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IN REMEMBRANCE

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Filippo Spagnolo
In Remembrance

This volume is dedicated to the memory of a very fond friend and colleague Filippo Spagnolo who left us so soon. I feel honoured to have had him as a very close personal friend and a close colleague. I would like to express my deep sadness for the loss of a beloved friend and colleague. Very recently he was at the University of Cyprus as a member of Annitas' Monoyiou doctoral dissertation committee. This dissertation dealt with the conceptual understanding of functions and Filippo played an instrumental role since the research was conducted simultaneously in Cyprus and Italy with Cypriot and Italian prospective teachers. By sheer coincidence one of his articles is published in the present volume.

I feel that he was and will always be an important part of my personal and professional life. Philippo was an easy going joyful person who was always in a good mood and with a great spirit of co-operation. I am sure that his loss will be felt by everyone, friends and colleagues alike. He will always be remembered with great nostalgia and we will continue his work to honour his memory.

Ed. in Chief

Athanasios Gagatsis

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Strategy Switch Cost in Mathematical Thinking: Empirical Evidence for its Existence and Importance

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ABSTRACT: *The present paper reviews the research that has recently been carried out with respect to the strategy switch cost phenomenon in young adults. The majority of these findings has been observed in the context of a numerosity judgment task. In a first part, an overview of the available evidence concerning the occurrence of a strategy switch cost is provided, whereas in a second part the findings regarding the impact of this switch cost on individuals' strategy selection process are discussed. In a third part, we draw a number of general conclusions and propose a number of mechanisms that could account for each of the two aspects of the strategy switch cost phenomenon. We end with a number of recommendations for further research.*

Key words: *Strategy switch cost, Strategy selection, Strategy execution, Strategy perseveration, Mathematics.*

INTRODUCTION

In the last 25 years numerous studies have shown that individuals exhibit a remarkable variability in their strategies for accomplishing various cognitive tasks. Strategy diversity is advantageous since it offers the potential to adapt to inherent problem characteristics, such as the difficulty of the problem at hand, but also to changing situational demands, such as the need to answer quickly or accurately, and to subject features, such as people's knowledge and mastery of particular solution strategies or their working-memory capacity (Siegler, 1996; Siegler & Lemaire, 1997; Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009).

Realizing this potential, however, requires making adaptive choices among the available strategies in one's strategy repertoire. The better one can adapt one's strategy use to the demands of the task and situation and to one's own general and specific competencies, the better the resulting performance in a specific task will be. Being adaptive, however, implies also that one can switch flexibly between strategies when appropriate. This involves, among other things, that one is able to disengage from the last activated strategy and select and execute another strategy that is more appropriate to resolve the current problem. A crucial question in this respect is whether changing from one strategy to another involves a cost – a *strategy switch cost* – and if so, what impact this cost may have on people's actual strategy choices.

The direct cause for raising and investigating this question are some anecdotal findings in our previous research on people's strategic behavior in the domain of numerosity judgment. We will describe these anecdotal findings in greater detail later on, but, generally speaking, they refer to the observation that people sometimes keep on using a particular strategy even if, from an objective point of view, a different strategy would have been slightly or even considerably more appropriate for solving the problem at hand. It seems plausible that, at least in some of the cases wherein this happens, it is, because of a strategy switch cost, actually more "adaptive" for a given person to continue applying a strategy that is less optimal for solving a particular problem than to switch towards the objectively more suited strategy for that problem. Stated differently, there may be cases wherein the *cost* of switching towards a more appropriate strategy turns out to be larger than the *gain* from this switch. Furthermore, it may be that people take – either consciously or subconsciously – that element into account when making a strategy choice. If this theoretical possibility would be supported by empirical evidence, then a revision of the available models of how people select and use cognitive strategies might be required. Indeed, none of the models like SCADS* (Siegler & Araya, 2005), ACT-R (Lovett & Anderson, 1996), the adaptive decision maker (Payne, Bettman, & Johnson, 1993), RCCL (Lovett & Schunn, 1999), or Rieskamp and Otto's (2006) SSL model, take into account the possibility that changing from one strategy to another might entail such a cost which may affect people's strategy choices. These models share several important assumptions about how people make selections among strategies, such as: (a) the assumption that choosing between different strategies involves some kind of mechanism that compares the relative costs and benefits of each strategy involved in the selection process and that selects the strategy that works best for a given problem on the basis of general as well as specific problem and/or subject characteristics, (b) the assumption that the ease of execution of a strategy can be defined in terms of the number and complexity of the steps that constitute that strategy, and (c) the assumption that this strategy selection process can take place both at a conscious and subconscious level (Lemaire & Lecacheur, 2010; Torbeyns, Arnaud, Lemaire, & Verschaffel, 2004). Although these models of strategy choices proved sufficient to account for a large number of findings about people's strategic behavior, they cannot account for why one would continue using a strategy when an alternative is "objectively" faster and/or more accurate. Such a phenomenon, whereby people continue applying a strategy even though there may be a slightly or considerably more efficient strategy available in their strategy repertoire, suggests that adopting a strategy is not just a product of considering all task, situation, and subject characteristics already involved in the above-mentioned models.

The complete absence of a "strategy switch cost" element in the above theoretical models is remarkable, because there are numerous studies in the experimental psychological domain of "task switching" that have extensively documented the existence of a cognitive cost (i.e., a *task switch cost*) when individuals have to switch between two cognitive *tasks*. Moreover, based on these studies a number of underlying mechanisms have been proposed that might explain this cost (see Logan, 2003; Monsell, 2003; Pashler, 2000; Vandierendonck, 2000 for reviews). This task switch cost can simply be described as follows: When individuals have to switch between two simple tasks (e.g., reading a word and naming a color), they will perform more slowly and less accurately on the trial immediately after a switch from one task to the other, than when they have to repeat one of these tasks on two consecutive trials. This task switch cost is a very robust phenomenon that has been observed in numerous studies involving a wide range of tasks, for example, switching between naming the color and reading the word in a Stroop task (Allport & Wylie, 2000), or switching between a parity judgment task and a magnitude judgment task (Verbruggen, Liefooghe, & Vandierendonck, 2006). Moreover, several authors have found that task switch costs are larger when participants switch from a hard to an easy task than when switching from an easy to a hard task (e.g., Allport, Styles, & Hsieh, 1994; Yeung & Monsell, 2003). Therefore, if task switch costs would generalize to strategy switch costs, we might find largest strategy switch costs when solving an easy problem after solving a hard one and/or when switching from a harder to an easier strategy.

However, these insights from *task switch* literature have barely influenced the research on *strategy choice* yet. Indeed, until very recently, there was, to the best of our knowledge, no empirical research that had systematically investigated the existence and the impact of such a strategy switch cost in mathematical tasks. Since a couple of years we have started to investigate the theoretical possibility of a strategy switch cost in the domain of numerosity judgment by gathering empirical evidence of the *existence* of such a cost as well as of its *impact* on people's strategy choice process. Interestingly, Lemaire and Lecacheur (2010) have – simultaneously but independently – started to study the same phenomenon in other task domains, namely computational estimation and two-digit addition. Therefore, we will briefly refer to this complementary line of research at the end of our article, and integrate their findings in the discussion section and conclusion.

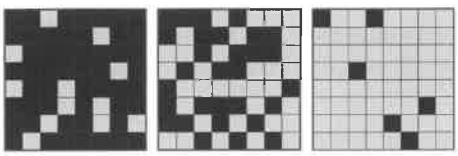

Figure 1. Examples of trials with 12, 34, and 58 colored blocks in an 8 x 8 grids.

OUR EXPERIMENTAL PARADIGM

In all our studies about numerosity judgment (e.g., Luwel, Verschaffel, Onghena, & De Corte, 2001; 2003) we have used a task in which participants have to determine numerosities of colored blocks in a grid (e.g., 10 × 10 or 8 × 8). Figure 1 presents some illustrative trials taken from an 8 × 8 grid. This task allows for the use of two main solution strategies: (a) an addition strategy, in which the total number of blocks is divided into several groups and the total number of blocks in each group is added, and (b) a subtraction strategy in which the number of empty squares is subtracted from the total number of squares in the grid. The adaptive use of both strategies results in a two-phase pattern of reaction times which is characterized by a linear increase as a function of the number of blocks due to the application of the addition strategy (= phase 1), followed by a linear decrease as a result of the use of the subtraction strategy (= phase 2) (see Figure 2). The trial on which participants switch from the addition towards the subtraction strategy is called the "change point". The location of the change point can be determined by applying a two-phase segmented linear regression model (Beem, 1995; Luwel, Beem, Onghena, & Verschaffel, 2001) on the individual reaction times. Our studies demonstrate that from a rather young age on people indeed quite systematically and adaptively switch between the addition and subtraction strategy depending on the "ratio of colored blocks to empty squares in the grid", but also that this strategy selection process is co-determined by various other task, subject and situational factors (for an overview see Luwel & Verschaffel, 2008).

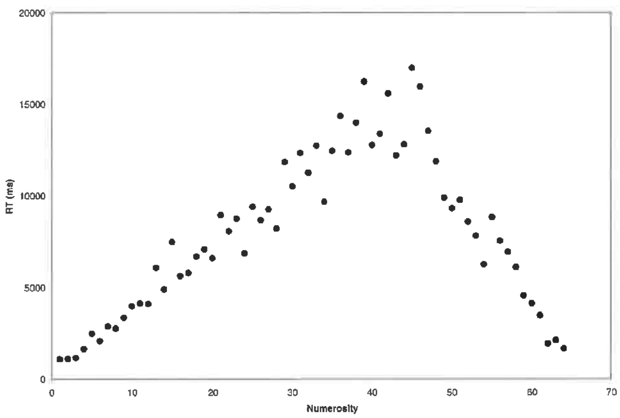

Figure 2. Example of an individual reaction time pattern from an 8 x 8 grid (from Luwel, Verschaffel, Onghena, & De Corte, 2001).

However, as said before, we found in some of our investigations indications of the possible existence of a strategy switch cost among people performing our numerosity judgment task. Indeed, participants sometimes applied the subtraction strategy on a trial that was located (far) before their change point and/or the addition strategy on a trial that was located (far) beyond their change point. When we further analyzed those seemingly inadaptive strategy choices as a function of the presentation order of the items during the experiment, we noticed that these specific trials often came after an – accidentally randomly generated – row of trials that all had been solved with the same strategy. That is, participants kept on using the same strategy on these specific trials as they had been using on the previous trials instead of switching to a somewhat more optimal strategy.

In sum, there are some theoretical as well as anecdotal empirical indications that, because of a strategy switch cost, people may – consciously or subconsciously – continue applying the same strategy as on the previous trial(s), even when another strategy is "objectively" more efficient. With our studies we wanted to test in a more systematic way the hypothesis that, at least in some cases, strategy execution and strategy selection on a given problem are influenced by the strategy that was applied on the previous trial. More specifically, we wanted to investigate, first, the *existence* of a strategy switch cost and, second, the *impact* of this cost on the strategy choice process, in a domain of mathematical strategies with which we had a lot of research experience, namely numerosity judgment in a grid. Moreover, we wanted to investigate whether this strategy switch cost is – like the task switch cost – asymmetrical and whether the size of this possible strategy switch cost *varies* with certain task characteristics, e.g. how strongly a certain problem evokes a particular strategy.

STRATEGY SWITCH COSTS DURING STRATEGY EXECUTION

Experiment 1

A first experiment (Luwel, Schillemans, Onghena, & Verschaffel, 2009) tried to provide an answer to the following three research questions. First, is there a cost when individuals switch between strategies on problems within the same task? Second, does the difference in relative strategy difficulty also result in an asymmetric switch cost? And if so, is the switch cost larger for switching to the easier strategy than for switching to the harder strategy – as is typically found in most task switching studies? Third, can the variability in associative strength between a strategy and the different problems in a task, result in a variation in the size of the switch cost? It seems plausible to assume that it will be easier to switch to a certain strategy when this strategy is very suitable to solve a particular problem (i.e., when it has a larger associative strength) than when it is less optimal to solve that problem (i.e., when it has a smaller associative strength).

Thirty-six university students participated in this experiment, which was run on a computer. Stimuli were numerosities of colored blocks that were presented in a 10 × 10 grid. Two types of problems were presented: target and buffer problems. The target problems were the stimuli containing 55–64 colored blocks. These problems were considered as more or less "strategy neutral" since previous research had shown that they elicited the addition and subtraction strategy about equally strongly in young performance on an addition item with 20 colored blocks was compared with the performance on a subtraction item with 20 empty squares (or 80 colored blocks). Second, we used a somewhat different paradigm, in which switch and non-switch trials were presented equally often in a completely randomized fashion.

