge that is required, but also meta-cognitive factors: the acceptance of the stimulus to use them, the will to do so, the will to complete the knowledge previously revealed as insufficient before evidence of the facts, and finally the very will to increase their own competence.

As mentioned in the section on the knowledge development for student teachers, for Llinares (2008) it is important that both the teacher and the student teacher be competent in the resolution of the problematic situations generated in the practice of mathematics teaching. This makes clearly that this document reflects and analyses in depth on the fundamental characteristics of the practice of teaching mathematics.

The Practice of Teaching Math and the Management of the Teaching-Learning Process

Jackson (1975) pointed out various phases in which teacher's practice activities take place. He named them as proactive phase (before the class), interactive phase (during the class) and postactive phase (after the class). Ponte (1995) refers to the teaching practice when he considers that knowledge into action relates to three areas: teaching practice, non-teaching practice and professional development. This knowledge is closely related to the reference knowledge (which include the knowledge related to contents from teaching, pedagogy, and curriculum), as well as with various processes of reflection (by, in, and on performance) (Ponte, 1995).

Regarding the performance of teaching practice, Ponte (1995) presents two distinct domains: the first is the knowledge about classroom management and the second is the didactical knowledge. These two domains are deeply intertwined, so that everything that happens in each of them reflects immediately in the other (Doyle, 1986). However, it is pertinent to classify them by taking into account the achievement of objectives, because in each domain those are different, they both have different logics and the relations of each one with the different reference fields, are also different.

For Ponte (1995), the knowledge of classroom management includes everything that allows the teacher to create an environment conducive to learning, establishing the rules for his work, implementing organization methods of the students, compared to the situations or behaviors consistent with his rules etc. On the other hand, Ponte states that regarding to didactical knowledge you can distinguish four fundamental aspects: a curriculum guide, a calendar, the monitoring, and the evaluation.

The considerations about the knowledge put into practice, and in particular on the practice made by Ponte (1995), can be related to the phases proposed by Jackson (1975) in the teaching practice and even more with the phases that Llinares (2000) established by taking Ponte's work as basis. For Llinares (2000), the first phase is the phase of planning and organization of the mathematics to be taught, that is to say, it is the phase in which decisions are made about what to teach and how to teach it. The second phase is the phase of management of the teaching-learning process, in which takes place the relationship between the problem exposed and the students in the classroom context, and the third phase is the stage of reflection and new understanding (Llinares, 1991), and the purpose thereof is to learn from their own experience.

Llinares (2000) states that some of the teacher's tasks in the management phase of the teaching-learning process are specific to the mathematical content, while others boasts a general nature (in the sense of Doyle, 1986) related to the organization of students, the handling of order and discipline, the tasks proposed, among others. In relation to the tasks of the mathematical content, this author believes that they are those linked to the management of the interaction between the students and the mathematical knowledge that underlies the mathematical problem proposed (Saraiva, 1996; Perrin-Glorian, 1999 and Llinares, 2000) and in the definition on classroom speech (Hache & Robert, 1997). This is, using the pedagogic discourse and the communication that is conducive.

However, in relation to the tasks associated with classroom management, Duke (1979) considered that all of those provisions and procedures required establishing and maintaining an environment in which is possible to deliver instruction and learning. In this regard, Emmer (1987) stated that the classroom management was a set of behaviors and activities by teacher, aimed to ensure that students adopt a proper conduct and that distractions come to a minimum.

Based on the considerations of Duke (1979) and Emmer (1987) on the classroom management, Davis and Thomas (1992) set out recommendations for such management. Management breaks down into four major categories, namely: those recommendations associated with the standards and expectations, other related with the organization of the classroom, other relating to the activities in the classroom and finally recommendations aimed to reduce bad behaviors or deviations. However, all these recommendations focus basically on maintaining the classroom order and discipline among students, as well as other provisions that allow you to have the control of the classroom. That is to say, they fall within the general teacher's tasks that Llinares (2000) established in the management phase of the teaching-learning process.

Some of the teacher's activities that Llinares (2000) identified within this phase, are: (1)

The management of the various phases or sections that make up each class, lesson, theme or unit of teaching and learning that constitute the math lesson; (2) Information's presentation; (3) The management of the work and the group discussion; (4) The interpretation and response given to the ideas of the students; (5) The management of the discussion in a large group; (6) The construction and use of representations; (7) The introduction of teaching materials or computing environments; and (8) Manage the construction of the new mathematical knowledge from interaction teacher-student-homework etc.

Regarding the design and management of teaching-learning situations and teaching materials, Niss (2003) identified it as a didactical and pedagogical competence of mathematics teacher. On the other hand, Rico (2004) established that the management of mathematical content in the classroom is one of the core competencies of the mathematics teacher. In other words, for these authors addressing the management of the teaching-learning process reflects a fundamental competence of the teacher, which should be developed in the student teachers.

However, in accordance with the above, it is reasonable. Marin (1997) and Zabalza (2004) state that management of the teaching-learning process in classroom is difficult, as it requires to take into account many aspects, which arise directly in the context of the classroom and therefore cannot always be planned in advance. However, according to Lupiáñez (2009), the management can be seen from the planning's point of view.

According to Gomez (2007), the classroom management seen as "the planning and the procedure for establishing and maintaining an environment in which teaching and learning can take place" (Duke, 1979, cited in Doyle, 1986, p. 394), implies that the teacher must not only act to "maintain discipline in the classroom". In fact, he should also plan and manage his performance in such a way that the students achieve the learning objectives. In this way, management can be considered as the competence that the teacher has to plan, act in the classroom and interact in a way that creates a conducive learning environment for mathematics.

Bohórquez, Bonilla, Narváez and Romero (in press) propose the tasks to be followed by teacher when he is performing in a learning environment based on the resolution of problems. One of the tasks that they consider as fundamental is that a large part of the teacher's performance must involve the redirecting of his prominent role in order to achieve communication among students, in both small groups and the whole class. For this reason, they state that the incorporation of methods that improve such communication is crucial for a change in the way mathematics classes are performed.

Some of the mechanisms that Bohórquez et al. (published) suggested are:

1. Organizing small groups to work on the addressing and resolution of the problems;
2. The presentation of resolution's progress achieved by group to the other groups in general;

3. The participation of the large group in the presentations of other groups;
4. The group participation in the virtual classroom;
5. The designing of the "resolver notebook", both for individual and collective use.

Taking into account the above, it is definitely mandatory for student teachers' development that teacher learns how to analyze the roles of the mathematical problems and the teacher's management once students face such problems. He is required to perform in such a way that he would be able to determine the characteristics of the problem that allow him to create a suitable environment for the mathematics learning.

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Perceptions of Teachers and Experts in Terms of the Investigation of Primary School 4th and 5th Grade Science and Technology and Mathematics Curricula Regarding Science and Mathematics Integration

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ABSTRACT: *The study focuses on the level of suitability according to teachers and experts in the integrations of objectives in Turkish science and technology, and mathematics curricula in primary school 4th and 5th grades and whether there are other objectives that can be integrated. The study group is composed of 12 fourth-grade teachers, 14 fifth-grade teachers. Two assistant professor doctors and a research assistant participated to the study as experts. Surveys, one for 4th grade and one for 5th grade, were reviewed by the experts. The aim of the surveys was to investigate the suitability level among objectives in 4th and 5th grade science and technology, and mathematics curricula. The obtained data showed that associations in objectives between science and technology, and mathematics curricula were suitable; however, there were other objectives that could be integrated besides these associations. Furthermore, the findings revealed the topics in the 4th and 5th grade science and technology, and mathematics curricula among which integration was possible.*

Key words: *Mathematics curriculum, Science and technology curriculum, Integration*.

INTRODUCTION

One of the common objectives of mathematics, science and technology curricula is the provision of meaningful learning. Meaningful learning can be provided by the application of information to different environments, by the formation of relationships among pieces of information and by the adaptation of information to various representations (Ministry of National Education [MoNE], 2009). Therefore, investigating the relationships between mathematics, science and technology lessons and providing integration of these classes are very important in the establishment of meaningful learning for students.

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Many studies in the field of mathematics education show that usage of real life representations in classes by teachers supports effective conceptual understanding by students and provides meaningful learning (Rogers, Volkmann & Abell, 2007). American Association for the Advancement of Science (AAAS) (1989) emphasizes the fact that science is related to real life subjects. Therefore, mathematics, science and technology classes can be integrated with each other because science and technology classes inherently consist of real life problems. Moreover, using real life problems in teaching mathematics supports the comprehension of concepts and plays an important role in teaching. From a different viewpoint, concepts and relationships may need to be symbolically represented since some of the concepts and relationships are abstract despite the real life content of science and technology (Roth, 2005). Since symbolic representation of abstract concepts and the relationships among these concepts are characteristics of mathematics (Ball, Bass & Hill, 2004), some of the concepts in science and technology classes can be expressed by using mathematical terms. In line with these viewpoints, it can be stated that there is a crucial relationship between science, technology and mathematics.

Science and technology classes are rich in content for the presentation of mathematical relationships and the provision of patterns among these relationships. Therefore, mathematical concepts and patterns are found in science and technology classes and students may need to associate what they learn in terms of concepts and patterns with mathematics. The investigation of mathematics curriculum also points to various subjects, concepts and data that can be integrated with science and technology such as statistics, probability, table and graphic formation, measurement and ratio and proportion (MoNE, 2009). As various concepts and data are found in mathematics classes that can be associated with science and technology, there are mathematical concepts and patterns in science and technology classes as well. In this regard, presenting the connections and relationships between science, technology and mathematics is crucial for students to make sense of subjects and concepts taught in science and technology and mathematics classes and to apply what they have learned to other problem situations.