Participants were 25 university students. The material and stimuli were identical to those used in experiment 1, except that we used a different set of numerosities and that the stimuli were presented without a cue. Cueing became obsolete, since we now presented univalent problems consisting of numerosities that very strongly elicit one specific strategy in adults: from 10 to 19 for the addition strategy and from 81 to 90 for the subtraction strategy. Except for the above-mentioned differences, the procedure for presenting the stimuli was the same as in experiment 1. One practice block was presented at the beginning of the experiment, followed by five experimental blocks that were separated from each other by a short break. Each block contained 40 [2 (switch status: switch vs. non-switch) × 2 (strategy: addition vs. subtraction) × 10 (numerosities per strategy)] target trials and one start-up trial. Thus each participant solved in total 200 target trials.

To compare both strategies properly, data were analyzed in terms of the number of to-be-counted units (or "counting load"). A 2 (switch status: non-switch vs. switch) × 2 (strategy: addition vs. subtraction) × 2 (counting load: 10–14 vs. 15–19 units) repeated measures ANOVA was conducted on the response times. This analysis revealed a main effect of switch status ($F(1, 24) = 18.64, p < .001$), indicating that the switch trials were solved significantly slower than the non-switch trials. This resulted in an average switch cost of 182 ms. We did not find any significant interaction effects, but the data exhibit a trend suggesting an increase in the switch cost for each of the two strategies with increasing counting load (from 134 to 235 ms for the addition strategy and from 120 to 236 ms for the subtraction strategy). Thus, in line with the results of experiment 1, it seems that it became increasingly difficult to switch towards a strategy as the associative strength between a particular problem and a specific strategy decreased. As in experiment 1, an ANOVA on the accuracy data did not yield any evidence for the impact of the strategy switch cost on response accuracy.

In sum, experiment 2 showed that a strategy switch cost also occurs when individuals have to switch between univalent instead of bivalent stimuli. This finding is in line with evidence from task switching studies, in which a switch cost for univalent stimuli has been found too (Rogers & Monsell, 1995; Ruthruff, Remington, & Johnston, 2001). Also in line with previous task switching research (e.g. Allport et al., 1994; Jersild, 1927; Rogers & Monsell, 1995; Spector & Biederman, 1976) is the finding that the size of the switch cost on the univalent stimuli from experiment 2 (on average 182 ms) is considerably smaller than on the bivalent problems from experiment 1 (on average 870 ms). And, also as in experiment 1, this cost was reflected in longer solution times immediately after the switch, but not in a reduced accuracy. Also similar to the results of experiment 1, was the absence of any differences in the size of the switch cost as a function of strategy difficulty.

Summarizing the results of experiment 1 and 2, we were able to show, first, that a cost is present when educated young adults are switching between cognitive strategies in a numerosity judgment task. This cost was reflected in significantly longer response times, but not in a reduced accuracy, on switch compared to non-switch trials, and occurred on bivalent problems as well as on univalent problems.

A second set of interesting findings pertains to the variation in the size of the switch cost. We found that the difference in cognitive resources required to execute either of the two strategies did not lead to an asymmetric strategy switch cost. It took about the same time to switch from the cognitively less demanding addition strategy to the cognitively more resource consuming subtraction strategy and vice versa. However, this does not mean that there was no variation at all in the size of the switch cost. Indeed, we observed a change in the size of the strategy switch cost as a function of the associative strength between a strategy and a particular problem. More specifically, the cost for switching towards a specific strategy decreased if that strategy became more suitable to solve a particular problem and, likewise, increased if that strategy became less appropriate.

Although the present findings convincingly show that there *is* a cost involved in switching between cognitive strategies (i.e., a strategy switch cost), at least for the task at hand, they do not yet demonstrate that this cost *influences* people's strategy selection process (i.e., a strategy switch cost *effect*). We addressed this issue in the next series of experiments. As explained in the introduction, in some cases it might be more efficient to keep on using a specific strategy, even if it is strictly speaking not the most suitable one, because switching towards that other, more appropriate, strategy might consume more cognitive resources than sticking to the less optimal strategy on the current item. It can be expected that the switch cost will only have an effect on individual's strategy choices on items where the difference in suitability between the competing strategies is relatively small (e.g., on bivalent items located in the middle range of the numerosity continuum, as in experiment 1). When the difference in suitability becomes too large, (as in the univalent items from experiment 2), it may be more efficient – from a cognitive resources perspective – to switch towards the most suitable strategy, since the resources one spares by switching to that alternative strategy will by far overrule the switching cost.

STRATEGY SWITCH COST IN STRATEGY SELECTION

Experiment 3

In this experiment, we wanted to determine the range in which the hypothesized strategy switch cost effect could occur. Thirty-one university students participated in this experiment (Schillemans, Luwel, Bulté, Onghena, & Verschaffel, 2009). Stimuli were basically the same as in the previous experiments, but this time we used rectangular 5 × 10 instead of 10 × 10 grids. Given the large number of trials that participants had to solve and given the long solution times of the trials in the middle range of a 10 × 10 grid, we decided to halve the size of the grid for this experiment.

Two types of items were presented: extreme items and target items. There were two kinds of extreme items: (a) addition items that comprised the numerosities at the lower end of the numerosity continuum (i.e., 1 to 10) and which were known to strongly evoke the addition strategy, and (b) subtraction items that consisted of numerosities at the upper end of the numerosity continuum (i.e., 40 to 49) and which are known to strongly evoke the subtraction strategy (Luwel et al., 2003). The target items were five numerosities that were selected at regular intervals from the range between the extreme items (i.e., 13, 19, 25, 31 and 37).

We created sequences of items that always consisted of a series of randomly chosen extreme items of the same kind (either addition or subtraction), followed by one target item. Four different lists containing 30 such sequences were generated with the following restrictions: (a) each target item had to be included six times, (b) half of the sequences had addition items as extreme items, and the other half subtraction items, and (c) half of the sequences contained five preceding extreme items, the other half six. The exact number of preceding extreme items was varied between five and six to obscure to some extent the typical pattern that arises in this kind of experiment. Thus, one list contained 195 trials in total.

Participants were randomly allocated to one of the four lists and were then tested individually. Before the start of the experiment a number of practice trials was given. Participants were instructed to determine the number of green blocks in each grid as quickly and as accurately as possible (using either an addition or subtraction strategy) and they were also asked to point on the computer screen at the type of cells they were counting (to allow a reliable registration of their strategy use). Each stimulus stayed on the screen until participants had made their numerosity judgment. As soon as participants gave their answer, the experimenter pressed the ENTER-key, which blanked the screen. Then the experimenter typed in the given answer as well as the type of strategy used, after which the next trial started. Participants were allowed a short break at three fixed moments during the experiment.

The analyses were conducted on the target items only. A 5 (numerosity: 13, 19, 25, 31, 37) x 2 (preceding strategy: addition vs. subtraction) repeated measures ANOVA was performed on the proportion subtraction strategy use. This analysis revealed a significant main effect of the type of preceding strategy ($F(1, 30) = 9.54, p = .004$). As expected, we observed a *strategy switch cost effect*: The subtraction strategy was used more frequently when the preceding trials were also solved with this strategy ($M = .45$) than when these previous items were solved with the addition strategy ($M = .40$). We also found a significant two-way interaction between numerosity and preceding strategy ($F(4 , 120) = 2.51, p = .045$). A Tukey test indicated that the strategy switch cost effect was restricted to the numerosity 31 ($p = .01$), and thus, that it did not occur on any of the other four target numerosities (i.e., 13, 19, 25 and 37).

So, experiment 3 revealed a strategy switch cost effect: Participants' strategy choices were influenced by the repeated use of a particular strategy on the preceding items. However, this effect was restricted to numerosity 31. At first glance, it seems rather surprising to observe this effect on this numerosity, and not on numerosity 25, which is, after all, the mathematical midpoint of the continuum. This is due to the fact that the subtraction strategy is more complex than the addition strategy because it involves an additional solution step (see Footnote 1), and therefore the numerosities around the midpoint of the continuum may still be somewhat more strongly associated with the addition than with the subtraction strategy. So, except for the (most neutral) numerosity 31, the associative strength between the problem features and the associated strategy may have been so strong that the expected impact of the context factor "previous strategy" was negligible.

Experiment 4

Given the seemingly rather limited range of numerosities on which the strategy switch cost effect can be observed, we decided to replicate experiment 3 on a narrower numerosity range (Schillemans et al., 2009).

Twenty-four university students participated in this experiment. The stimuli in this experiment were similar to those in experiment 3, except that the range of the target items was narrowed down seriously by using only the three numerosities preceding and following the item on which we had observed the strategy switch cost effect in experiment 3 namely 31. As such, this yielded the target items: 28, 29, 30, 31, 32, 33, and 34.

Following the same restrictions as in experiment 3, we created four lists of item sequences that always consisted of a series of five or six randomly chosen extreme items of the same kind followed by one target item. Since each target item was now presented eight times instead of six times (as in experiment 3), each list contained 56 item sequences instead of 30. As such, each participant solved 364 trials. The procedure was exactly the same as in experiment 3.

A 7 (numerosity: 28, 29, 30, 31, 32, 33, 34) x 2 (preceding strategy: addition vs. subtraction) repeated measures ANOVA, performed on the proportion subtraction strategy use, revealed a significant main effect of preceding strategy ($F(1, 23) = 12.30$, $p = .002$): The proportion of subtraction strategy use was higher after the use of the subtraction strategy on the preceding trials ($M = .97$) than after the use of the addition strategy ($M = .85$). The interaction between those two variables was also significant ($F(6, 138) = 7.00, p < .001$) and a Tukey test indicated that the difference in proportion subtraction strategy use was only significant for some of the numerosities, namely 28 ($d = .32, p < .001$), 29 ($d = .15, p = .007$), and 30 ($d = .13, p = .05$).[2]

Summarizing the results of experiment 3 and 4, we can conclude that they both showed that educated young adults' strategy choices on a numerosity judgment task were influenced by the type of strategy being repeatedly executed on previous trials. As expected, participants were more inclined to reuse the addition strategy on a target item when that item was preceded by a series of addition items than when it was preceded by a sequence of subtraction items and vice versa. However, it was also found that this strategy switch cost effect was limited to a rather small range of (the most neutral) target items that elicited the two strategies about equally strongly. For the other (less neutral) target items, the impact of the problem characteristic "ratio of colored versus empty cells" was apparently already so strong that the strategy switch cost effect became insignificant.

An important feature of the design of experiments 3 and 4 is that the target items were always preceded by several extreme items. This raises the question whether the strategy switch cost effect would already occur after a single application of a strategy. This question was addressed in our fifth and last experiment.

Experiment 5

This last experiment (Schillemans, Luwel, Ceulemans, Onghena, & Verschaffel, submitted) had three goals. First, we wanted to replicate the occurrence of a strategy switch cost effect as in experiment 3 and 4. Second, we wanted to test whether this effect would also occur after a *single* application of a strategy. Third, if this effect would already appear after a single strategy application, we wanted to examine whether it would be larger after a *repeated* than after a *single* previous strategy application. To achieve these three research goals, we conducted an experiment that consisted of two conditions: a *repeat* condition in which a neutral target item was preceded by either five addition or subtraction items and a *single* condition in which only one addition or subtraction item preceded the neutral target item.

Sixty university students participated in this study. Again two types of items were presented: target items and extreme items. Due to the small inconsistency between experiment 3 and 4 in terms of the numerosities on which the strategy switch cost effect occurred, we conducted a preparatory study to look for the most neutral items before conducting experiment 5. This study revealed that the numerosities 25 to 29 were the most optimal neutral target items. The extreme items consisted again of two types,

[2] Although the results of experiment 4 were in line with the results of experiment 3, we did not find the effect on exactly the same numerosities, namely on the numerosity 31 in experiment 3, and on#–ª ˇ# numerosities 28 to 30 in experiment 4, most probably because of individual differences in the associative strength between the strategies and the numerosities (Verschaffel et al., 1998), which – due to the rather small sample sizes of these experiments – may have led to that difference in the numerosity/ies where the effect was found.

namely addition items and subtraction items. In the repeat condition, a sequence consisted of a target item that was always preceded by either five addition or five subtraction items. In the single condition, the target item was always preceded by only either one addition or one subtraction item. To ensure comparability between conditions in terms of the number and type of trials solved, the target item in the single condition was always followed by four extreme items of the same type as the single item that preceded the target item. For each condition, 50 different sequences of five extreme items and one target item were created. To obscure the typical pattern of the sequences and to neutralize influences from a previous sequence to the next one, we took several additional measures that are described in detail in Schillemans et al. (submitted). Participants were randomly allocated to either the repeat condition or the single condition. The procedure was the same as in experiments 3 and 4.