Since the early 1900s, the integration of science and mathematics has been suggested as a way that enables students to understand science and mathematics, improve their performance and point of views (Berlin & White, 2001). Various international organizations have expressed opinions in presenting the relationships among science and technology, and mathematics. With the help of international organizations such as National Council of Teachers of Mathematics (NCTM), National Research Council (NRC), and AAAS, integration of science and mathematics has gained a central focus at schools (Berlin & White, 2001; Furner & Kumar, 2007). These international organizations support the integration of the two disciplines by emphasizing the relationships between science and mathematics. According to NCTM (2000), it is "important to integrate and associate mathematics with other fields such as science and arts" (p. 212). Similarly, NRC (1996) suggest that "the science program should be coordinated with the mathematics program to enhance student use and understanding of mathematics in the study of science and to improve student understanding of mathematics" (p.14), while AAAS (1989) states the importance of science and mathematics in "the understanding of the processes and meaning of technology" (p. 9) and emphasize the vitality of their integrations with the technology education curricula.

While international organizations such as NTCM, NRC, and AAAS emphasize the relationships among science and technology, and mathematics, it is also thought that integrating science and technology, and mathematics classes is important in terms of student motivation. Generating student motivation in classes is regarded as an important objective in all disciplines for teachers (Haigh & Rehfeld, 1995) who are expected to create meaningful problems that will increase students' motivation (Ball, 1990; Brookhart, Walsh, & Zientarski, 2006; La Turner, 2000; Schulman & Schulman, 2004). Student interests, motivation and willingness to learn will increase when students make associations between *what they have learned* in the class with *how and where to use the knowledge in their daily lives* and with the areas in which the knowledge can be beneficial and when they realize the relationships between classroom objectives and daily life objectives. Identifying the relationships between science and mathematics and integrating these two disciplines are believed to increase student motivation. Therefore, integration of science and mathematics is important to increase motivation, develop positive attitudes towards science and mathematics classes and enhance student achievement (Furner & Kumar, 2007; Watanabe & Huntley, 1998).

Student interest and motivation are expected to increase along with their achievements under conditions where associations between science and mathematics classes are generated in an appropriate manner and these two classes are integrated because mathematics supports the comprehension of science concepts by explaining the relationships among subjects and concepts in science and by providing means to identify the quantity of these subjects or concepts (McBride & Silverman, 1991). The reverse is true for mathematics integrated with science since science supports student learning by generating content based on real life and problem based learning (Kumar & Sherwood, 1997).

In addition to the importance of the approach of integration of mathematics, science and technology in enhancing student interest, motivation and achievement, this integration is also important for teachers who follow the curricula (Watanabe & Huntley, 1998) because "they need to comprehend how a given idea is related/associated with other ideas in the same context and ideas in other contexts" (Shulman, 1987, p. l4). Teachers' ideas about the integration of mathematics, science and technology classes provide important tips for us regarding the integration of these classes because one of the most effective factors in the success of curricula is the teacher (Howson, Keitel & Kilpatrick, 1981). The ideas and beliefs of teachers either facilitate or complicate the application of curricula (Sosniak, Ethington & Varelas, 1991; Koehler & Grouws 1992). In this context, teachers' ideas and beliefs should be taken into consideration to achieve success in the proposed teaching programs (Knapp & Peterson, 1995).

In Turkey, studies regarding the integration of science and technology, and mathematics have started to emerge recently. Within this respect, as well as the need to work on the integration of science and technology, and mathematics in the world (Ferrini-Mundy, 2013), there is also a need to work on this field in Turkey (Corlu, 2014). Analyzing the studies regarding the integration of science and technology, and mathematics in Turkey; in his study, Taşdemir (2008) has stressed that constructivist teaching that included mathematical thinking activities in science and mathematics courses has a significant impact on students' academic achievement, attitudes and problem solving skills. In his study, Obalı (2009) has stated that the academic achievement in science and technology courses has increased in accordance with the achievement on reading comprehension in Turkish and natural numbers, fractions and decimal fractions in mathematics courses. Additionally, in their studies, Taşdemir and Taşdemir (2011) revealed that the correlation between science and technology, and mathematics has the highest correlation among 4^{th}, 5^{th}, 6^{th}, 7^{th} and 8^{th} grades final achievement scores in terms of Turkish, Mathematics, Science and Technology and Foreign Language Courses. In their study that aimed to investigate the impact of the integration of science and mathematics, Kaya, Akpınar and Gökkurt (2006) have concluded that the integration of science and mathematics has an impact on students' achievement. Besides, Kıray (2010) has revealed a model inspired by several models in the literature regarding the integration of science and mathematics. In their study, Corlu, Capraro and Capraro (2014) have tried to introduce STEM education within the theoretical framework. Şahin, Ayar, and Adıgüzel (2014) have demonstrated a framework regarding the evaluation of STEM activities. Through the studies, it is seen that the studies regarding the integration of science and technology, and mathematics in Turkey have started to emerge recently. The call letter included in the study conducted by Corlu (2014) regarding the studies on STEM education is an indicator of this situation as well.

By taking experts' views into consideration, the current study investigates whether the integration of science and mathematics has been obtained in a satisfactory level in the objectives in science and technology, and mathematics classes in primary school 4^{th} and 5^{th} grades, whether the integration that exists in 4^{th} and 5^{th} grade science and mathematics programs is satisfactory and whether there are other objectives that can be integrated.

Since similar concepts or methods in science and technology, and mathematics classes serve the objectives of both disciplines, there are fields in science and mathematics that cannot and should not be separated from each other (AAAS, 1989; NCTM, 1989). However, by themselves, students cannot integrate these fields that are connected in science and technology, and mathematics curricula and as a result of this, they believe that these two disciplines are unrelated and disconnected (Wicklein & Schell, 1995). In the absence of student-centered integration, it is important to provide the relationships between science and technology curricula in an active manner. In their study, Uzun, Bütüner and Yiğit (2010) stated that it is required to present the relationships between science and mathematics fields and to provide an effective spiral structure between these two.

It is believed that the findings obtained in the current study which investigates the primary school 4^{th} and 5^{th} grade science and mathematics objectives in line with science and mathematics integration will be important in presenting the relationships between these two curricula. Results of studies also show that teachers believe in the necessity of integration but they are not sufficiently informed about integration (Başkan, Alev & Karal, 2010). Therefore, presentation of the relationships among the objectives in primary school 4^{th} and 5^{th} grade science and mathematics curricula and identification of other objectives in the fields outside the integrated domain are expected to provide new ideas about integration and are beneficial in the field for teachers who implement the curricula, for program developers and scientists who work in the field.

Problem Statements

Are there any sufficient connections and integration among the objectives in primary school 4^{th} and 5^{th} grade Science and Technology, and Mathematics curricula in line with science and mathematics integration?

1. What is the level of suitability of the integration between primary school 4^{th} and 5^{th} grade Science and Technology, and Mathematics curricula according to Science and Technology and Mathematics objectives?
2. Are there any different objectives that can be integrated other than the integration among the objectives of science and technology, and mathematics in primary school 4^{th} and 5^{th} grade Science and Technology, and Mathematics curricula?

METHOD

Research Model

The current study which aimed to investigate the relationships among the primary school 4^{th} and 5^{th} grade objectives in science and technology, and mathematics curricula in the context of science and mathematics integration employed a quantitative research method. As a quantitative research method, survey model was preferred that aims to identify a current or past event, individual or the object in question as is without any changes in the situation (Karasar, 2005).

Study Group

Since the study aimed to describe the relationships of objectives in primary school 4^{th} and 5^{th} grade science and technology, and mathematics curricula in terms of science and mathematics integration, the participants of the study were composed of experts and teachers. Experts were two assistant professors and a research assistant with proper qualifications to design 1-8^{th} grade Science and Technology, and Mathematics curricula. Twelve 4^{th} grade teachers and 14 5^{th} grade teachers, who used Science and Technology, and Mathematics curricula during the 2011-2012 educational year, were participated in the study.

Table 1 presents the distribution of frequencies, maximum and minimum years of service and average years of service for teachers who taught 4th and 5th grade science, technology and mathematics classes.

Table 1

Characteristics of Participant Teachers

Teacher	Frequency (f)	Min. year of service	Max year of service	Average year of service
4th grade teachers	12	4	33	16.3
5th grade teachers	14	3	31	22.3

Table 1 displays a total of 12 teachers who taught 4th grade and answered the survey. The teachers with the minimum years of service were found to have 4 years of experience and the teachers with maximum years of service were found to have 33 years of experience with an average of 16.3 years of experience. A total of 14 teachers who taught 5th grade answered the survey. The 5th grade teachers with the minimum years of service were found to have 3 years of experience and the teachers with maximum years of service were found to have 31 years of experience with an average of 22.3 years of experience.

Instruments

Separate surveys were developed for 4th and 5th grades in the study to present the relationships of objectives in primary school 4th and 5th grade science and technology, and mathematics curricula in terms of science and mathematics integration. Prior to the development of surveys to collect data from 4th and 5th grade teachers and the experts, 4th and 5th grade science and technology, and 4th and 5th grade mathematics textbooks were examined. Literature reviews were also undertaken regarding the topics that can be integrated in the field of science and mathematics integration. These investigations and reviews provided the topics that can be integrated and the related objectives. Later, objectives in 4th and 5th grade science and technology, and mathematics curricula were examined and the draft survey was developed which included the objectives in 4th and 5th grade science and technology, and mathematics curricula and the objectives that are thought to be integrated. Subsequent to the development of the draft survey, expert views were taken, pilot implementation was carried out and the survey was finalized after required adjustments and modifications.

The survey prepared for 4th and 5th grades included 11 items for 4th grade and 27 items for the 5th grade. Introduction of the surveys for the 4th and 5th grades included the request that participants should not be biased towards the survey and should answer the items comfortably and emphasized that the data would be used only for scientific purposes so that unbiased responses could be obtained.

Data Analysis

The sections in the surveys that corresponded to the 1st and 2nd sub-problems of the first research question that required "yes/no" options were analyzed with frequencies and percentages. Then, the data was tabulated.

FINDINGS AND INTERPRETATIONS

Findings and Interpretations Regarding the Research Questions 1.1 and 1.2.

In this section which provides the findings regarding the suitability of integration among the objectives of science and technology in 4th and 5th grade science and technology, and mathematics curricula and the existence of other possible objectives which can be integrated, 4th and 5th grade surveys were examined separately. The frequencies and percentages of the Yes/No sections are revealed in tables.

Findings and Interpretation Regarding the 4th Grade Science and Technology and Mathematics Curricula

The Table 2 presents the percentages and frequencies of the responses provided by the experts and the teachers.

Table 2

Frequencies and Percentages of 4th Grade Survey

						Teacher (12)		Expert (3)	
Unit	No	Obj.	Science and Technology Objectives	Mathematics sub learning domain	Mathematics Objectives	Yes % *f*	No % *f*	Yes % *f*	No % *f*
1. Let's solve the puzzle of our bodies	1	3.4	Takes own pulse and the pulses of others	4th grade: multiplication with natural numbers	7- Forms or solves problems that require multiplication with natural numbers.	91.7 *11*	8.3 *1*	66.7 *2*	33.3 *1*

	2	4.2	Records and interprets data obtained for the pulse after exercise	4th grade: bar chart	1- Creates bar charts	100 *12*	0 *0*	100 *3*	0 *0*
	3	4.3	Realizes through observation about the effect of exercise on the frequency of ventilation	4th grade: bar chart	2- Interprets bar charts.	100 *12*	0 *0*	100 *3*	0 *0*
	4	1.2	Classifies matter according to characteristics identified by five senses	5th grade: tables and diagrams	2- Arranges data by creating diagrams	100 *12*	0 *0*	100 *3*	0 *0*
	5	2.7	Classifies matter according to solid, liquid and gas forms	5th grade: tables and diagrams	2- Arranges data by creating diagrams	100 *12*	0 *0*	100 *3*	0 *0*
2. Let's Learn about the matter	6	3.3	Converts mass units (kg-g/g-kg)	4th grade: weighing	2- Identifies relationships between Ton-kilogram, kilogram-gram and gram-milligram	91.7 *11*	8.3 *1*	100 *3*	0 *0*
	7	3.5	Converts volume units (L-mL/mL-L)	4th grade: measuring the liquids	2- Conversions between liter and milliliter	83.3 *10*	16.7 *2*	100 *3*	0 *0*
	8	5.1	Measures the temperature of different matter with thermometer and states them with the use of ^{0}C	4th grade: bar charts	1- Creates bar charts	100 *12*	0 *0*	100 *3*	0 *0*
	9	5.1	Measures the temperature of different matter with thermometer and states them with the use of ^{0}C.	4th grade: natural numbers	6- Counts up to six digit numbers	75 *9*	25 *3*	0 *0*	100 *3*

3. Movement and Force	10	1.3	Compares and classified beings according to characteristics of movement (changing direction, acceleration, slowing down)	5th grade: tables and diagrams	2- Arranges data by creating diagrams	91.7 *11*	8.3 *1*	33.3 *1*	66.7 *2*
4. Our Planet, the Earth	11	1.1	States that the shape of the Earth is a sphere	3rd grade: planes and platforms	2- Knows the areas of cube, tetragonal prism, rectangular prism, triangular prism, cylinder, conic and spherical models	100 *12*	0 *0*	100 *3*	0 *0*

Objectives in Table 2 such as “Records and interprets data obtained for the pulse after exercise” (Objective 2), “Converts mass units (kg-g/g-kg)” (Objective 6) and “Converts volume units (L-mL/mL-L)” (Objective 7) are the objectives in 4th grade science and technology curriculum integrated with the objectives in 4th grade mathematics curriculum. All of the experts that participated in the study stated that these objectives could be integrated with the related mathematics objectives. All the teachers thought that Objective 2 could be integrated with the related mathematics objective. For objectives no. 6 and 7; 91.7% of the teachers believed that Objective 6 could be easily integrated with mathematics objectives and 83.3% of the teachers believed that Objective 7 could be easily integrated with mathematics objectives as well. Since the majority of the teachers and all of the experts thought that objectives no. 2, 6 and 7 could be integrated with related mathematics objectives, it can be said that the integration of the mathematics objectives with the objectives in the 4th grade science and technology, and mathematics teaching programs is suitable. Therefore, the integration that exists between science and technology and mathematics objectives in the 4th grade science and technology and mathematics curricula is suitable.

The objectives in Table 2 other than objectives 2, 6 and 7 are the objectives that are not integrated in the 4th grade science and technology, and mathematics curricula. The percentages and the frequencies of the teachers’ and experts’ responses in these objectives show that both the teachers and the experts believe in the possibility of integration for these objectives such as “Realizes through observation about the effect of exercise on the frequency of ventilation” (Objective 3), “Classifies matter according to characteristics identified by five senses” (Objective 4), “Classifies matter according to solid, liquid and gas forms” (Objective 5), “Measures the temperature of different matter with thermometer and states them with the use of ^{o}C” (Objective 8) and “States that the shape of the Earth is a sphere" (Objective 11). Therefore, it can be thought that these objectives can be integrated with the related mathematics objectives in the 4^{th} grade science and technology curriculum. However, integration of the mathematics objectives in no. 4 and 5 can be problematic since they are objectives that belong to grade 5. The problem can be solved by including these 2 objectives within the 4^{th} grade mathematics objectives or necessary adjustments can be undertaken for these objectives. One of the experts has negative ideas about the integration of the following objective with mathematics objective: "Takes own pulse and the pulses of others" (Objective 1). Only one teacher out of 12 stated that this objective cannot be integrated with mathematics objective. About 92% of the teachers and two experts out of three believe that this objective can be integrated within the mathematics objective. Therefore, this objective can also be integrated with the related mathematics objectives in the 4^{th} grade science and technology curriculum. Hence, the objectives no. 1, 3, 4, 5, 8, and 11 can be regarded as objectives that can be integrated with the related mathematics objectives in the 4^{th} grade science and technology curriculum and there may also be other objectives that can be integrated. Also, based on the findings on Table 2, the objectives no. 1, 3 and 4 are thought to be suitable for integration in the 4^{th} grade mathematics curriculum. In this case, it can be stated that other objectives exist in the 4^{th} grade mathematics curriculum in addition to the existing ones.

Examination of objectives "Measures the temperature of different matters with thermometer and states them with the use of ^{o}C" (Objective 9) and "Compares and classifies beings according to characteristics of movement (changing direction, acceleration, slowing down)" (Objective 10) show that Objective 9 is the least preferred choice as an objective that can be integrated compared to other objectives. None of the experts believed that Objective 9 could be integrated with the related mathematics objective. However, 75% of the teachers were found to believe the reverse. Similarly, all three experts stated that Objective 10 could not be integrated with the related mathematics objective whereas about 92% of the teachers stated the opposite. A difference is observed in the views of teachers and experts in objectives no. 9 and 10. Therefore, a clear point of view has not been presented about the integration of these objectives. More detailed studies can be undertaken regarding the integration of these objectives to have a clearer idea.

Findings and Interpretation Regarding the 5^{th} Grade Science and Technology and Mathematics Teaching Programs

The table below indicates the percentages and frequencies of the responses provided by experts and the teachers.

Table 3

Frequencies and Percentages of 5th Grade Survey

Unit	No	Obj.	Science and Technology Obj.	Math. Sub Learning Domain	Math Obj.	Teacher (14) Yes % *f*	Teacher (14) No % *f*	Expert (3) Yes % *f*	Expert (3) No % *f*
1. Let's solve the puzzle of our bodies	1	1.8	Knows the importance of sell-by dates on packaged products	5th grade: measuring time	1- Forms and solves problems regarding time measurement units	100 *14*	0 *0*	33.3 *1*	66.7 *2*
2.Change and identification of states of matter	2	2.2	Shows through experiments that the same matter heats up less when given less heat and more when given more heat.	6th grade: ratio-proportion	1- Explains proportion and the direct proportions between quantities.	78.6 *11*	21.4 *3*	100 *3*	0 *0*
	3	2.3	Shows through experiment that the same amount of heat will heat up less matter more and more matter less.	7th grade: ratio-proportion	1- Explains the relationship between quantities with direct proportions and inverse proportions.	71.4 *10*	28.6 *4*	100 *3*	0 *0*
	4	2.9	Explains the quantity of 1 joule and 1 calorie with examples from daily life.	5th grade: multiplication with natural numbers	3- Determines the unknown multiplier in a multiplication with four digits.	85.7 *12*	14.3 *2*	66.7 *2*	33.3 *1*
	5	2.10	Converts energies given as joule and calorie into each other.	5th grade: multiplication with natural numbers	7- Forms and solves problems that require the use of multiplication with natural numbers.	78.6 *11*	21.4 *3*	100 *3*	0 *0*
	6	2.10	Converts energies given as joule and calorie into each other.	5th grade: division with natural numbers	5- Forms and solves problems that require the use of division with natural numbers	85.7 *12*	14.3 *2*	100 *3*	0 *0*