The analyses were conducted on the target items only. Two participants were removed from the analysis, one because she frequently solved the subtraction items with the addition strategy, and the other because it was impossible to accurately identify her strategy use. We conducted a 2 (condition: single vs. repeat condition) x 2 (preceding strategy: addition vs. subtraction) x 5 (numerosity: 25-29) ANOVA with repeated measures on the last two variables and with the proportion subtraction strategy use on the target items as the dependent variable. The analysis revealed a main effect of preceding strategy ($F(1, 56) = 57.96$, $p < .001$). As expected, participants applied the subtraction strategy significantly more frequently after having executed the subtraction strategy ($M = .68$) than after having used the addition strategy ($M = .42$). There was also a significant interaction between preceding strategy and numerosity ($F(4, 224) = 7.37$, $p < .001$), which indicated a variation in the size of the strategy switch cost effect across the different numerosities, however, without exhibiting a clear trend. The crucial test for the third research question whether the strategy switch cost effect differs between the single and the repeat condition, namely the interaction between condition and preceding strategy, failed to reach significance, $F(1, 56) = 3.22$. As such, we found evidence for the strategy switch cost effect both in the single condition and the repeat condition, but we did not find evidence for an increased strategy switch cost effect after a larger number of strategy repetitions.

Because we observed that some students seemed to exhibit the strategy switch cost effect to a larger extent than others, we conducted an additional cluster analysis on the numerosity × preceding strategy data to investigate whether groups of participants with different response patterns could be distinguished. This analysis yielded three clusters, only one of which comprised participants who used both strategies frequently and who did show the strategy switch cost effect ($n = 22$). The members of the two other clusters showed little or no strategy switch cost effect and either used the addition ($n = 16$) or the subtraction strategy ($n = 20$) very often.

So, with this final study, we were able to replicate the strategy switch cost effect found in experiment 3 and 4 by showing that the repeated use of a strategy has an influence on the subsequent strategy choice. Indeed, participants from the repeat condition chose on the target items more often the strategy that they had used on the previous trials.

Second, this experiment extended the findings from experiment 3 and 4 by showing that the strategy switch cost effect already occurs after a single application of a strategy, but we did not find evidence for a larger strategy switch cost after the repeated use of the previous strategy compared to the single use of the previous strategy. Additionally, a cluster analysis revealed large individual differences in the occurrence of the effect, in the sense that only one third of the participants demonstrated the strategy switch cost effect.

A STUDY ON STRATEGY SWITCH COSTS IN ARITHMETIC PROBLEM SOLVING

As stated above, Lemaire and Lecacheur (2010) also conducted a study to test whether switching between strategies involves a cost. In two experiments (experiment 1 and 2), participants had to give approximate products to two-digit multiplication problems (e.g., 47 x 76), whereas in a third experiment they had to provide exact answers to two-digit addition problems.

In experiments 1 and 2 a distinction was made between two main strategies to solve these computational estimation problems: a rounding-down strategy, which involves rounding both operands down to the closest decades (e.g., 70 x 40 to solve 78 x 43), and a rounding-up strategy, which involves rounding both operands up to the closest decades (e.g., 80 x 50). The problems to be solved could be small-unit problems (i.e., unit digits of both operands were smaller than 5; e.g., 42 x 73) or large-unit problems (47 x 58). Previous studies have shown that participants' performance was poorer with the rounding-up strategy and with the large-unit problems (e.g., LeFevre, Greenham, & Waheed, 1993; Lemaire, Arnaud, & Lecacheur, 2004; Lemaire, Lecacheur, & Farioli, 2000). Participants were always told which strategy to use (as in our experiments 1 and 2). In line with our results for strategy execution, participants in their experiment 1 obtained poorer performance on a given trial when they executed different strategies on two consecutive trials than when they executed the same strategy. But such strategy switch costs were only found when participants had to switch to the easy, rounding-down strategy and had to solve the easy small-unit problems. Remarkably, Lemaire and Lecacheur (2010) obtained these switch cost effects not only on participants' reaction times but also on their accuracy scores. Experiment 2 replicated the most important findings from experiment 1, while controlling for potential additional variance in the reaction times.

Because Lemaire and Lecacheur (2010) were also interested in the question whether these switch cost effects will also be observed when they *select* among available strategies, their third experiment tested strategy switch costs when participants were free to choose among strategies (as in our experiments 3-5). For their third study, they used a two-digit addition task for which the two strategies involved had an equal level of difficulty: a full-decomposition strategy, in which one first adds the decades, then the units, and finally the two partial sums, and a partial-decomposition strategy, in which one first adds the decades of the second operand to the first operand and then the units. Participants had to solve a series of problem pairs. On the first problem of each pair either of the two strategies was imposed, whereas on the second problem of each pair participants were free to choose one of the two strategies. So, the design of this experiment is similar to the "single" condition of our experiment 5. The basic finding from Lemaire and Lecacheur's (2010) experiment 3 was that participants repeated the same strategy over two consecutive problems more often than they switched to the other strategy.

GENERAL DISCUSSION

When we put the results of our studies and those by Lemaire and Lecacheur (2010) together, we can conclude that these studies have yielded parallel and complementary empirical evidence for the following conclusions about the *existence* of a strategy switch cost and its *impact* on people's strategy choices:

1. Switching a strategy involves a certain cost, which has shown to be expressed in higher reaction times and/or reduced accuracies on switch trials compared to non-switch trials.
2. The size of the cost of a strategy switch (in terms of reaction time and/or accuracy) decreases as a function of the associative strength between the strategies and the particular item.
3. Items following a series of items that are (typically) solved by a given strategy are more frequently solved with that strategy than with the other strategy; in other words, there is evidence for a strategy switch cost effect.
4. However, the probability of observing a strategy switch cost effect tended to decrease as the distance between a specific item and the neutral item increases, and it has not been found yet on extreme items being strongly associated with a particular strategy.
5. The strategy switch cost effect is already apparent after a single application of a previous strategy and the available research evidence suggests that the effect does not increase with the frequency with which that strategy has been applied previously.
6. Whereas the asymmetrical nature of task switch cost (i.e., task switch costs are larger when participants switch from a hard to an easy task than vice versa) has been amply documented in the task switch cost literature (e.g., Allport et al., 1994; Yeung & Monsell, 2003), the available evidence suggests a more complex pattern for strategy switch cost.
7. Individuals differ in the extent to which they exhibit a strategy switch cost effect.

Of course, with this small set of pioneering studies, research on strategy switch cost in mathematical thinking has not come to an end. A first important task for future research on the strategy switch cost is to replicate both kinds of documented phenomena among various types of mathematical tasks, under various types of situational demands, and with various groups of people. As far as tasks are concerned, all evidence collected so far comes from three types of tasks, namely numerosity judgment, computational estimation and two-digit addition. Moreover, in all three cases people only had the choice between two strategies each of which has a strong association with a specific type of problems. It is therefore important to investigate the strategy switch cost effect with other types of tasks involving more than two strategies. With respect to situational demands, all available data were collected under conditions that equally stressed the speed and accuracy of the response. It would be interesting to analyze whether and how the strategy switch cost changes under different situational demands, e.g., when one strategy performance parameter (speed or accuracy) is given more importance than the other or when other aspects of people's strategy performance (such as the elegancy or originality of the solution process) are emphasized more strongly. Finally, participants were always educated young adults. Future studies should replicate the above experiments about the existence and the impact of a strategy switch cost in other age groups, including children and older adults. Moreover, one could replicate the studies with groups who differ in terms of IQ or cognitive rigidity. Based on the observation that rigidity exhibits a U-shaped relationship with age (see Schultz & Searleman, 2002, for a review), we expect a similar relationship between age and the strategy switch cost effect.

Besides studies aimed at replicating the observed strategy switch cost phenomena and defining the conditions of their occurrence, a second important goal is to *further our understanding of the underlying processes*. In the final discussion sections of the original reports, the authors (Lemaire & Lecaucheur, 2010; Luwel et al., 2009; Schillemans et al., 2009) discuss various possible processes underlying switch costs and their implications for strategy choice. We will not repeat these discussions here, but only briefly discuss some recurrent elements.

As far as the processes and mechanisms underlying *strategy execution* are concerned, both Lemaire and Lecacheur (2010) and Luwel et al. (2009) argue that since the strategy switch costs bear some resemblance with task switch costs reported in the task switch cost literature (see Introduction), strategy switch costs may involve the same processes and mechanisms that have been proposed to account for these task switch costs. Consequently, two types of mechanisms have been proposed to account for strategy switch costs, namely set reconfiguration processes and inhibition effects.

Similar to the task-set reconfiguration account in task switching (De Jong, 2000; Rogers & Monsell, 1995; Rubinstein, Meyer, & Evans, 2001), a strategy switch cost can be interpreted as the time taken by active control processes to reconfigure the cognitive system for the execution of another strategy. More specifically, when a problem is solved with a given strategy, and the next problem has to be solved with another strategy, the cognitive system must be reconfigured. This set reconfiguration involves changing a complex of numerous settings, such as shifting attention (from the just executed strategy to the new strategy), retrieving task goals, rules, or procedures (of the new strategy) into working memory, activating, ordering, and executing component procedures of the new strategy, as well as deleting and/or inhibiting irrelevant strategies (like the just executed strategies) from working memory. This enables a different response set and an adjustment of the response criteria.

In line with the task-set interference account (Allport et al., 1994; Allport & Wylie, 1999) in the domain of task switching, inhibition effects can also be responsible for the occurrence of a strategy switch cost. More specifically, the recent retrieval and execution of a strategy could facilitate the subsequent availability for retrieval and execution of that strategy and hamper the availability of another strategy. According to this account, the switch cost reflects the amount of inhibition that needs to be exerted to deactivate the recently executed strategy and/or the amount of resources necessary to overcome the inhibition of the suppressed strategy.

In the domain of task switching there seems to be a general agreement that the switch cost reflects both the time taken by control processes and inhibition effects. Whether this should be the case for strategy switch costs as well, should be the subject of further research.

As far as the processes and mechanisms underlying *strategy choice* are concerned, the hypothetical explanation that has been put forward in the present article is that items following a series of problems that are (typically) solved by a given strategy will be more frequently solved with that strategy than with another strategy being used before, *because people – consciously or subconsciously – take into account that switching a strategy involves a certain cost*. According to this hypothesis, it can be expected that the switch cost will only have a significant effect on an individual's strategy choice on items where the difference in suitability between the competing strategies is relatively small (e.g., on more or less neutral items). When the difference in suitability becomes too large, that is, on extreme items, it may be more efficient to switch towards the most suitable strategy, since the resources one gains by switching to that alternative strategy will by far overrule the switching cost. And this is exactly what was observed in Schillemans et al. (2009). On the other hand, it should be clear that neither Lemaire and Lecacheur (2010) nor Luwel et al. (2009) and Schillemans et al. (2009, submitted) have provided direct evidence that people stick to a previously used strategy because they take into account that strategy switch cost when choosing for the previously used strategy.

Therefore, we argue that, besides the strategy switch cost effect, there may be alternative explanations for the phenomenon that people select a strategy because that strategy had been used on previous items, namely the Einstellung or set effect and the priming effect.

A first mechanism is the occurrence of an Einstellung or set effect (see e.g., Luchins, 1942). According to this explanation, the repeated application of one specific strategy may cause a set effect which biases participants' strategy selections in the direction of the most recently used strategy. Stated differently, this set effect makes participants "blind" for the possibility of applying the other strategy that might have been equally or even slightly more efficient for the item at hand. There are, however, some critical differences between the design and the results of the studies reported in this article and those from the Einstellung literature, which jeopardize to some extent the relevance of the Einstellung or set concept as the explanation of the findings about the strategy switch cost effect. For instance, in the Einstellung literature participants typically have not yet applied both strategies when they are confronted with the critical target item; they still have to discover the (applicability of the) other, simpler strategy when getting the target item (for a detailed discussion of these differences, see Schillemans et al., 2009; submitted).

A second possible alternative mechanism that could account for the available findings about the impact of the strategy switch cost effect is priming. This priming mechanism refers to a temporary increase in the associative strength of the last applied strategy, which in its turn increases the probability that this strategy will be chosen on the following trial. Thus, for items for which the two strategies are more or less equally efficient, the primed strategy will be favored in the selection process at the expense of the other one. But for items that clearly favor one strategy (i.e., items for which the associative strength of one strategy is much higher), the boost in the strength of the weaker strategy due to the priming process may not be large enough to overcome the existing strength of the stronger strategy.

So, a serious task for future research is to test the explanatory power of the different above-mentioned hypothetical explanations.

Assuming that future research will confirm the main results concerning strategy switch costs found so far and will reveal which (combination of) hypothetical explanations have most explanatory power, this research will point to the need to elaborate existing theories of strategy selection. As argued before, none of the available models like SCADS* (Siegler & Arraya, 2005), ACT-R (Lovett & Anderson, 1996), the adaptive decision maker (Payne et al., 1993), RCCL (Lovett & Schunn, 1999), or SSL (Rieskamp & Otto, 2006), assume that strategy choices on a given trial are influenced by using and executing a strategy on a previous trial. Such an assumption could be included in all models of strategy choices. Enriching these models with a strategy switch cost component might help to further increase their predictive power and to explain why certain strategy choices are in fact more adaptive than they appear at first sight.