7	4.3	Uses experiment results to reach the conclusion that vaporization accelerates when more heat is given.	6th grade: ratio-proportion	1- Explains proportions and the relationship between quantities with direct proportion.	78.6 *11*	21.4 *3*	100 *3*	0 *0*
8	4.4	Expresses observations while a liquid is boiling.	5th grade: line chart	1. Forms a line chart. 2. Reads a line chart.	100 *14*	0 *0*	33.3 *1*	66.7 *2*
9	5.1	Designs an experiment that shows that pure substances have a stable boiling point.	5th grade: tables and diagrams	1- Forms and reads a table based on two properties.	85.7 *12*	14.3 *2*	66.7 *2*	33.3 *1*
10	5.2	Realizes that liquids can be recognized by their boiling points.	5th grade: tables and diagrams	1- Forms and reads a table based on two properties.	92.9 *13*	7.1 *1*	100 *3*	0 *0*
11	6.2	Shows through experiment that pure substances have a stable melting and freezing points.	4th grade: bar chart	1- Forms a bar chart.	92.9 *13*	7.1 *1*	66.7 *2*	33.3 *1*
12	6.3	Learns through experiments that freezing and melting points of the same substance are very close.	4th grade: bar chart	2- Reads a bar chart.	100 *14*	0 *0*	66.7 *2*	33.3 *1*
13	7.5	Knows the definition and unit of density.	5th grade: ratio-proportion	1- Expresses the relationship between two quantities as a proportion.	78.6 *11*	21.4 *3*	33.3 *1*	66.7 *2*
14	7.7	Compares the mass of objects of equal volume based on a list of densities.	5th grade: decimal fractions	3- Ranks three decimal fractions from smallest to biggest and biggest to smallest.	85.7 *12*	14.3 *2*	100 *3*	0 *0*

3.Electricity in our lives	15	1.7	Expresses how the brightness of a light bulb in a circuit where the number of light bulbs remains the same and the number of batteries increases or decreases.	6th grade: ratio-proportion	1- Explains proportions and the relationship between quantities with direct proportion.	100 *14*	0 *0*	100 *3*	0 *0*
4. The Earth, The Sun and the Moon	16	1.3	Ranks the Sun, the Earth and the Moon by size.	4th grade: natural numbers	6. Ranks six digit numbers.	92.9 *13*	7.1 *1*	100 *3*	0 *0*
	17	2.2	Expresses that the time it takes for the Earth to complete one circulation around itself is accepted to be a day.	5th grade: measuring time	1- Forms and solves problems about time units.	100 *14*	0 *0*	100 *3*	0 *0*
	18	2.6	Expresses that the time it takes for the Earth to make one full circulation around the Sun is accepted to be a year.	5th grade: measuring time	1- Forms and solves problems about time units.	100 *14*	0 *0*	100 *3*	0 *0*
	19	3.5	Reaches the conclusion that the phases of the moon are regularly repeating natural events through observation.	5th grade: measuring time	1- Forms and solves problems about time units.	92.9 *13*	7.1 *1*	0 *0*	100 *3*
5. Let's explore the world of living beings	20	6.3	Emphasizes the methods used over time to protect food from the detrimental effects of microscopic life forms.	5th grade: measuring time	1- Forms and solves problems about time units.	85.7 *12*	14.3 *2*	0 *0*	100 *3*
	21	7.2	Guesses about animals and plants that can adapt to a habitat in the vicinity.	5th grade: probability	Makes guesses about the chances of events occurring.	92.9 *13*	7.1 *1*	100 *3*	0 *0*

22	1.1	Realizes that light coming from a source spreads along lines.	6th grade: lines, line segments and closed half lines	2- Explains and uses symbols to represent line segments and closed half lines.	100 *14*	0 *0*	100 *3*	0 *0*
23	1.3	Uses lines to show the path that light follows between two points.	6th grade: lines, line segments and closed half lines	2- Explains and uses symbols to represent line segments and closed half lines.	100 *14*	0 *0*	100 *3*	0 *0*
24	3.4	Uses simple closed half line drawings to show the forming of shadows	6th grade: lines, line segments and closed half lines	2- Explains and uses symbols to represent line segments and closed half lines	100 *14*	0 *0*	66.7 *2*	33.3 *1*
25	4.3	Correctly measures the shadow length of a stick at different times of the day.	4th grade: measuring length	Forms and solves problems using length measurement units.	100 *14*	0 *0*	100 *3*	0 *0*
26	4.3 4.4	Correctly measures the shadow length of a stick at different times of the day. Records findings on a table.	5th grade: tables and diagrams	1- Forms and reads a table based on two properties.	100 *14*	0 *0*	100 *3*	0 *0*
27	4.5	Forms a bar graph showing the relationship between shadow-length and time.	4th grade: bar chart	1- Forms a bar graph	100 *14*	0 *0*	100 *3*	0 *0*

Objectives in Table 3 such as “Explains the quantity of 1 joule and 1 calorie with examples from daily life” (Objective 4), “Expresses that the time it takes for the Earth to complete one circulation around itself is accepted to be a day” (Objective 17), “Expresses that the time it takes for the Earth to make one full circulation around the Sun is accepted to be a year” (Objective 18) and “Reaches the conclusion that the phases of the Moon are regularly repeating natural events through observation” (Objective 19) are the objectives integrated both in 5th grade science and technology curriculum and in 5th grade mathematics curriculum.

Investigation of the findings in Table 3 shows that both the experts and the teachers believed that these objectives could be integrated with related mathematics objectives. Therefore, we can claim that objectives no. 17 and 18 can be suitable to be integrated with mathematics objectives. For Objective 4, two of the three experts and about 86% of the teachers believed that this objective could be integrated with related mathematics objective; hence, Objective 4 can be suitable to integrate with mathematics objectives.

None of the experts believed that Objective 19 could be integrated with related mathematics objectives. However, only one of the 14 teachers stated the same point of view supporting the ideas of the experts. While Objective 19 was regarded as an objective that could be integrated with mathematics objectives, experts stated the opposite. Therefore, since there are differences of opinion regarding this objective, it is thought to be best not to use a clear expression regarding the suitability of integration of this objective with the related mathematics objective. A more comprehensive research can be carried out in this context.

Examination of the frequencies and percentages (provided in Table 3) of the objectives other than 4, 17, 18 and 19 whose integration was implemented in 5th grade science and technology and mathematics curricula shows that all the experts and the teachers believed that integration with the related mathematical objectives was possible for the following objectives: “Expresses how the brightness of a light bulb in a circuit where the number of light bulbs remains the same and the number of batteries increases or decreases” (Objective 15), “Realizes that light coming from a source spreads along lines” (Objective 22), “Uses lines to show the path that light follows between two points” (Objective 23), “Correctly measures the shadow length of a stick at different times of the day” (Objective 25), “Correctly measures the shadow length of a stick at different times of the day, records findings on a table” (Objective 26) and “Forms a bar graph showing the relationship between shadow-length and time” (Objective 27). Therefore, these objectives can be integrated with the related mathematical objectives in science and mathematics curricula. All the teachers believed that objectives such as “Learns through experiments that freezing and melting points of the same substance are very close” (Objective 12) and “Uses simple closed half line drawings to show the forming of shadows” (Objective 24) could be integrated with the related mathematical objectives whereas two of the experts out of three believed the same. In this context, since most of the experts and all of the teachers believed that these objectives could be integrated with the related mathematical objectives, we can claim that these objectives are needed to be integrated with the related mathematics objectives in the science and mathematics curricula.

Objectives such as “Shows through experiments that the same matter heats up less when given less heat and more when given more heat” (Objectives 2 and 3), “Converts energies given as joule and calorie into each other” (Objectives 5 and 6), “Uses experiment results to reach the conclusion that that vaporization accelerates when more heat is given” (Objective 7), “Realizes that liquids can be recognized by their boiling points” (Objective 10), “Compares the mass of objects of equal volume based on a list of densities” (Objective 14), “Ranks the Sun, the Earth and the Moon by size” (Objective 16) and “Guesses about animals and plants that can adapt to a habitat in the vicinity” (Objective 21) were found suitable by the experts to be integrated with the related mathematical objectives. When we examine the responses provided by teachers regarding the integration of these objectives with related mathematical objectives, we see that the objective with the lowest percentage is no. 3 with 71.4% “yes” answer. Therefore, since most of the experts and all of the teachers believed that these objectives could be integrated with the related mathematical objectives, we can claim that these objectives are needed to be integrated with the related mathematics objectives in the science and mathematics curricula.

The objectives “Designs an experiment that shows that pure substances have a stable boiling point” (Objective 9) and “Shows through experiment that pure substances have a stable melting and freezing points” (Objective 11) were found suitable respectively by 85.7% of the teachers and 66.7% of the experts and by 92.9% of the teachers and 66.7% of the experts for integration with the related mathematics objectives. Since most of the experts and all of the teachers believed that these objectives could be integrated with the related mathematics objectives, we can claim that these objectives are needed to be integrated with the related mathematics objectives in the science and mathematics curricula.

Objectives no 2, 3, 5, 6, 7, 9, 10, 11, 12, 14, 15, 16, 21, 22, 23, 24, 25, 26, and 27 were not integrated with the related mathematics objectives in the 5th grade science and technology curriculum. However, in line with the teacher and expert views, it is thought that these objectives should be integrated with the related mathematics objectives. Therefore, in line with the findings obtained in Table 3, we can claim that other objectives exist in addition to the objectives integrated with the science and technology and mathematics objectives.

Additionally, it was seen that mathematics objectives no. 2, 3, 7, 15, 22, 23 and 24 are objectives that belong to higher grades than the 5th grade. Therefore, these objectives can be problematic for 5th graders. Grade levels of these objectives can be readjusted or the objectives can be modified to be included in the 5th grade mathematics teaching curriculum in order to alleviate the problems that may ensue.

All of the teachers believed that integration with the related mathematics objective would be suitable for objectives “Knows the importance of sell-by dates on packaged products” (Objective 1) and “Expresses observations while a liquid is boiling” (Objective 8); 78.6% of the teachers believed that integration with the related mathematic objective would be suitable for objective “Knows the definition and unit of density” (Objective 13) and 85.7% of the teachers believed that integration with the related mathematic objective would be suitable for objective “Emphasizes the methods used over time to protect food from the detrimental effects of microscopic life forms” (Objective 20). When percentages of expert views are examined for these objectives, 33.3% of the experts believed integration was possible for objectives 1, 8 and 13 and none of the experts (0%) believed integration was possible for Objective 20. For objectives 1, 8, 13 and 20, the majority of teacher believed integration with the relayed mathematic objective was necessary whereas the majority of the experts defended the opposite case. Therefore, since a difference is observed in the views of teachers and experts; a clear point of view has not been presented about the integration of these objectives. A more comprehensive research can be carried out in this context.

DISCUSSION

In line with the findings obtained from the 4th and 5th grade surveys, it is believed that the current integration in the 4th and 5th grade Science and Technology, and Mathematics curricula is suitable. However, it has been shown that there are other objectives that can be integrated in addition to science and technology, and mathematics objectives in the 4th and 5th grade Science and Technology, and Mathematics curricula. The fact that there are other objectives that can be integrated in addition to science and technology, and mathematics objectives in the 4th and 5th grade Science and Technology, and Mathematics teaching curricula clearly shows that sufficient integration between mathematics program and science and technology curricula has not been undertaken yet. This finding is parallel with the ideas of Çeken and Ayas (2010), Kıray (2010), and Taşdemir and Taşdemir (2011) who stated that associations between science and technology, and mathematics curricula are done superficially and in limited areas.

In line with the responses provided by the teachers and the experts to the 4th and 5th grade survey questions, it is thought that science and technology topics on the 4th grade level that can be integrated with mathematics are “Living Beings and Life”, “The Earth and The Universe”, “Matter and Change” and mathematics topics that can be integrated with science and technology are “Tables and Graphics and “Planes/platforms”. Science and technology topics on the 5th grade level that can be integrated with mathematics are “Living Beings and Life”, “The Earth and The Universe”, “Electricity”, “Light and Sound” and mathematics topics that can be integrated with science and technology are “Ratio and Proportion”, “Tables and Graphics” and “Measurement”. This finding is supported by the findings in Douville, Pugalee, and Wallace’s (2003) study that reveals the science and mathematics topics that can be integrated in the 4th and 5th grade levels in the context of teacher views. Besides, the finding indicating that topic of ratio and proportion is among the topics that can be integrated with the science and technology topics in the 5th grade is parallel to the finding of Çeken and Ayas (2010) who stated that the topic of ratio and proportion is among the subjects that can be integrated in the 5th grade with science and technology topics.

The findings obtained from the 4th and 5th grade surveys also show that some objectives can be integrated with the objectives of a grade higher than the current grade. This result may have been due to disregard to the parallelism and integrity of related classes with the curriculum during program development. Therefore, this finding and view are found to be contradictory with the supposition that MoNE (2006) considered the parallelism and integrity with the related classes while developing the teaching programs in question.

In summary, it is believed that integration in the 4th and 5th grade science and technology, and mathematics curricula is insufficient and there are a rather large number of objectives that can be integrated. It is known that achievement levels of the students in international exams such as PISA and TIMSS are rather low (Topbaş, Elkatmış, & Karaca, 2012). Investigation of the content of questions in international exams such as TIMSS and PISA points to the fact that some of the exam questions are interdisciplinary and based on real life (The Organization for Economic Co-operation and Development [OECD], 2007; MoNE, 2010). Since the integration of science and technology, and mathematics classes carries an interdisciplinary quality and will increase student interest, motivation and achievement (Watanabe & Huntley, 1998; Ryan & Deci, 2000; Furmer & Kumar, 2007), the integration of objectives in the 4^{th} and 5^{th} grade science and technology and mathematics curricula will increase student achievement in exams like TIMSS and PISA in countries like Turkey with very low rankings.

Suggestions

In line with the findings obtained from the surveys, it was seen that associations and integration in the 4^{th} and 5^{th} grade science and technology, and mathematics teaching curricula is superficial. Therefore, it is believed that objectives in science and mathematics teaching curricula should be revised in the context of science and mathematics integration. A combined commission composed of science and mathematics experts should be formed for the objectives that are needed to be integrated so that appropriate associations and integration will be possible. Additionally, findings show that some objectives should be integrated with higher level grades. These types of objectives should be readjusted and they should be either moved up to the grade level they belong to or they should be modified and adjusted to the grade level they are represented in. If the objective is not problematic in the grade level it is presented, its integration can be undertaken as suggested by the surveys.

Appropriate materials can be developed to reinforce integration among objectives which are emphasized in the surveys in the current study. In-service training courses can be provided about the integration of science and technology, and mathematics classes so that teachers can have more comprehensive information about this integration. Science and technology, and mathematics teacher candidates can be provided with courses about the integration of science and technology and mathematics classes prior to starting their service.

Mathematics and science and technology groups can be combined at schools in order to ensure integration between science and technology, and mathematics topics and to prevent the provision of information regarding mathematical concepts in science and technology classes before they are taught in mathematics classes. Besides, commissions about the integration of programs can be established at schools to research the subjects that can be integrated. More comprehensive studies can be undertaken about the objectives whose possibility of integration is not clearly defined. There are limited number of studies regarding the integration of science and mathematics. The number of studies in Turkey regarding the integration of science and mathematics should be increased to express the importance of integrating science and mathematics and to stress the need for this integration.

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The Dual Aspect of Natural Numbers Bias in Arithmetic Operations

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ABSTRACT: *Students struggle to understand rational numbers. This paper focuses on two certain aspects of this difficulty, namely doing operations with numbers and counting on certain (natural number) values of missing numbers. The empirical study reported here offers quantitative data in support of the hypothesis that a natural number bias influences students' assumptions about the results of arithmetic operations between numbers that are not represented with specific numerals and acts in two main ways: Not only it forms students' intuitions about the general effect of each arithmetic operation but also affects students in a way that makes them think that the missing numbers can only be natural numbers. A paper and pencil test was administered to 111 fifth and sixth grade students. The test included tasks specifically designed to capture the dual effect of the natural number bias in arithmetic operations between missing numbers: the tasks were either in-line with students' intuitions about the general effect of each operation, i.e. that multiplication always makes bigger (congruent tasks), or not in-line (incongruent tasks) and the missing numbers were either natural or rational numbers. The results showed that the students scored significantly higher in the congruent tasks that involved natural numbers as missing numbers than in the congruent tasks that involved rational numbers. The lowest performance appeared in the incongruent tasks that involved rational numbers as missing numbers. Further theoretical and educational implications are discussed.*

Key words: *Arithmetic operations, Intuitions, Missing numbers, Multiplication makes bigger, Natural number bias, Whole number bias.*

INTRODUCTION

The difficulties students face understanding rational numbers appear in many different domains of reasoning with numbers. This study focuses on two aspects of this difficulty, namely the results of operations and the way students think of numbers when they are not represented with specific numerals. The natural number bias, which characterizes students' tendency to use characteristics of natural numbers when reasoning about non-natural numbers (Christou, 2012; Ni & Zhou, 2005; Vamvakoussi, Van Dooren, & Verschaffel, 2012), may influence both these aspects, however in quite different ways.

The purpose of this study is to reveal the ways the natural number bias may influence this specific domain, of learning rational numbers.

The effect of the number bias may result in misconceptions and mistakes, which appear due to fundamental differences between natural numbers and non-natural numbers. It would be expected that arithmetic operations might also be subject to the natural number bias because the effects of arithmetic operations among natural numbers differ from the effects of operations between non-natural numbers. More specifically, it would be expected that students have the tendency to think that the result of addition or multiplication between two natural numbers is always a number bigger than the two initial numbers (unless 0 or 1 are involved). Similarly, they assume that the result of subtraction or division between two natural numbers is a number smaller than the minuend and the dividend, respectively. This is, however, not necessarily true for rational numbers, for which the effects of operations depend on the numbers involved. For instance, 7: 0.2 is bigger than 7 and 8*0.5 is smaller than 8. An influence of the natural number bias would lead students to think that addition and multiplication always produce a larger number and that subtraction and division always produce a smaller number, regardless of the kinds of numbers involved.

The Natural Number Bias Perspective

The natural number bias, a term initially introduced by Ni and Zhou (2005), is a well-documented phenomenon, which can explain mistakes that students make when reasoning about non-natural numbers. Students tend to apply their prior knowledge and experience about natural numbers even when a task requires reasoning that is not in line with this prior knowledge, as in cases of reasoning with fractions or decimal numbers (Moss, 2005; Ni & Zhou, 2005; Smith, Solomon, & Carey, 2005; Vamvakoussi & Vosniadou, 2010).