From an educational point of view, strategy adaptivity is seen as an important characteristic in most reform-based approaches of mathematical education (Verschaffel, Greer, & De Corte, 2007). Mathematics educators define and operationalize strategy adaptivity mostly (purely) in terms of preferential links between particular types of tasks, on the one hand, and particular strategies on the other hand, without paying much attention to various other kinds of factors that may complicate the process of strategy choice (Verschaffel et al., 2009). The observed strategy switch cost effect is an example of such a complicating factor, since it refers to a characteristic of the context wherein the task is embedded that co-determines in a subtle but significant way the learners' strategy choice process, namely the strategy that was used on the previous trial(s) and the cost that is involved in switching to another strategy. The demonstration of the importance of this context factor on learners' strategy choices may help curriculum developers, textbook authors, and teachers to design proper series of mathematical exercises for their learners and/or to give feedback about the adaptive nature of their strategy choices.

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A Secondary Analysis from a Cognitive Load Perspective to Understand Why an ICT-based Assessment Environment Helps Special Education Students to Solve Mathematical Problems

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ABSTRACT: *This study is a continuation to two earlier studies we carried and which revealed that special education students showed a higher performance in mathematics in an ICT-based dynamic assessment including auxiliary features than in the regular standardized paper-and-pencil test. The focus in both studies was on subtraction problems up to 100 with 'crossing the tens'. In the present study we tried to find an a posteriori explanation for these finding by adopting a Cognitive Load Theory perspective. We conducted a secondary analysis in which we related the features of the ICT-based assessment to the responses of the students. The results of this analysis were varied. For some features we found evidence that they might have influenced student responses, for other features this was not clear. Nevertheless, the analysis gave us a better understanding of how we can develop assessment tools that offer weak students in mathematics opportunities to show what they are capable of.*

Key words: *Assessment, ICT (Information and Communication Technology), Special education, Subtraction problems, CLT (Cognitive Load Theory).*

TWO EARLIER STUDENTS AS START

Revealing Special Education Students' Mathematical Abilities

Children in special education (SE) schools have a severe delay in their mathematical development compared to their peers in regular primary schools. In the Netherlands, at the end of special primary school, SE students' scores are between one to four years behind their peers in regular primary schools (Kraemer, Van der Schoot, & Engelen, 2000; Kraemer, Van der Schoot, & Van Rijn, 2009). A difficulty of these scores is that they are obtained from standardized paper-and-pencil tests, which do not allow children to use auxiliary resources. As a consequence, these tests may not be really appropriate to reveal possibly hidden abilities of SE students. Actually, this static way of testing does not give access to what Vygotsky (1978) called the students' zone of proximal development. This is a serious shortcoming when using test results for informed educational decision making. Therefore, we think it is important that assessment instruments are available that give insight in weak students' zone of proximal development in mathematics. In this matter we are following the footsteps of Feuerstein (1979) and Campione (1989) who developed the idea of dynamic assessment by means of which it can be investigated whether students are able to solve problems with some help and can be observed how this help is used. This makes dynamic assessment pre-eminently suitable for revealing the latent mathematical talents of SE students.

To exploit the dynamic approach to assessment, we see an important role for Information and Communication Technology (ICT). Firstly, ICT makes it possible to get access to the students' solution processes by registering detailed information on students' strategies (Woodward & Rieth, 1997). In this way, ICT offers teachers and researchers 'windows to the mind' of students (Clements, 1998). For example, Barmby, Harries, Higgins, and Suggate (2009) made screen videos of students working in an ICT environment in which they used an array representation for solving multiplication problems.

Secondly, ICT can make problems more accessible for students. Bottge, Rueda, Kwon, Grant, and LaRoque (2009) showed that ICT can eliminate some of the cognitive demands for low achieving students, which enables them to more fully demonstrate their understanding of the mathematical concepts they have learned. With respect to this reduction of cognitive demand, positive results were also found in several studies on computer-based assessment in which students with disabilities were read aloud mathematics problems on the computer by means of a digital aid (Elbaum, 2007; Helwig, Rozek-Tedesco, & Tindal, 2002; Trotter, 2008).

A third advantage of ICT is that it can include optional auxiliary tools which may assist students in cognitive structuring, that is providing them with a structure for thinking and acting when solving problems (Bottino & Chiappini, 1998; Clements, 2002; Clements & McMillen, 1996). For example, digital base-ten blocks can provide flexible and manageable manipulatives which 'snap' into rods of five blocks, that might put students on the track of finding a solution by structuring instead of counting blocks one-by-one.

Based on the above, we see ICT-based dynamic assessment which includes auxiliary resources as a promising avenue to proceed for disclosing SE students' so far hidden mathematical abilities. To test this assumption, we carried out two studies in which SE students' performance in mathematics was assessed in two ways.

Methods of the Two Studies

In the first study (Peltenburg, Van den Heuvel-Panhuizen, & Doig, 2009), we assessed 37 children. The second study (Peltenburg & Van den Heuvel-Panhuizen, 2009) included 43 children (not the same as in the first study). The children were all attending a school for SE and were 8 to 12 years old. Their IQ scores were between 60 and 90. They all had cognitive deficits, often in combination with having concentration and motivational problems. Their development delay in mathematics was between one to four years.

In each study, two different assessment formats were used. First, the children did an ICT-based assessment. This assessment was completely new for them. After the ICT-based assessment the children took the CITO Monitoring Test for Mathematics End Grade 2 (CITO-ME2) (Janssen, Scheltens, & Kraemer, 2005). All children in the two studies were familiar with the CITO tests. These tests are standardized paper-and-pencil tests that are widely used to measure students' mathematical performance in both regular and special primary education in the Netherlands. The test for end grade 2 consists of two parts which should be administered in two successive days. Part I contains problems that are read out by the teacher. In part II, the instruction text is included in the test sheet. The students have to read this instruction by themselves.

The focus in the studies was on subtraction problems up to 100 that require crossing the tens, i.e. problems in which the value of the ones-digit of the subtrahend is larger than the ones-digit of the minuend (e.g., 62–58). This type of problems was selected because it is generally recognized as especially difficult for weak students (e.g., Kraemer, Van der Schoot, & Engelen, 2000; Kraemer, Van der Schoot, & Van Rijn, 2009). A frequent mistake when solving subtraction this kind of problems is reverse processing of the digits (in the case of 62–58, subtracting 2 from 8 instead of 8 from 2).

The set of problems used in both studies and included in the ICT-based assessment was the complete collection of seven subtraction problems up to 100 with crossing the tens that is in the CITO-ME2 test. Four of the problems are in part I of the test and three problems are in part II, which include the instruction text. All except one, the problems are word problems. Figure 1 shows such a problem which belongs to part II of the test.

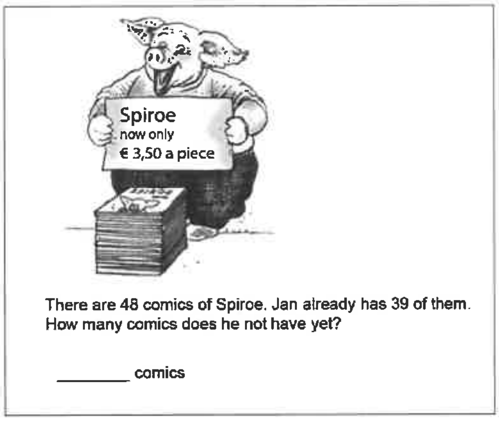

Figure 1. Spiroe problem in standardized paper-and-pencil test (CITO-ME2).

The ICT-based assessment was developed to offer students optimal opportunities to show their mathematical abilities. This means that the assessment has a number of auxiliary features which may give children better access to problems than regular standardized test do and may even help them to solve the problems. The features include: a read aloud function, a flexible answer field, and a mathematical auxiliary tool.

The read aloud function enables the children to not only read the problems, but to hear them as spoken text as well. The flexible answer field makes it possible that children can fill in the numbers of an answer in the order they like. Moreover, with this answer field they can easily correct their answers. However, of all included features the optional mathematical auxiliary tool is the most important one. The tool can give students support in modeling the problem situations, carrying out operations at a concrete level and keeping track of solution processes. Because the students are free to use the tool or not, the ICT-based assessment has a dynamic, adaptable nature.

For the two studies we developed two different mathematical auxiliary tools, based on the two main models that are considered to be helpful when children solve calculations up to 100: the group model (i.e., manipulatives) and the line model (i.e., empty number line) (Van den Heuvel-Panhuizen, 2001). Both models are commonly used in Dutch mathematics textbook series and the children involved in the studies were familiar with these models.

The first study (Peltenburg, Van den Heuvel-Panhuizen, & Doig 2009), here after called the 'manipulatives study', included an optional digital manipulatives tool in the ICT-based assessment.

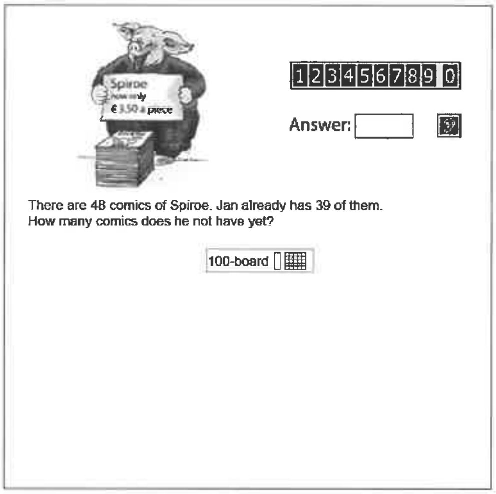

Figure 2. Spiroe problem in ICT-based assessment with tool button for manipulatives tool.

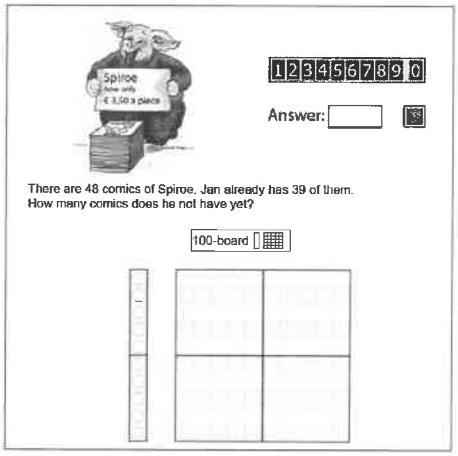

Figure 3. Spiroe problem in ICT-based assessment with digital manipulatives tool.

This tool consists of a digital 100-board and a storage tray with counters. Any number of counters from 1 to 10 can be dragged in one movement to the board. The board features a 10 by 10 grid, with a 5-5 structure, which means that the board is divided in four parts. The students can use the tool to make a visual representation of the numbers involved by putting them on the board and can carry out the required operations by moving or removing the counters. They can work with the tool like they do with wooden manipulatives on their desk. Whether the students use the auxiliary tool or not is their own decision. If they want to use it, they can activate the tool by clicking the tool button (see Figure 2). When the button is clicked, the tool pops up (see Figure 3).

In the second study (Peltenburg & Van den Heuvel-Panhuizen, 2009), here after called the 'number line study', we included a digital empty number line as a mathematical auxiliary tool in the ICT-based assessment (Figure 4). This tool features a horizontal line, a pencil, an eraser, and a clear-all button. The students can use the pencil to carry out operations by putting markers and numbers on the number line and making jumps backwards and forwards. This means that, like in the manipulatives study, a flexible and easily manageable tool can be used. The students can work with the tool in the same way as with a pencil on a piece of scrap paper.

Figure 4. Digital empty number line.

Although the children in both studies had experience with working with manipulatives and number lines when solving calculations up to 100, the ICT-based environment with the digital version of these tools were new for them.

In both studies, data were collected on whether the students could solve the problems and how they solved them. The data collected with the paper-and-pencil test consisted only of the students' answers on paper. In the case of the ICT-based assessment capturing software was used to record screen videos of the students' working. In order not to place extra load on the students' working memory we did not ask them to explain their strategies or to think aloud when solving the problems.

Results from the Two Studies

A comparison of the data from the two test formats showed that the ICT-based assessment is a more appropriate instrument than the standardized test to reveal what the children are really capable of. In both studies, the children solved more problems correctly in the ICT-based assessment than in the standardized test (see Peltenburg, Van den Heuvel-Panhuizen, & Doig 2009; Peltenburg, Van den Heuvel-Panhuizen, & Robitzsch, in press). To test for significant differences between the percentages of correct answers in the two test formats, we used a t-test for paired samples. We did this for the manipulatives study and the empty number line study separately.