The origins of the natural number bias and its relation to the way we perceive and develop the number concept is still debated (Rips, Blomfield, & Asmuth, 2008). However, there seems to be a consensus that, from early on, long before they engage in systematic instruction, students already have a rich understanding of numbers that resembles the mathematical concept of natural numbers (Ni & Zhou, 2005; Smith et al., 2005). In line with other researchers who argue that this bias is a product of students' initial conception for numbers which resembles the mathematical concept of natural numbers (Smith et al., 2005; Vamvakoussi, Vosniadou, & Van Dooren, 2013) the term *natural number bias* is used as a more appropriate way to refer to what Ni and Zhou (2005) initially characterized as *whole number bias*. This initial conception for numbers is supported by children's everyday experience with counting numbers (Gelman, 2000), under the influence of lay culture and its representational tools (Carey, 2004). This conception is further strengthened and confirmed by early instruction in the first years of primary school, when students are taught the natural numbers and their properties in a systematic way. In this way, they end up to have shaped a conclusive body of knowledge, often interpreted to share the characteristics of *a framework theory for* *number* (Vamvakoussi, Vosniadou, et al., 2013), which may undermine students' ideas about what a number is supposed to look like and how it is supposed to behave (Smith et al., 2005; Vamvakoussi & Vosniadou, 2010). Based on this early conception of number, students tend to think that numbers in general are discrete, meaning that every number has a unique successor and that there is no number between two pseudosuccessive numbers, for example 0.5 and 0.6 (Vamvakoussi & Vosniadou, 2010). Students also tend to think that numbers can be ordered by means of their position on the count list (Gelman, 2000), that longer decimals are bigger (Nesher & Peled, 1986), or that the bigger the numerator and denominator of a fraction, the bigger the fraction (e.g., 2/5<2/7) (Mack, 1990).

Quite recently, research has also shown that there is a natural number bias in the way students interpret literal symbols when used as variables in algebra. Christou & Vosniadou (2012) showed that, counter to their instruction that variables can stand for any real number, students up to 10th grade tend to interpret literal symbols in algebraic expressions as representing only natural numbers. That is, students tend to interpret g to represent only natural numbers, 4g to represent positive integer multiples of 4 and k+3 to always be bigger than 3. The third case, which motivates the current study, suggests that the natural number bias phenomenon influences students' ways of interpreting arithmetic operations that include quantities not represented by specific numerals.

The Natural Number Bias in Arithmetic Operations

Research on the influence of the natural number bias has recently revisited a phenomenon already well known from the 80s, considering the results of arithmetic operations between rational numbers (Fischbein, Deri, Nello, & Marino, 1985; Greer, 1987). Initially, the research showing students' misconceptions about the effect of arithmetic operations came from studies on students' choices of operations in word problems that involved multiplication or division. Students tended to think that multiplication always produced a larger number and that division always produced a smaller number (Bell, Swan, & Taylor, 1981; Fischbein et al., 1985; Greer, 1987). Regarding addition and subtraction, there is some evidence that students associate more with addition and less with subtraction when solving word problems (De Corte, Verschaffel, & Pauwels, 1990), which may support the claim that students hold beliefs such as that addition always makes bigger and subtraction always makes smaller (Bishop et al., 2014; Green, Piel, & Flowers, 2008; Tirosh, Tsamir, & Hershkovitz, 2008). However, very little research has investigated students' intuitions specifically concerning addition and subtraction.

Fischbein (1985) was likely the first to have noticed this phenomenon, namely students' intuitions about arithmetic operations and he conjectured that students have primitive, implicit models of the operations, such as the model of addition as putting together, subtraction as taking away, multiplication as repeated addition and division as partitive or quotative division. As Vamvakoussi, and her colleagues (2013) have argued, these primitive models are compatible with - and based on - natural number operations, in particular, with the characteristic that their effects (i.e., whether the result will be larger or smaller) depend merely on the operation and not on the numbers involved (p. 324).

Vamvakoussi and her colleagues (Vamvakoussi et al., 2012; 2013) have pioneered work on this phenomenon by testing intuitive reasoning about operations in adult participants with measuring participants' reaction times while reasoning about arithmetic operations with given numbers and algebraic expressions with literal symbols (i.e., 5+2x). The results provided evidence of a strong tendency for participants to think that addition and multiplication always result in bigger numbers, while subtraction and division result in smaller numbers. This tendency was clear not only to secondary students but also to adult participants, indicating that it is deep rooted and difficult to emend. The same phenomenon appeared in a recent paper and pencil study with middle grade students (Van Hoof, Vandewalle, Verschaffel, & Van Dooren, 2015). However, Vamvakoussi and her colleagues (2013) only partly interpreted their results as the influence of the natural number bias. They argue that students' tendencies to think, for example, that 5+2x is always bigger than 5, is due to participants' preconceptions about what the specific operation sign represents, in other words, it is due to participants' intuitions about the effect of arithmetic operations and in this case that *addition always makes bigger*. Even though they acknowledge that students may use a particular strategy, namely, trying specific numbers in order to check the results of the given operations and that this strategy could also be affected by the natural number predisposition, they do not investigate this possibility.

As mentioned earlier, this possible strategy on the behalf of the students is supported by evidence that they indeed tend to interpret missing numbers when presented as literal symbols in algebraic expressions to stand mostly for natural numbers (Christou & Vosniadou, 2005, 2012). Those studies showed that when presented with algebraic expressions which involve operations between numbers and literal symbols (i.e., 4g), students tended to substitute mostly natural numbers for the literal symbols in order to decide about the numerical value of the algebraic expression. Most revealing were the results of an interview study with 10th graders, in which the students were asked a series of questions such as whether they think that 5d is always bigger than 4/d. The majority of the students responded that 5d is always bigger than 4/d and most often they supported their decision by substituting specific numbers for the literal symbols. In fact, the majority of them repeatedly used natural number substitutions, despite of the hints provided by the interviewer to think about other kinds of numbers (Christou & Vosniadou, 2012). Further support of this finding was offered by Van Hoofs and her colleagues' (2015) research study, which provided more data from students' individual interviews in which they explicitly referred to general natural number rules, or substituted natural numbers in order to come to an answer. Thus, qualitative data occur, showing that students tend to use two strategies when making assumptions about the results of operations between missing numbers: they either count on the general rules about the effect of arithmetic operations, or, using a trial and error process, they mentally substitute certain numbers, which, in most cases, are natural numbers. However, it is essential that this issue should be approached from a perspective that uses quantitative methodology as well. That way a wider sample of students may be involved and the results of the study may be generalized, an important premise for suggesting certain educational implications. This gap is hopefully at least partly filled by the present research study. In addition, implementing a paper and pencil test for examining students' achievement in operations with missing numbers can be a useful tool for mathematics teachers to assess their students' knowledge of the number concept in this specific domain.

To sum up, it is expected that the natural number bias would act in two ways when students are asked to make assumptions about arithmetic operations with numbers that are missing. As mentioned above, the natural number bias shapes students' conceptions about what counts as a number and how numbers are supposed to behave. Thus, in the case of operations between missing numbers (i.e., 3:_), the natural number bias would lead students to think that multiplication and addition produce bigger numbers, while subtraction and division produce smaller numbers, since that is what natural numbers do. Students would also tend to think that missing numbers stand for natural numbers primarily, because, given the more general natural number predisposition, only natural numbers count as numbers. In order to test this explicit hypothesis and the more general, underlying issue of the natural number bias phenomenon detected in arithmetic operations, specifically designed tasks that capture the dual aspect of the natural number bias were administered to primary students. The tasks were either in-line with students' intuitions about the results of arithmetic operations (i.e., multiplication always results in bigger numbers), or counter to those intuitions; they were also involving either natural or rational numbers as missing. It is assumed that, if the students would only use the general rules about the effect of each operation in order to determine the results of the operations (i.e. bigger result in multiplication, smaller result in division) without caring about the numbers involved, there would be no differences in their responses on the different sets of congruent and incongruent items.

METHOD

Participants

One hundred and eleven fifth and sixth graders from three public primary schools in Greece participated in the study; 50 were boys and 61 were girls; 33 were from fifth grade and 78 from sixth grade; 62 from school A and 49 from school B. Students in Greece are introduced to rational numbers in third grade and, by fifth grade, they have gained a large experience in operations between rational numbers, which are the numbers that initially violate their intuitions about the effects of multiplication and division. However, one issue with choosing students at this age is that they lack knowledge of negative numbers, which are introduced in seventh grade. It is specifically those kinds of numbers that violate students' intuitions that addition always produces bigger numbers and subtraction always produces smaller numbers [i.e., 4+(-2)=2, 4-(-2)=6]. Furthermore, the tasks involved operations with a given and a missing number because students at this age have not yet been introduced to literal symbols as symbols that stand for numbers. The students of the sample have extensively used missing number tasks to practice calculations and they are very familiar with this kind of notation.

Materials

A paper and pencil test with 34 forced choice questions was administered to the students. In the first part of the questionnaire, the students were given 28 questions which presented arithmetic operations between one given number and one missing number and the results of the operation as well (e.g., 2:_=5). Examples of the specific tasks that were administered to the students are presented in Table 1. The students were asked to decide whether it is possible for a given relation representing an operation between one number and one missing number to be true and to choose between two given alternatives: "it is possible" and "it is not possible."