In the manipulatives study, our observations included a total of 259 cases (37 students did seven problems each). A case covers two scores: one for the ICT test format and one for the standardized test format. Table 1 shows the percentages of correct and incorrect answers for all the cases in for both formats. The percentage of correct answers was significantly higher ($t(36)= 3.67$, $p<.01$, $d=.71$) for the ICT version of the problems (54%) than for the problems in the standardized test format (34%).

Table 1
Cross Tabulation of Correct and Incorrect Answers in Both Test Formats in Manipulatives Study

		ICT test		
		Correct answer	Incorrect answer	Total
Standardized test	Correct answer	24% (61)	11% (28)	34% (89)
	Incorrect answer	30% (78)	36% (92)	66% (170)
Total		54% (139)	47% (120)	100% (259)

In the empty number line study, our observations included a total of 301 cases (43 students did seven problems each). As in the manipulatives study, a case covers two scores: one for the ICT test format and one for the standardized test format. Table 2 displays the results for the 301 cases. It appears that the percentage of correct answers was significantly higher ($t(42)=4.77$, $p<.01$, $d=.75$) for the ICT version of the problems (55%) than for the problems in the standardized test format (36%).

Table 2
Cross Tabulation of Correct and Incorrect Answers in Both Test Formats in Empty Number Line Study

		ICT test		
		Correct answer	Incorrect answer	Total
Standardized test	Correct answer	28% (83)	9% (26)	36% (109)
	Incorrect answer	28% (83)	36% (109)	64% (192)
Total		55% (166)	45% (135)	100% (301)

CONSIDERING THE ICT-BASED ASSESSMENT ENVIRONMENT AND ITS RESULTS FROM A CLT PERSPECTIVE

The findings from these two earlier studies raised the question whether the features in the ICT-based assessment made that this assessment is more suited than the standardized paper-and-pencil test to reveal what students are capable of in mathematics. To find an answer to this question is the main purpose of the present study.

Because there is much evidence that SE students have working memory deficits (see, e.g., Gathercole & Pickering, 2001) the auxiliary features within the ICT-based environment could have influenced the students' ability to process information. This consideration led us to Cognitive Load Theory (CLT) and to take this perspective as a basis for identifying and understanding the features of the ICT-based environment that were effective in bringing the students to a higher level of performance.

Research inspired by CLT aims to develop instructional design guidelines that lead to instructional materials that foster students' learning by offering possibilities for reducing working memory load (Sweller, Van Merriënboer, & Paas, 1998). Although some researchers (Beddow, Kettler, & Elliott, 2008) have emphasized that CLT-based design principles can also be used for improving assessment, research in this area is scarce. Moreover, besides the fact that CLT is not often used explicitly to design and improve tests, studies on CLT have rarely been concerned with SE students. This is rather remarkable, because design guidelines based on CLT may be crucial especially for students with deficits in their working memory. In other words, it seems there would be much to gain for these students if instructional materials and tests were designed based on CLT principles. The result of such an approach to assessment may give SE students opportunities to show their competence.

Therefore, in the present study we adopted a CLT perspective and subjected data from the two earlier studies that we carried out to a secondary analysis. As Richey stated (1998), secondary analysis can be a suitable methodology to enhance the usability of instructional technology research. The secondary analysis addressed in this study focuses on characteristics of the ICT-based assessment environment and the achieved results in the two test formats. The goal of this secondary analysis is to find out whether the results are in agreement with what CLT principles predict.

Research on CLT has shown that the quality of an instructional (and assessment) task can be improved by reducing extraneous (i.e., ineffective) load on working memory and increasing germane (effective) load (Sweller et al., 1998), for example by avoiding redundancy in the task presentation (Sweller, 2005) and allowing students to represent knowledge externally to off-load their working memory (Beers, Boshuizen, Kirschner, Gijselaers, &Westendorp, 2008). To test whether these principles were operational in the ICT-based assessment, we subjected the following features to a secondary analysis: (1) the read aloud function and the optional read aloud function; (2) the flexible answer field; (3) the optional mathematical tools (i.e., the digital manipulatives and the digital empty number line).

RESULTS FROM SECONDARY ANALYSIS FROM A CLT PERSPECTIVE

The Read Aloud Function and the Optional Read Aloud Function

Many students in SE have reading difficulties which may influence their ability to understand mathematical problems in a paper-and-pencil test. This is especially true for word problems. To overcome these difficulties, it has been argued (Smolkin & Donovan, 2003) that struggling readers could benefit from hearing spoken text as a scaffold to eliminate the need to focus on decoding, allowing them to concentrate on constructing meaning from text. Moreover, giving students control over this function offers them the possibility to adapt the use of the read aloud options to their own needs (Reinking & Schreiner, 1985). Therefore, the ICT-based assessment environment was equipped with a read aloud function. This means that the visual presentation of every new test problem was accompanied by an oral presentation. After that, the students could listen to the text again by clicking a button depicting an ear. This optional read aloud function may make the text of the problems even more accessible for students (Beddow, Kettler, & Elliott, 2008; Elbaum, 2007; Helwig, Rozek-Tedesco, & Tindal, 2002; Trotter, 2008).

From the CLT perspective, it can be argued that presenting written text on the screen together with reading the text aloud is not recommended because the redundancy effect can lower the performance (Mayer & Moreno, 2003; Diao & Sweller, 2007). Processing the on-screen text and connecting it to the spoken text requires cognitive capacity that decreases the cognitive power that is left for solving the problems. What supports the presence of this mechanism is that we observed that the students paid attention to both sources of information.

Nevertheless, it is unclear whether, in the case of the read aloud functions, the redundancy effect will be shown in the results of the students. Because of their low reading ability, the read aloud function can help the students to ‘outsource’ their reading by listening to the spoken text. In other words, what might be a redundancy effect for capable students without reading difficulties could actually lower extraneous cognitive load for SE students. Consequently, the read aloud function would contribute to the students' performance.

To find out whether and how the read aloud functions may have influenced student scores, we compared the percentage of correct problems that the students obtained in the ICT-based assessment with the percentage correct problems that they obtained in the standardized test. This comparison was based on the three problems from part II of the standardized test for which the written text was included in the test sheet. In the ICT-based assessment, these problems also have the text included and in addition they have the read aloud functions. To make the comparison as pure as possible, we excluded temporarily all problems for which the students used the mathematical tool in the ICT-based assessment (i.e., the digital manipulatives or the digital empty number line), as the use of these tools could have influenced the students' scores as well.

Table 3 shows that 67 students completed one, two or all three problems in which they heard the spoken text once in addition to the written text (thus, these students did not use the optional read aloud function). The 67 students completed 132 problems in total. Applying a two-way repeated measures ANOVA using the difficulty of the three items as a fixed factor and the students' ability as a random factor showed that the percentage of correct problems (61%) on the ICT-based assessment did not differ significantly from that (58%) on the standardized version of the test problems (F (1,83.571) = .06; p=.798). Table 3 also shows that 25 students used the optional read aloud function at least once (total of 29 cases). In this group, also no significant difference was found between the percentage of correct problems (55%) on the ICT-based assessment and the percentage of correct problems (59%) on the standardized version of the test problems (F (1,24) = .009; p = .93).

Table 3
Percentages of Correct Answers in Standardized and ICT-based Test Format

Test format	Read aloud function	Optional read aloud function	Cases* (Students)	% Correct answers
ICT	Yes	No	132 (67)	61
Standardized	No	No	132 (67)**	58
ICT	Yes	Yes	29 (25)	55
Standardized	No	No	29 (25)**	59

* Total number of problems made by the students
** Same cases (and students) as selected from the ICT-based assessment

Although there is quite a number of students who used the optional read aloud function – which may suggest that there is a need to hear the text – we did not find evidence that it influenced the score of the students. Similarly, we did not find support for this when we compared the ICT-based assessment in which the problems were only read aloud once with the standardized test in which they were not read aloud. Therefore, we could not determine an impeding role of the read aloud function (redundancy effect) nor a compensating role. A tentative explanation for this might be that a possible positive effect of hearing spoken text compensating for SE students' low reading ability is cancelled out by a possible negative redundancy effect.

The Flexible Answer Field

When students have to put a multi-digit number in an answer field on a computer or calculator, they generally have to do this by filling in the digits from left to right. This means that they have to start with the digit that stands for the largest value. In English, this way of writing down numbers is in agreement with the pronunciation of the numbers. However, in some other languages, including the Dutch language, the ones digit in a two-digit number is pronounced before the tens digit (e.g., in Dutch, 68 is verbalized as 'eight-and-sixty'). Consequently, some students may experience a discrepancy between the order of pronouncing a number and the order of writing or typing the digits of that number. Zuber, Pixner, Möller, and Nuerk (2009) found that this language-specific property of the number word system strongly influences primary school students' performance in writing two-digit numbers. For example, Dutch students may easily end up writing 86 instead of 68 if they write down first what they hear first. To write down 68, and especially to type 68 from left to right directly in the correct order would mean that the students have no choice but to change the pronunciation (and thinking) order into a writing order, which is rather hard for weak learners. To avoid this difficulty, a flexible answer field was included in the ICT-based assessment. This answer field contains a drag-and-drop answer box that allows students to fill in first the ones digit and then fill in the tens digit to the left of the ones digit (see Figure 5).

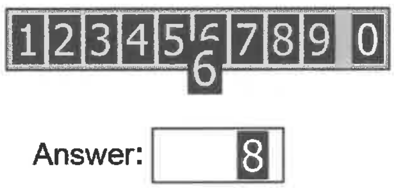

Figure 5. Student is writing down 68 from right to left, filling in ones digit first.

Moreover, the flexible answer field gives students who incorrectly filled in 86 instead of 68, the possibility to correct this by taking the 6 and 'pushing' the 8 aside (see Figure 6).

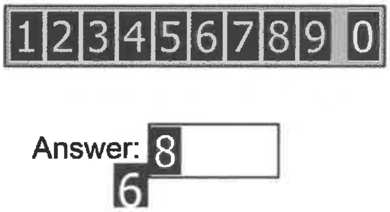

Figure 6. Student is correcting the 86 into 68 by 'pushing' the 8 to the right.

Finally, this flexible answer field offers the students the opportunity to 'park' the part of the answer they already have figured out.

Considering this flexible answer field from the perspective of CLT, it is clear that it offers ample opportunities to reduce the load on working memory for children who speak a language in which the pronunciation of numbers differs from the writing order. With the flexible answer field it is not necessary that students keep the answer in their working memory while at the same time mentally changing the pronounced order to adapt it to the writing order. Similarly, the easy way of pushing away numbers may reduce the children's working memory load because they can start filling in what they already know of the answer.

To investigate how the students used this flexible answer field while solving subtraction problems up to 100, we analyzed screen video data from their work in the ICT-based assessment environment. For this analysis, we only used the five test problems that had a two-digit answer. This resulted in 400 cases; 80 students did five problems each. Table 4 provides an overview of the relative frequencies of how the students filled in their answer.

Table 4
Relative Frequency of Type of Use of Flexible Answer Field

Type of use	Frequency (%) of use based on 400 cases (N=80)
Student filled in the tens digit first	78
Student filled in the ones digit first	17
Student did not fill in an answer	3
Student only filled in one digit	2
Total	100

As Table 4 shows, in most cases the students preferred to fill in the tens digit first, followed by the ones digit. However, in 17% of the cases the students made use of the possibility to fill in the ones digit first. This type of use was most frequently applied (39%) in the problem which has 68 as an answer and less frequently (8% and 13% respectively) in the two cases where the answer is 12. This result may imply that the need for a flexible answer field increases when the numbers that have to be filled in become larger. A small two-digit number like 12 may function more as a visual *Gestalt* than a larger two-digit number like 68. Another related explanation could be that 12 is pronounced as one word 'twelve' and not as 'ten-two' (or as 'two-ten'). For both reasons it is plausible that writing down 12 may require less cognitive capacity than 68.

Compared to the option of filling in the ones digit first, the option of correcting the order of the digits after filling in both digits appeared to be less popular: in only 2% of the total cases did the students make use of this possibility. The option that we called 'parking', in which the students can write down that part of the answer they already know, was applied more frequently. By timing the students' work while filling-in their answer, we found that in 11% of the total cases the students paused for three or more seconds between filling-in the two digits.

The Optional Mathematical Tools

From a didactical perspective, the two tools each have particular 'affordances' (Gibson, 1977) that prompt particular actions. The 100-board with the manipulatives will encourage the representation of numbers by decomposing them into tens and ones (in the case of 62–58, the students might put six tens and two ones on the board and then take away five tens followed by removing two ones and another six ones from the remaining ten). The digital number line will support keeping the starting number as a whole, positioning it somewhere on the number line and then applying a stringing strategy (making jumps on the number line) to move backward (62 ..., 12, 10, 4) or forward (58, 60, 62, ... 4) in order to find the answer. Although both tools can help students to solve the given subtraction problems, they differ in the structure they provide, the availability of processing traces, the representation of numbers, and the range of calculation procedures supported by the tool (see Table 5).

Table 5
Differences in Didactical Characteristics of the Optional Mathematical Tools

100-board and manipulatives	Empty number line
Structure is provided	No structure provided
No processing traces are available	Processing traces are available
Making number values concrete	Positioning numbers
Favors direct subtraction	Favors direct subtraction and indirect procedures

From a CLT perspective, both tools can be considered as an aid for reducing extraneous load by offering the opportunity of knowledge externalization (Gathercole, Lamont, & Alloway, 2006). This means that both the 100-board and the number line tool allow students to make an external representation that they can work with instead of solving the problems mentally, which may offload working memory (Beers et al., 2008). Externally representing information with the tools involves creating an on-screen visual representation of the subtraction operation that has to be carried out. Compared to the

100-board, the number line tool has two advantages regarding knowledge externalization: (1) the number line leaves visual traces of the actions that have been carried out so far, which may lead to a further reduction of extraneous load, whereas no traces are left behind on the 100-board, i.e., removed counters are gone, and (2) the 100-board, although more concrete, requires more actions (counting, dragging and dropping) to represent a number, which may impose extraneous load, whereas representing a number on the empty number line only includes writing the number.

In addition to considering the tools as aids for reducing extraneous load by externally representing information, they can also be considered as aids for increasing germane cognitive load (Sweller et al., 1998). In fact, working with the tools may help students to gain a better understanding of carrying out subtraction problems. In the case of the 100-board, the students are more or less guided in following a taking away procedure, i.e., taking away the subtrahend from the minuend. Compared to the 100-board the number line has less performance constraints (Beers et al., 2008), which means students are not so much guided towards one procedure. Therefore, the number line allows a larger variety of subtraction procedures, including both direct and indirect procedures. As a consequence, working with the number line forces students to come up with a subtraction procedure themselves, which may serve as an additional trigger for advanced ways of working, and as such increase germane cognitive load.

To investigate whether these CLT mechanisms are recognizable in the students' responses, we looked at the problems with the largest gain in correct answers in the ICT-based assessment compared to the standardized test. We chose these problems, because we expected that the described CLT mechanisms manifest themselves best in test problems where the students benefited the most from using the mathematical tools in the ICT-based assessment.

In the 100-board study, the students benefitted most from using the tool in two problems where they had to solve the subtractions 50–38 and 50–37. For both problems, 26% of the students (11 out of 43 students) had an incorrect answer in the standardized test and answered the test problem correctly in the ICT-based assessment. Based on this finding, it can be argued that representing a number on the 100-board imposes less extraneous load when the number only consists of tens.

The situation in the number line study was different. Here, the test problem with the largest gain was the subtraction 37–9. For this problem, 41% of the students (15 out of 37) had an incorrect answer in the standardized test and a correct answer in the ICT-based assessment. The context of this test problem prompts the use of an indirect procedure (8 out of 15 students applied this procedure). This means that the problem is solved not by taking away the 9 from the 37, but by bridging the difference by jumping from 9 to 37. This way of calculating the difference can be easily carried out on the number line, as shown by the work of Martin and Rosa (fictive names). They both applied an indirect procedure, which matches with the context in which the particular problem was presented; see also Torbeyns, Verschaffel, and Ghesquière (2006) for students' adaptive strategy choice. In general, indirect procedures appeared to be more frequently applied with the empty number line than with the manipulatives (see Peltenburg, Van den Heuvel-Panhuizen, and Robitzsch, in press). Because students in the empty number line study made more use of the inverse relation between addition and subtraction in which they followed an indirect procedure, it could be argued that the empty number line encouraged the students to apply cognitively more sophisticated strategies than the manipulatives.

The two examples of student work also demonstrate that the number line does hardly have any performance constraints, which makes it at the same time an accessible tool and one that stimulates sophisticated strategies.

Martin (see Figure 7) used an indirect addition procedure and solved the problem by starting with nine. Then he made three jumps of ten and after that he jumped back two ones to find the answer.

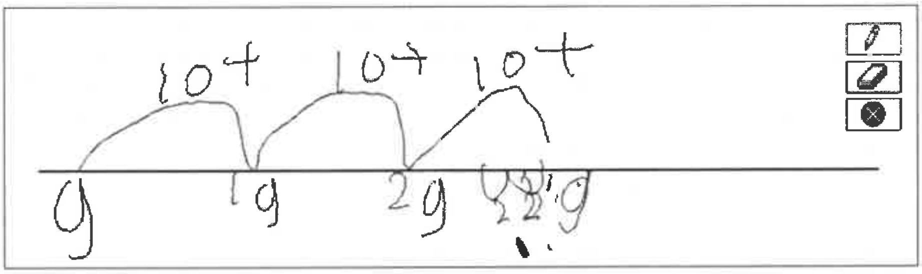

Figure 7. Caption from screen video showing Martin's work when solving 37-9.

Rosa (see Figure 8) used the number line in her own way. She applied an indirect subtraction procedure by jumping from 37 until she reached 9. However, instead of working from right to left she started with 37 on the left and counted backwards while jumping to the right.

Figure 8. Caption from screen video showing Rosa's work when solving 37–9.

These examples show how few performance constraints the number line has. This characteristic of the number line explains why embedding this tool in the ICT-based assessment may help SE students to solve mathematical problems.

CONCLUDING REMARKS

In two earlier studies (Peltenburg & Van den Heuvel-Panhuizen, 2009; Peltenburg et al., 2009), it was found that an ICT-based assessment – compared to a standardized test – gave SE students a better opportunity to show their mathematical ability. By adopting a CLT perspective, we tried to find an a posteriori explanation for these findings. In our analysis we focused on the following significant features of the ICT-based assessment: the read aloud function, the flexible answer field, and optional mathematical auxiliary tools.

Concerning the read aloud function no evidence could be found for a redundancy effect, but the results made us aware of the fact that this effect may behave differently in SE students who lack particular abilities than in regular students. A possible positive effect of hearing spoken text that compensates for students' low reading ability could be cancelled out by the negative (redundancy) effect of getting an overload of information.

The results for the flexible answer field reveal that quite a number of students make use of its flexibility, which may imply that a flexible answer field could help a particular group of students to overcome obstacles and thus can reduce the load on working memory when writing down their answers.

Finally, the results for the mathematical tools show that such tools are not restricted to provide memory aids, but may also offer strategy support. This means that, due to a reduction of extraneous load, students can devote more cognitive resources to identifying or sequencing the solution steps necessary for achieving the goal, which may contribute to performance (i.e., germane load).

However, all above results should be considered with prudence. Our analysis was based on a small sample of students and we only had a limited amount of student variables available. More in particular, in our analysis we missed a direct measurement of the cognitive load for students, which is essential when building knowledge about the role of cognitive load in multimedia learning (Brünken, Plass, & Leutner, 2003). Despite these shortcomings, this secondary analysis from the CLT perspective has been a fruitful enterprise for us, since it gave us more insight in possible positive and negative effects of particular features of the ICT-based assessment environment.

Based on the findings regarding the mathematical tools one may ask whether it should be recommended to provide students with blocks or a number line as an auxiliary tool when they do a paper-and-pencil test. Of course, such a recommendation does not directly follow from our studies. To know more about this, data should be collected with a paper-and-pencil test including these auxiliary resources. However, we doubt whether we consider this as a next step in our study. Note that it was not without a reason that we chose to assess students' abilities in an ICT-based assessment environment. As mentioned before, ICT has some unique advantages, such as the possibility to register detailed information on students' steps in their solution process, which allows their strategies to be assessed in more precise ways than in a paper-and-pencil test.

The CLT perspective could help to design an assessment environment that makes this possible. However, this is not as obvious as one may think. Although multimedia learning is a well-established research domain in CLT, this is not the case for assessment design or, in general, for instructional design to be used in SE. Especially the latter is remarkable. Given the fact that SE students have learning difficulties and often deal with working memory deficits, we think CLT-inspired research can be of invaluable help for improving SE students' learning and assessment environment. Moreover, testing CLT-based instructional design principles more systematically with SE students might be an interesting avenue to expand the CLT theory.

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Attitudes Toward Statistics: How do they Evolve during Students' Curriculum and What is the Relation with Students' Evaluation of their Course?

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ABSTRACT: *Attitudes toward statistics are considered an important factor in enhancing students' chances on developing useful statistical thinking skills. This study uses the Survey of Attitudes Toward Statistics (Schau, Dauphinee, and Del Vecchio, 1992) to (1) map students' evolution of attitudes over a period of three years using five measurement moments and (2) link students' evaluation of course characteristics to their evolution in attitudes toward statistics. We found that students' attitudes are stable over time, although small but statistically significant differences between administration times were found, and that students' evaluation of their statistics course contributes to a limited extent in predicting the evolution of attitudes toward statistics.*

Key words: *Attitudes toward statistics, Longitudinal study, Evolution, Subjective course characteristics, Higher education.*

INTRODUCTION

Statistical literacy is an important competence in modern society. Not only in academic education, but also in daily life people are regularly confronted with statistical information. Students learn statistics at all educational levels, but often the subject is a stumbling block for these students. As does the mathematics education literature (e.g., McLeod, 1992), the statistics education literature stresses the importance of attitudes in learning (e.g., Schau, 2003). Positive attitudes toward statistics are believed to improve chances of students in developing useful statistical thinking skills.

By presenting two separate but related studies, the present paper aims at extending the existing evidence on the evolution of attitudes toward statistics and the relationship with students' evaluation of their course. First, while earlier studies on the evolution of attitudes toward statistics only included two measurement moments, a longitudinal study with five measurement occasions is presented. Second, a study of students' evaluation of their course and the relation with their attitudes toward statistics is discussed.

Because of the assumed importance of students' attitudes toward statistics, high-quality instruments have to be available to assess these attitudes. One increasingly used instrument is the Survey of Attitudes Toward Statistics scale (SATS; Schau, Dauphinee, & Del Vecchio, 1992). The SATS is a 36 item Likert-type scale with seven response alternatives from 'strongly disagree' to 'strongly agree'. The paper-and-pencil questionnaire consists of six subscales: Affect, Cognitive competence, Value, Difficulty, Effort, and Interest. The Affect subscale asks about the positive and negative feelings students have toward statistics. Cognitive competence relates to the feeling of students that they are cognitively able to learn and do statistics. Value is about the perceived relevance, value and usability of statistics. The subscale Difficulty asks students how difficult they think statistics is. Interest measures the interest of students in statistics and the subscale Effort asks about the amount of time and effort students plan to put into their statistics course. The last two subscales, Interest and Effort, were added to the SATS later (C. Schau, personal communication, May 8, 2008). The internal consistency of the first four subscales is reported to be fairly high (Cronbach's α between .64 and .85; Schau et al., 1992; Schau, Dauphinee, & Del Vecchio, 1993; Faghihi & Rakow, 1995; Schau, Stevens, Dauphinee, & Del Vecchio, 1995; Schutz, Drogosz, White, & Distefano, 1999; Watson, Lang, & Kromrey, 2002; Hilton, Schau and Olsen, 2004; Tempelaar, Van Der Loeff, & Gijselaers, 2007). Tempelaar et al. (2007) found high internal consistency scores for the new subscales: Cronbach's $\alpha = .80$ for the Interest subscale and Cronbach's $\alpha = .76$ for the Effort subscale.

LITERATURE OVERVIEW

Many studies show that students' attitudes become more positive during a first statistics course, while others show no difference or a negative evolution of (components of) the attitudes toward statistics (see Hilton et al., 2004 for an overview). Because of these different evolutions it is believed that characteristics of the course may have an influence on the attitudes of students. Evidence for course and/or teacher effects is found in various studies. Waters, Martelli, Zakrajek, and Popovich (1988) for instance studied the attitudes of nine groups of students, enrolled in nine different statistics courses, taught by different teachers. The attitudes of the students of five of these groups improved during the course, while the attitudes of the other students did not improve or even worsened. Schau (2003) found, in 11 different student groups, lower attitudes after a statistics course. Furthermore, the differences between these groups were larger after the course than at the start of the course, suggesting a teacher effect. Unfortunately, these two studies did not explicitly investigate the instructional differences between the courses, but Schau did report students' emphasis on the importance of teachers regarding their attitudes.

A myriad of instructional factors is believed to positively influence students' learning and attitudes concerning statistics. For example: Examples and exercises in statistics courses should be authentic, recognizable, and relevant to students (Cobb, 1987; Singer & Willett, 1990), while collaboration and projects are believed to further enhance students' attitudes and learning opportunities (e.g., Potthast, 1999; Carnell, 2008). Students reported, in a study of Lee, Zeleke, and Meletiou-Mavrotheris (n.d.), similar characteristics when they were asked what would positively contribute to their learning process.