There were three main categories of tasks, either congruent (that is in-line with students' intuitions about the results of arithmetic operations) or incongruent (counter to these intuitions): first, there were congruent tasks which involved operations with a natural number and a missing natural number, e.g. 7×_=21. Second, there were congruent tasks that involved rational numbers bigger than one as the missing numbers, e.g., 6×_=11. Students' intuitions about the results of the arithmetic operations were not violated in either case, however, to successfully complete the second type of congruent task, students were required to think beyond the set of natural numbers. The third set of tasks was incongruent, as it involved rational numbers smaller than 1 as missing numbers and violated students' intuitions about the results of operations (e.g. for 2:_=5, division results in a larger number).

Although the tasks involved all four operations, there were not incongruent tasks involving addition or subtraction, since, as explained above, students had not learned yet about negative numbers, the kind of which violate students' knowledge that addition makes numbers bigger and subtraction makes them smaller. Therefore, for tasks involving negative numbers, the expected response was "it is not possible." These tasks were used as buffer questions in order to avoid the possibility that the expected response is always "it is possible." The items were presented in mixed order in the paper and pencil test. Some of the tasks also included rational numbers, specifically positive decimals, either as one of the given factors or as the given result. In that way students were exposed to non-natural numbers and this could act as an additional hint to think with non-natural numbers. Note that it would be impossible to cover all the possible combinations of different kinds of given and the missing numbers in one paper and pencil test (i.e., the different combination of natural numbers, fractions, decimals, given either as factors or as results).

Table 1

Overview of Experimental Items by Kind of Operation and Type (Congruent and Incongruent)

Tasks	Congruent with natural numbers	Congruent with rational numbers	Incongruent (rational numbers smaller than 1)	Buffers
Multiplication	7×_ = 21	6×_= 11	3×_= 2	
Division	8: _ = 2	8: _ = 5	6: _ = 14	
Addition	3+_ = 8	_+ 3 = 4.7		_+5 = 2
Subtraction	5−_= 1	13−_= 7.5		5−_= 9

On the second part of the questionnaire, the students were given six inequalities, with the left side of the inequality sign presented an operation between two numbers, which omitted the operation sign. The right side presented a number equal to the operant number that was presented on the left side of the inequality. Two congruent tasks were assigned, involving natural numbers, giving results in line with students' intuitions about the results of operations (e.g., 3 _ 10 > 3). The remaining four tasks were incongruent, involving one natural number and one rational number smaller than one (e.g., 6_0.2<6), carefully selected in order for the operations to give results counter to students' intuitions. The students were asked to choose the missing operation between two given alternatives, "multiplication" or "division" that would give the correct result. Those tasks offered an opportunity to test students' intuitive responses concerning the results of operations that could act supplementary to the equation-like tasks used in the first part of the questionnaire. Tasks that include the inequality sign are most often used in the studies that examine intuitions about the size of the result of each operation (Van Hoof et al., 2015; Vamvakoussi et al., 2012, 2013).

Procedure

The students participating in the research completed the tests in their classroom during their mathematics course with the presence of their teacher and the researcher. For the first part of the questionnaire, students were told to choose one of the two given alternatives that best represents their opinion. In the missing number tasks, the students often asked what kinds of numbers they could think of. They were told that they could use any kind of number they know and have used in mathematics so far, regardless of its form. In the second part of the questionnaire, the students were also given the correct meaning of the signs for bigger (>) and smaller (<), in order to avoid mistakes that arise from an erroneous use of these symbols.

RESULTS

Students' responses were scored on a right-wrong basis (0 for incorrect and 1 for correct). Overall, the questions used in the questionnaire showed high reliability (Cronbach's Alpha= .749). Students' total score analysis of variance in the test showed that neither gender [$F(1, 103) = .131$, $p = .718$, $\eta_p^2 = .001$], nor school [$F(1, 103) = 1.712$, $p = .194$, $\eta_p^2 = .016$] had any effect. However, grade had a main effect [$F(1, 103) = 16.742$, $p = <.001$, $\eta_p^2 = .140$], which indicated that sixth grade students performed higher (M=15.3, SD=3.94) than fifth grade participants (M=12.64, SD=4.37).

Natural number bias in multiplication and division

In order to test the main hypothesis of the study, student's mean score was calculated for each category of tasks that appear in Table 1, giving the descriptive results presented in Table 2. As expected, students in both grades showed the highest performance on congruent tasks that involved natural numbers as missing and the lowest on the incongruent tasks that violated students' intuitions about the effect of arithmetic operations. The very few mistakes that appear in the congruent tasks indicate that the given tasks were well within the students' abilities.

The results of paired-samples t-tests supported our main hypothesis concerning the dual effect of the natural number bias. The fifth graders performed significantly higher on the congruent tasks that involved natural numbers as missing than on the congruent tasks that involved rational numbers $t(32) = 11.51$, $p<.001$, and they also scored significantly higher on the congruent tasks that involved rational numbers than on the incongruent tasks $t(32) = 4.842$, $p<.001$. The results were quite the same for sixth graders, who performed significantly higher on the congruent with natural numbers tasks than on the congruent with rational numbers tasks $t(77) = 19.6$, $p<.001$ and they also scored significantly higher on the congruent with rational numbers tasks than on the incongruent tasks $t(77) = 12.58$, $p<.001$.

Interestingly, students responded in a very similar way in the buffers tasks and the incongruent tasks; there were no significant differences in the incongruent tasks and in the buffer tasks $t(32) = .564$, $p=.576$ for the fifth graders, but there were small but significant differences for the sixth graders $t(77) = 2.536$, $p =<.05$. This indicates that it is almost as difficult for students to accept counter intuitive results such as, for example, that multiplication makes numbers smaller, as it is to think with negative numbers that they have not even learned yet. As expected, the vast majority of students responded negatively in the five buffer tasks and therefore they were excluded from the analysis that follows; this exclusion raised the overall reliability of the test to Cronbach's Alpha= .773.

Table 2

Mean Scores for Each Category of the Missing Number Items

Tasks	Minimum		Maximum		Mean		Std. Deviation	
	5th (N=33)	6th (N=78)	5th (N=33)	6th (N=78)	5th (N=33)	6th (N=78)	5th (N=33)	6th (N=78)
Congruent with natural numbers	0.29	0.14	1	1	0.714	0.873	0.257	0.206
Congruent with rational numbers	0	0	0.88	1	0.333	0.461	0.245	0.202
Incongruent (rational numbers smaller than 1)	0	0	0.63	1	0.106	0.141	0.162	0.194
Buffers	0	0	0.6	0.6	0.09	0.074	0.15	0.155

For the second part of the questionnaire, where students were asked to choose the operation (i.e., multiplication or division) that could make the inequality true, the results showed again that the students in fifth grade did significantly better on the congruent tasks (M = .60, SD = .348) compared with the incongruent tasks (M = .39, SD = .33), $t(32) = 2.78$, $p<.001$. In a similar way, sixth graders performed significantly better on the congruent tasks (M = .71, SD = .36) compared with the incongruent tasks (M = .40, SD = .33), $t(77) = 5.022$, $p<.001$. These results from this sample of students further support the hypothesis that there are certain intuitions concerning multiplication and division, namely that multiplication produces bigger numbers and division produces smaller ones.

The Natural Number Bias in Each of the Given Operation

In a subsequent analysis, it was investigated the way students responded in each category of congruent/incongruent tasks for each of the four arithmetic operations used. For this reason, mean scores were calculated for the cases including more than one item in each category, giving the descriptive results for each grade presented in Table 3. It should be reminded that incongruent tasks did not include addition and subtraction.

As a general remark, the results show that the students of the sample performed lower in multiplication and division than in addition and subtractions and this contradicts to Van Hoof and her colleagues' (2015) earlier findings from older students (eighth to tenth graders). In addition it appeared that students had a higher mean accuracy on congruent addition and subtraction tasks that involved decimal numbers as missing numbers (e.g., _+3=4.7), than on congruent multiplication and division tasks that involved decimal numbers as missing numbers (e.g., 6×_=11). A possible explanation for these differences is that it is more difficult to find the missing rational number in the case of multiplication or division than in the case of addition or subtraction. Despite the fact that students were specifically told that they did not have to find the missing number, upon easily finding the missing number it is probably also easier to respond that such a number exists and thus that the specific result is possible. However, more research with different number sentences is needed to further clarify this issue.

Table 3

Mean Scores for Each Category by Operation

Category of item	Operation	Minimum		Maximum		Mean		Std. Deviation	
		5^{th} N=33	6^{th} N=78	5^{th} N=33	6^{th} N=78	5^{th} N=33	6^{th} N=78	5^{th} N=33	6^{th} N=78
Congruent with natural numbers	Addition	.5	0	0	1	.98	.96	.08	.16
	Subtraction	0	0	0	1	.64	.89	.49	.31
	Multiplication	.5	0	0	1	.88	.95	.22	.16
	Division	0	0	0	1	.56	.80	.41	.34
Congruent with rational numbers	Addition	0	0	0	1	.56	.82	.46	.35
	Subtraction	0	0	0	1	.54	.79	.49	.35
	Multiplication	0	0	0	1	.23	.38	.31	.31
	Division	0	0	0	1	.14	.18	.26	.30
Incongruent (numbers <1)	Multiplication	0	0	.75	1	.05	.12	.16	.22
	Division	0	0	.75	1	.16	.16	.24	.26

As far as the students from fifth grade are concerned, paired-samples t-test indicated that, as it was the case before, the students did significantly better in the congruent tasks that presupposed natural numbers as missing than in those that presupposed rational numbers: that was the case for addition $t(32) = 5.382$, $p<.001$, multiplication $t(32) = 10.944$, $p<.001$ and also division $t(32) = 5.014$, $p<.001$, but in the case of subtraction the difference was not significant $t(32) = 1.099$, $p=.280$. Furthermore, the participants did significantly better in the congruent tasks that presupposed missing rational numbers bigger than 1, than in the incongruent tasks that presupposed missing rational numbers smaller than 1, thus violating their expectation about the result of multiplication $t(32) = 2.768$, $p<.05$. Nevertheless, in the case of division, the students performed slightly better in the incongruent tasks than in the congruent tasks that presupposed missing rational numbers bigger than 1, a difference which, however, was not statistically significant $t(32) = .620$, $p = .540$.