The influence of course characteristics on the evolution of attitudes toward statistics has been regularly studied by quasi-experimental studies in which objective differences between courses are tested in relation to attitudes toward statistics. Only in a few studies subjective course characteristics, as assessed by the students, were used to learn more about the relation between course characteristics and the evolution of attitudes. In what follows we provide an overview of the results of these two types of studies.

The results of quasi-experimental studies vary and the emphasis in most studies lies on the use of modern technology in statistics classrooms. The use of computers for example has been regularly studied. While Ware and Chastain (1989) and Kennedy and Broadston (2003) did find more positive attitudes when a computer was used, Elmore, Lewis, and Bay (1993) and Rhoads and Hubele (2000) were not able to replicate these results. The same holds for the student-paced computer-based courses that were studied by Faghihi and Rakow (1995) and Suanpang, Petocz, and Kalceff (2003). Also the use of videos in class (Alldredge, Johnson, & Sanchez, 2006) and multimedia lectures (Hilton & Christesen, 2002) do not seem to improve students' attitudes toward statistics. It is thus likely that the mere use of modern technology in a statistics classroom is not enough to improve students' attitudes.

Other instructional methods, like the use of projects and group work, might be able to realize more clear improvements in attitudes toward statistics. D'Andrea and Waters (2002) found a lower attitude toward the course when short stories were used in class, but stable attitudes toward the field. While Carnell (2008) was not able to find any differences between the experimental group that did projects in their statistics course and a control group concerning their attitudes toward statistics, Chadjipadelis and Andreadis (2006) did find significant differences between a group of students who were enrolled in a project-based course and a control group, with the students in the project-based group showing more positive attitudes. Tsoa (2006) found more positive attitudes after a statistics course which was constructivist-based, but no control group was used in this study.

It is clear that objective course characteristics fail to conclusively explain differences in evolutions of attitudes toward statistics. Another way to study course characteristics, is to use students' evaluation of their course, which is considered an important method when studying teaching effectiveness (Kulik, 2001). These student ratings prove to agree well with other measures of teaching effectiveness as well as with student achievement. Student ratings also seem to help teachers improve their teaching (Kulik, 2001). Only two studies that tried to relate student evaluations to attitudes toward statistics were found. Roberts and Saxe (1982) found a positive correlation between attitudes toward statistics and course and instructor evaluation (r = .18 and r = .25, after the course), but the contents of these evaluations were not reported. Mvududu (2003) asked two groups of students to assess the level of constructivism in their statistics courses and correlated this measure with their attitudes toward statistics. A statistically significant relation (r^2 = .12) was found between two constructivist measures (personal relevance and student negotiation) and attitudes toward statistics.

Concluding, we can state that evolutions of the attitudes toward statistics of students differ over studies and instructional methods, but that no clear evidence is available from the literature which course related factors do and which factors do not contribute to a positive evolution of attitudes. Student evaluations of their statistics course are only rarely used in studies on attitudes toward statistics, while the results of the few studies who do use them seem promising. Furthermore, all the above studies administered an attitudes toward statistics questionnaire only at the beginning and at the end of a statistics course. In some of these studies, students already took exams and knew their mid-term or even final results at the moment of the post-test, while other students did not know their results or did not have any kind of evaluation at the time of the second administration. We believe these differences can account for some of the differences between studies, as evaluations or knowing one's course results, can have an impact on statistics attitudes. Furthermore, no studies have been found that administered the SATS or any other survey designed to assess students' attitudes toward statistics more than twice, while this could tell us more about the nature and (in)stability of students' attitudes.

The goal of the following two studies is to address some of the above-mentioned remaining issues of previous research on the evolution of attitudes toward statistics and the relationship with students' evaluation of their course.

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STUDY 1: EVOLUTION OF STUDENTS' ATTITUDES TOWARD STATISTICS

In this first study we intensively studied the evolution of attitudes toward statistics over a period of three years. This longitudinal study can tell us more about the evolution of attitudes toward statistics. The research question to be answered here is how the attitudes of students evolve during three years, with special attention for meaningful moments during these three years, like the start and end of the statistics courses and examinations. The surplus value of our study in contrast with previous studies is that we use more than two measurement moments. This makes it possible to assess the evolution of students' attitudes for several consecutive courses and before and after an exam period, while in previous studies only the attitudes in introductory courses were studied.

Participants

A total of 784 bachelor students (750 female, 34 male), taking the same series of three statistics courses, were followed for three years. The overrepresentation of females is due to the fact that the curricula in which these students are enrolled (educational sciences, $n = 521$, and speech pathology and audiology, $n = 263$) predominantly appeal to female students. Of these students, 581 students were involved from the start of their first statistics course (the 2005 and 2006 generation), while 203 students only started participating in the study at the third administration moment (the 2004 generation) because of practical issues. Because of the frequent occurrence of bias due to drop-out in longitudinal studies, special attention was paid to reducing this drop-out: students were contacted repeatedly by e-mail when they did not fill in each of the five questionnaires in class. Of the 2004 generation, 142 students (70%) filled in all three questionnaires, 46 students (23%) filled in the questionnaire twice and 15 students (7%) participated only once. Of the 2005 and 2006 generation, 265 students (46%) filled in all five questionnaires and 126 students (22%) participated four times. Another 91 students (16%) participated three times, 65 students (11%) two times and 34 students (6%) only once. This means that 579 students (75%) participated in all questionnaires possible in their generation or only missed one administration moment.

All students of educational sciences and speech pathology and audiology are obliged to follow the same sequence of three statistics courses during their first three years at university. Their first statistics course is taught in the first semester of the first year. This course covers introductory methodology and descriptive statistics. The second course, in the first semester of the second year, covers research design and sampling, probability and sampling distributions and an introduction to statistical inference. The third and last course is taught in the second semester of the third year and covers more advanced statistical techniques like regression analysis and analysis of variance. All three courses require a minimum of mathematical background and have an emphasis on understanding statistics.

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Method

A Dutch translation of the SATS was administered five times during three years to the two groups of students described above. The translation from English into Dutch was done following a three-step approach. First, the third author and an expert translator with a Master's degree in Germanic languages translated the items independently from each other. These two translations were compared and differences and possible ambiguities were discussed. This resulted in a Dutch version of the (pre-test version of the) SATS-36. Second, a back-translation technique (Brislin, 1970) was used to validate this translation. Statistics experts translated the items of the Dutch version back into English. By comparing this version with the original English version, the quality of the translation was assessed. Again, differences and possible ambiguities were discussed. Third, the Dutch version of the SATS-36 was administrated to a small number of people ($n = 6$) with a diverse statistical background. Participants were instructed to write down suggestions while filling out the instrument. The latter merely resulted in some minor comments about the wording of the statements, suggesting that participants understood the content of the items.

The academic year consists of two semesters of 13 weeks, and each semester is followed by two weeks of study time and three weeks of exams. The questionnaire was first administered at the beginning of the first course and the second questionnaire was administered at the end of the first course, but before the students took their exam. The third administration took place at the end of the first year, three months after the second administration, when the students had received their exam results. In the second year the questionnaire was again administered after the (second) statistics course. The fifth and last administration of the SATS was after the methodology lectures of the third course. This administration schedule enabled us to collect data from the students before and after each course as well as before and after they received their exam results, with a minimum of administration times. The administration schedule is presented in Figure 1.

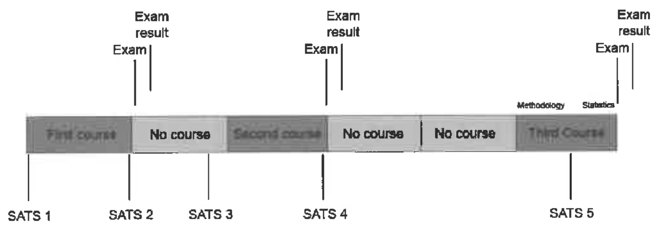

Figure 1. Administration schedule of the SATS.

Results

The internal consistency of the SATS was high for all subscales and administration moments. The lowest Cronbach's α-coefficients were found for the Difficulty subscale (between .65 and .72). Cronbach's α was .79 or higher for all other subscales, with a maximum at .89 (Affect subscale, third administration). Table 1 and Figure 2 show the evolution of the mean SATS subscores over time. Most attitude subscale scores show only slight undulation over time, but the subscale 'Effort' shows more clear differences between administration times. We see that students expect to put more effort into their statistics course than that they afterwards report to have put into their course. The subscale Affect shows a positive trend (although there is a decline at moment four), while students seem to get less interested in statistics over time, but the differences between administration times are overall quite small. If we take a score of '4' as a neutral score with all higher scores representing a positive attitude, we can say that these students show rather positive attitudes toward statistics, but that they do believe that statistics is a difficult course (a high score on the Difficulty subscale stands for low difficulty).

Table 1
Attitude Scores at the Five Administration Times

	SATS 1 Mean (SD) *N*	SATS 2 Mean (SD) *N*	SATS 3 Mean (SD) *N*	SATS 4 Mean (SD) *N*	SATS 5 Mean (SD) *N*
Affect	3.64 (1.01) 552	4.08 (1.18) 519	4.26 (1.26) 617	3.91 (1.15) 582	4.34 (1.20) 499
Cognitive competence	4.22 (0.82) 551	4.48 (0.91) 519	4.64 (0.95) 618	4.48 (0.85) 589	4.82 (0.89) 497
Value	4.73 (0.74) 548	4.69 (0.77) 521	4.78 (0.77) 618	4.68 (0.79) 584	4.85 (0.78) 496
Difficulty	3.38 (0.63) 546	3.45 (0.71) 518	3.52 (0.71) 617	3.44 (0.63) 589	3.65 (0.68) 500
Interest	4.53 (1.01) 551	4.15 (1.01) 521	4.27 (1.02) 619	4.02 (1.03) 587	4.10 (1.04) 503
Effort	6.29 (0.59) 554	5.47 (0.84) 515	5.92 (0.88) 618	5.41 (0.97) 576	5.82 (0.95) 501

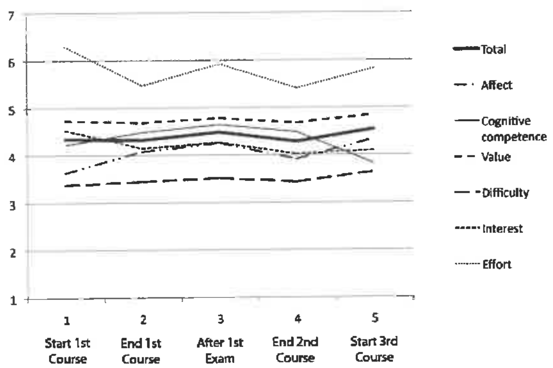

Figure 2. Evolution of the Mean SATS scores.

Correlations between subscale scores of subsequent administration moments are computed to test the stability of students' attitudes. These correlations reveal strong bivariate correlations that are statistically significant different from 0 (all well above .55, $p < .0001$) between subsequent attitude scores, suggesting stability of the attitudes of students. In order to test for each subscale whether the attitudes measured are the same at each administration moment, repeated measures ANOVA is used. According to these analyses, the attitudes toward statistics are not the same at each administration moment (Cognitive competence: $F(4,251) = 45.86$, $p < .0001$; Difficulty: $F(4,247) = 22.28$, $p < .0001$; Effort: $F(4,241) = 68.39$, $p < .0001$; Interest: $F(4,253) = 23.13$, $p < .0001$; Value: $F(4,248) = 7.77$, $p < .0001$; Affect: $F(4,246) = 56.50$, $p < .0001$). A set of post-hoc tests in the form of paired t-tests per subscale is performed in order to test for significant differences between subsequent administration moments. The Bonferroni-method for multiple comparisons is used, resulting in an alpha-level of .0125 (Moore & McCabe, 2001). The results, together with the effect size (Cohen's *d*), are presented in Table 2.