Quite similar were the results for the participants from sixth grade. Those students performed significantly better in the congruent tasks that presupposed natural numbers as missing than in those that presupposed rational numbers, for tasks that involved addition $t(77) = 3.76$, $p < .001$, multiplication $t(77) = 14.477$, $p < .001$, division $t(77) = 11.937$, $p < .001$ and subtraction $t(77) = 2.189$, $p < .05$ as well. Also, those students did significantly better in the congruent tasks that presupposed missing rational numbers bigger than 1, compared with the incongruent tasks with missing rational numbers smaller than 1 when multiplication was involved $t(77) = 7.713$, $p < .001$, a difference which was not significant in the case of division $t(77) = 0.74$, $p = .462$.

Surprisingly, students did better in the incongruent division tasks than in incongruent multiplication tasks, a difference which was statistically significant for fifth graders $t(32) = 2.435$, $p < .05$, but not for sixth graders $t(77) = 1.395$, $p = .167$. This indicates that the fifth graders of the sample appeared more willing to accept that division could make numbers bigger than to accept that multiplication could make numbers smaller. Further research with individual interviews especially with students who tend to give those kinds of responses could shed some light to this quite puzzling phenomenon.

DISCUSSION

The present study further investigates the influence of the natural number bias in students' ways of understanding arithmetic operations with missing numbers, providing quantitative data in support of the hypothesis that a natural number bias affects students' assumptions considering the results of arithmetic operations between missing numbers. In a paper and pencil test, a sample of fifth and sixth grade students were given congruent and incongruent tasks, specifically designed to capture the dual effect of the natural number bias in operations between given and missing numbers. According to this dual effect hypothesis, the natural number bias affects students to anticipate certain results from each operation (i.e., multiplication to always enlarge the numbers) and also to tend to think of missing numbers as natural numbers only.

The results showed that the participants hold certain intuitions considering the general result of each operation, such as that multiplication always makes numbers bigger, while division makes them smaller. In the first part of the questionnaire, which contained operations between numbers and missing numbers, the students performed significantly better in the tasks that were in-line with their intuitions about the result of each operation, compared with the incongruent tasks that violated them. Additionally, in the second part of the questionnaire, in which students were administered with pairs of operand numbers and were asked to choose the proper operation that would bring the given result, again there was a strong tendency on the part of the students to rely on their intuitions about the results of operations. This indicates that, counter to their instruction and experience with non-natural numbers, students tended to believe that operations between numbers in general provide results that resemble those of natural numbers. These results provide further support in earlier findings that such intuitions appear to be strong, not only in primary, but also in secondary students and adults (Fischbein et al., 1985; Harel & Confrey, 1994; Vamvakoussi et al., 2012; Vamvakoussi et al., 2013; Van Hoof et al., 2015).

The innovative finding of this study is that the participated students reacted in quite different ways in the two kinds of congruent tasks, those that involved natural numbers and those that involved rational as missing, even though in both cases students' intuitions about the effect of the operations were not violated. The congruent tasks that involved rational numbers as missing (e.g., 8:_=2) elicited more incorrect responses than the congruent tasks that involved natural numbers as missing (e.g., 8:_=5), and this was apparent in all kinds of operations. This finding indicate that, in operations between missing numbers, students may not necessarily decide about the outcome of an operation by counting exclusively on their preconceptions about the general effect of each arithmetic operation, i.e., by focusing exclusively on the operation sign, without considering the numbers involved. Instead, students may take into account the numbers involved and they may decide about the outcome of an operation after trying (mentally) some numbers. The results show that the numbers for which students would tend to mentally substitute the missing numbers, under the more general natural number bias predisposition, are mostly natural numbers.

Thus, the above findings from the specific sample of students provide quantitative data in support to the main hypothesis of the study, that the natural number bias would affect students' assumptions about arithmetic operations between missing numbers in two complementary ways: First, it supports and also probably shapes students' tendencies to intuitively associate each operation with specific results, i.e., the result of multiplication is always bigger than the numbers involved and the result of division is always smaller. Second, in the case of operations between numbers that are not presented with specific numerals, the natural number bias would cause students to mentally substitute only with natural numbers and decide about the general results of the operations based on the results of those trials.

This later conclusion is also supported by earlier qualitative data from individual interviews with secondary school students, showing a tendency to rely on both strategies when asked to reason about the results of arithmetic operations between missing numbers and their tendency to substitute only natural numbers for the literal symbols when they needed to argue that 5d is always bigger than 4/d (Christou & Vosniadou, 2012), or when asked if x<x*4 can be true (Van Hoof et al., 2015). Over and above, research on students' interpretations of the kinds of numbers represented by literal symbols in algebraic expressions showed that students give priority to natural numbers (Christou & Vosniadou, 2005, 2012). However, this study adds to previous findings by approaching this issue from a perspective that could offer quantitative data, and their statistical analysis that comes from a large group of participants responding in a series of mathematical tasks, in support of the above claims. Those tasks were developed in a certain way, in order to be either in line or counter line to students' intuitions about the results of operations and also to differentiate between the kinds of

numbers involved. In that way, they managed to capture and reveal the duality aspect of the effect of the natural number bias in this mathematics-learning domain. However, further research in a larger sample of students, which would not only ask students to respond to a series of questions but also to provide exclamations for their responses would be necessary for generalizing the above mentioned findings.

Some educational implications

In order to be able to reason with numbers other than natural numbers and overcome their intuitions about the effects of the arithmetic operations (i.e., to accept that multiplication can also produce smaller numbers), students should develop a conception for number that transcends the limits of their initial conceptions. The students should develop a more sophisticated conception for number, closer to the mathematical concept of numbers, such as incorporating the rational and the real numbers. Developing such conception of numbers requires students to revise their prior knowledge, which is more time consuming and involves substantial cognitive effort (Vosniadou et al., 2008). Students should be strongly supported in this direction from mathematics teachers and the designers of mathematics curriculum. It is argued before that the origins of the natural number bias could be traced on students' initial conception of number which resembles the mathematical concept of natural numbers. Such understanding of numbers appear very early in childhood and is further strengthened not only in the first years of systematic instruction but probably throughout the years of systematic instruction, where students are overexposed to natural numbers even many years after they have been introduced to rational numbers. A recent study on the use of natural numbers versus other kinds of numbers in the mathematics textbooks of 10th to 12th grade in Greece showed that natural numbers are the category of numbers that most often appear either as the given or as the unknown number in exercises, word problems, given examples, etc. (Dimitrakopoulou & Christou, 2014).

The cognitive conflict strategy, a term coined by Inhelder, Sinclair and Bovet (1974), can be used as a means to provoke and falsify students' anticipations about the results of arithmetic operations which stem from their existing viewpoint. One way to use this strategy would be to demonstrate the effects of operations using non-natural number substitutions which would act as counterexamples, in order to provide results that are contrary to students' intuitions. However, for students to experience the conflict in a constructive way and to challenge their intuitions (e.g., multiplication produces bigger numbers) in a productive way, they would need richer learning environments that move beyond simple learning activities that falsify their initial conceptions (Onslow, 1990; Vosniadou, Ioannides, Dimitrakopoulou, & Papademitriou, 2001). To this end, the it could be suggested that the refutational argumentation methodology should be fruitfully used as a means not only to challenge students' erroneous beliefs but also to offer students with alternative conceptions to adopt (Christou, 2012). In addition, the tasks concerning operations with missing numbers that where developed for testing the main hypothesis of this study could be easily adopted and implemented from mathematics teachers not only as material for assessing students' understanding the number concept, but also as a teaching material. Such tasks are easy for students to engage, and they could be used to start a class discussion where students may expose their ideas and learn from their classmates. Additionally the difficulty of these tasks could be easily raised to also include the real numbers. These tasks could be easily solved with the use of a calculator even by primary school students, allowing them to see the range of numbers that could be represented with missing number symbols.

Any instruction that focuses on learning problems, such as the natural number bias, should account for the fact that students not only carry intuitive ideas about numbers and the way those numbers act on operations, but they might also not be conscious of them (Ni & Zhou, 2005; Vamvakoussi et al., 2012). Strategies such as trying with at least one non-natural number - a negative for older students or a number smaller than 1 for younger ones - are indispensable for a students' mathematics toolkit because they can act as a stop and think strategy (Vamvakoussi et al., 2013) or as alarm device (see Fischbein, 1990) that would stop the student from thinking automatically of only natural numbers. These strategies may in turn cultivate metacognitive strategies for students that may raise the necessary awareness for dealing with issues of intuitive reasoning. However, in order for students to acquire and implement such strategies, it is necessary that they not only become aware of their intuitions but also to be in the process of developing a deeper conception of number, beyond the dominant category of natural numbers.

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