Table 2
Paired T-test Results and Cohen's d

	Adm1-Adm2 *t (p) d*	Adm2-Adm3 *t (p) d*	Adm3-Adm4 *t (p) d*	Adm4-Adm5 *t (p) d*
Affect	-7.40 (< .0001) -0.40**	-7.76 (< .0001) -0.15	9.49 (<.0001) 0.29*	-10.15 (< .0001) -0.36*
Cognitive competence	-6.74 (< .0001) -0.30*	-8.12 (< .0001) -0.17	6.27 (<.0001) 0.18	-8.99 (< .0001) -0.39*
Value	3.01 (.0027) 0.05	-3.58 (.0004) -0.12	1.92 (.0560) 0.13	-4.01 (< .0001) -0.22
Difficulty	-3.39 (.0007) -0.10	-5.88 (< .0001) -0.10	3.82 (.0002) 0.12	-7.72 (< .0001) -0.32*
Interest	11.26 (< .0001) 0.38*	0.84 (.4035) 0.12	4.24 (<.0001) 0.24*	-1.65 (.09970) 0.08
Effort	21.10 (< .0001) 1.13***	-9.15 (< .0001) -0.52**	5.59 (<.0001) 0.55**	-7.79 (< .0001) -0.42*

NOTE: * small effect ** medium effect *** large effect

According to the effect sizes, the largest differences during the first statistics course occur with regards to the Effort subscale: Students report to have put less effort into the course than they expected at the start of the course. The students furthermore get somewhat more positive feelings toward statistics and feel somewhat more competent to learn and use statistics. These tests also enable us to answer the question whether taking an exam and knowing the result influences students' attitudes, by looking at the paired t-tests that were performed on the attitude scores of the second and third administration. These tests reveal a statistical significant improvement after the exam for all subscales, except for the Interest subscale. This confirms our hypothesis that the evaluation and/or knowing the exam results, may have an influence on students' attitudes. The effect sizes however are medium or less than small following the classification of Cohen (1992).

Between the end of the second year and the end of the second course, only the Affect and Interest subscale show a small to medium effect size: Students have less positive feelings about statistics after the course, but are more interested in statistics. These feelings get more positive however during the months in which they do not attend any statistics lectures, but do attend a number of methodology lectures. During this period, also the feeling of cognitive competence and the appreciation of the value of statistics get more positive. Students think they need to put more effort into their last statistics course than they put in their second statistics course, this is in accordance with their more negative feeling about the difficulty of statistics.

A more informal way to assess the evolution of students' attitudes is to look at the absolute differences between the mean scores at subsequent administration moments. These (absolute) differences are between .01 and .72. The mean difference is .23. As SATS scores can vary between 1 and 7, this means that the differences between measurement moments are rather small.

STUDY 2: STUDENTS' EVALUATION OF COURSE CHARACTERISTICS

After studying the evolution of attitudes toward statistics over a period of three years, we focus in this second study on the relation between course characteristics as identified by the students of three statistics courses and the evolution of their attitudes toward statistics in order to learn more about instructional factors as experienced by students that influence this evolution. In this study, 549 students from three different courses were included and a pretest-posttest design was used. Besides the SATS, we used in this study a questionnaire that asks students about their course in order to be able to relate these subjective course characteristics to the attitudes of students. The research question to be answered in this study is whether the subjective evaluation of the course by the students is related to the evolution of attitudes toward statistics and the attitudes after the course.

Participants

Three groups of university students were involved in this study. One group majored in industrial engineering (n = 361; 56 female, 304 male, 1 unknown), one in business studies (n = 165; 62 female, 87 male, 16 unknown) and the third in language (n = 23; 13 female, 10 male). The industrial engineering students were all enrolled in the same course, but the group was divided into two groups, each group was taught by another professor, using the same textbook and exam. Of the 549 participants, 284 students (52%) filled out both questionnaires.

Method

Again, our Dutch translation of the SATS-36 was administered two times in each group: at the beginning of the first class and at the beginning of the last class. For the second questionnaire eight questions about the course were added to the SATS. These eight questions had three answer alternatives: not at all, to some extent, to a large extent. The questions were based on factors that are believed to improve statistics education (Cobb, 1987; Harwell, Herrick, Curtis, Mundfrom, & Gold,1996).

The questions were the following:

1.To what extent did the lectures assist you in your learning process?

2.To what extent were the examples in the lectures clarifying?

3.To what extent were the examples in the lectures recognizable?

4.To what extent did the course material assist you in your learning process?

5.To what extent were the examples in the course material clarifying?

6.To what extent were the examples in the course material recognizable?

7.To what extent was the content of the lectures similar to that of the course material?

8.To what extent was the arrangement of the lectures similar to that of the course material?

Question 1 and 4 are asked because the answers can tell us something about the overall quality of the course as perceived by the student. Question 2, 3, 5 and 6 are about the recognizability of the examples and about the extent to which these examples are clarifying to the student. According to various authors, it is important that examples in statistics courses are clear and relevant (e.g. Cobb, 1987). The last two questions, about the accordance between the course material and the classes were included because similarity in content and presentation order between the course material and the classes can make students feel more secure.

Results

The SATS again proved to provide reliable measures of students' attitudes toward statistics. The lowest Cronbach's α coefficients were again found for the Difficulty subscale (.60 at the first administration and .67 at the second administration), while all other subscales had an internal validity of .71 or higher, with a maximum of .86 (Interest subscale, second administration). Table 3 shows the mean SATS-scores for the subscales at both measurement moments for the four research groups separately. Again, we see that the attitudes scores are in general rather positive, while these students also think that statistics is a difficult subject. The students of business studies and languages however are quite negative regarding both Interest and Affect. The difference on the subscale Effort shows a similar trend as in the previous study: Students report to have studied not as much for their statistics course as they expected before the start of the course. The other subscales show smaller differences for both measurement moments, but they all show (except for the subscale Cognitive competence for all students and the subscale Interest for the Language students) a negative trend. Paired t-tests show significant differences between both measurement moments for all subscales (effect sizes: .16 for Affect, .20 for Value, .78 for Effort, and .32 for Interest), except for the Cognitive competence subscale and the Difficulty subscale (effect sizes -.06 and .08 respectively).

Table 3
SATS Scores for Each Group and Administration Moment Together with Standard Deviation and N

		Languages Mean (SD) *N*	Engineering A Mean (SD) *N*	Engineering B Mean (SD) *N*	Business studies Mean (SD) *N*
Affect	Adm1	3.86 (1.45) 22	4.30 (0.77) 168	4.33 (0.84) 182	3.78 (1.12) 153
	Adm2	3.63 (1.44) 15	4.08 (0.78) 108	4.35 (0.89) 115	3.54 (1.15) 120
Cognitive competence	Adm1	4.43 (1.18) 23	4.58 (0.70) 174	4.65 (0.76) 184	4.19 (0.81) 160
	Adm2	4.40 (1.34) 16	4.67 (0.74) 114	4.81 (0.76) 116	4.27 (0.97) 121
Value	Adm1	4.20 (1.17) 23	4.77 (0.71) 169	4.89 (0.65) 180	4.46 (0.76) 148
	Adm2	3.99 (1.51) 15	4.56 (0.81) 114	4.90 (0.66) 115	4.29 (0.80) 124
Difficulty	Adm1	3.54 (0.75) 23	3.79 (0.51) 170	3.79 (0.55) 179	3.34 (0.68) 157
	Adm2	2.90 (0.74) 16	3.76 (0.54) 112	3.72 (0.59) 113	3.32 (0.71) 123
Interest	Adm1	3.76 (1.34) 23	4.37 (0.91) 169	4.46 (0.85) 184	3.92 (1.14) 162
	Adm2	4.11 (1.91) 16	3.89 (0.98) 114	4.30 (0.94) 117	3.59 (1.11) 123
Effort	Adm1	5.62 (1.01) 23	5.43 (0.82) 167	5.53 (0.88) 184	5.15 (1.03) 157
	Adm2	4.97 (1.34) 16	4.43 (0.89) 111	4.85 (0.85) 115	4.92 (1.11) 119

The internal consistency of the course characteristics questions is .78. In order to relate the course characteristics to the evolution in attitudes, linear models were formulated using the Mallow's C_p selection procedure in SAS. Table 4 shows which course characteristics contribute significantly in predicting the evolution of each subscale (i.e., posttest score – pretest score) and how much of the variation in the evolution is explained by these course characteristics. The explained variances tell us that the course characteristics are only able to explain a limited amount of variation in the evolution of attitudes. Only one of the course characteristics does not contribute at all in predicting any evolution: The extent to which the lectures assisted students in their learning process. Of the other course characteristics, especially the recognizability of the examples in the lectures and the extent to which the content of the lectures and the course material are similar seem to contribute to a positive evolution of attitudes. With one exception, attitudes tend to get more positive when the answer to the related course question is more positive. The exception is that with more clarifying lectures, students report to have put less effort into their course than they predicted. This is explainable by the assumption that students need to put less effort into their course when the lectures are already clarifying enough.

Table 4
Course Characteristics which Contribute Significantly in Predicting the Evolution in Each Subscale Score

	Learning process lectures	Clarifying lectures	Recognizable lectures	Learning process text	Similar content	Similar arrangement	Clarifying course	Recognizable course	Adjusted R^2
Affect			x		x				.05
Cognitive competence			x				x		.07
Value					x		x		.03
Difficulty			x					x	.03
Interest				x	x				.03
Effort		x				x			.03

Table 5 shows how the evaluation of the course by the students contributes in predicting the attitude sub scores after the course. In predicting these scores, more course characteristics have a significant influence than in our previous models. The proportions of explained variance are also higher then when predicting the evolution, except for the Difficulty subscale. All course characteristics now contribute in predicting at least one subscale score.

Table 5
Course Characteristics which Contribute Significantly in Predicting the Subscale Scores After the Course

#	Learning process lectures	Clarifying lectures	Recognizable lectures	Learning process text	Similar content	Similar arrangement	Clarifying course	Recognizable course	Adjusted R^2
Affect		x	x	x					.11
Cognitive competence		x	x	x		x	x	x	.09
Value					x		x		.05
Difficulty			x					x	.02
Interest		x		x	x				.10
Effort	x	x			x		x		.12

CONCLUSION AND DISCUSSION

As attitudes toward statistics are considered to be an important factor in the learning process of students, extending our knowledge about these attitudes is important if we want to improve students' learning. This study was designed to both map the evolution of students' attitudes toward statistics and to investigate the relation between this evolution and students' evaluation of their course and between the evaluation and the attitudes after the course. These studies are different from other studies about attitudes toward statistics, as we did not limit ourselves to administering an attitudes questionnaire before and after one course, but administered the SATS five times over a period of three years. In relating course characteristics to attitudes, we did not, as most other studies, focus on objective course characteristics, but we asked students to evaluate their own course. In the next two sections we will discuss the results of both studies separately, after which we will integrate these results to formulate some general conclusions.

Study 1

In study one we administered the SATS-36 five times to a large but homogeneous group of students following the same sequence of three statistics courses. Correlation analyses showed the evolution of attitudes can be considered to be stable: All subsequent attitude scores are significantly correlated. The subscale scores were, however, not the same at each administration moment according to repeated measures analysis of variance, but effect sizes revealed that the differences between administration times were overall quite small. This is similar to most other studies that report (enough information to calculate) effect sizes (e.g. Robers & Saxe, 1982; Elmore et al., 1993; Faghihi & Rakow, 1995; Schultz & Kosino, 1998). Looking at the mean scores, we see that our scores and scores of other studies reporting both pre- and post-test scores, rarely differ more than .3 on the 7-point scale of the SATS or more than .3 point on 5-point scales of other attitudes toward statistics questionnaires. In general, we also found differences between attitudes on subsequent administration times similar in size as those reported in (quasi-)experimental studies. We found the largest differences between administration moments for the Effort subscale: Students predict to put more effort into their statistics course than they report to actually have done after the course.

A weakness in longitudinal studies is of course the occurrence of drop out of students out of the study, which is likely to cause some form of bias. Although we put a lot of effort in keeping all students involved in the study, some students dropped out, due to various reasons like stopping their studies. The dropout in this study was rather limited (75% participated in all administration moments possible in their generation or only missed one administration moment). Comparisons between dropouts and students who participated at all measurement moments reveal some small but statistically significant differences concerning their attitudes at the first measurement moment (Affect, Cognitive competence, Difficulty and Effort, with dropouts having somewhat more negative attitudes). A limitation of this first study is the homogeneity of the sample (all Educational Sciences and Speech Pathology and Audiology students; mainly female participants). The intensive administration scheme made it impossible to study more student groups in this study. Further research on the evolution of attitudes toward statistics could focus on other or more heterogeneous student populations in order to see whether our conclusions also hold for other students and/or courses.

Concluding we can say that we found differences between attitudes measures at subsequent administration moments, but that these differences are statistically, rather than practically significant. The large sample size in this study can account for the statistical significance of relatively small differences. Although our results are not conclusive, attitudes toward statistics seem to vary only to a very limited extent, both during courses and before and after exams. This could shed a new light on studies in which changing the attitudes of students is the main focus: One could question whether it is possible to alter attitudes during a short period of time and after so many years of experiences with statistics. McLeod (1992) includes stability even as one of the most important characteristics of attitudes, in his definition of attitudes. It might be the case that small and practically unimportant differences found in other studies are exaggerated because of their statistical significance.

Study 2

In our second study we asked students following different statistics courses to evaluate their course in order to relate this evaluation to the evolution of their attitudes and their attitudes after the course. By involving three quite different courses and groups of students in our second study, we were able to study a greater variety in attitudes, evolutions in attitudes and subjective course characteristics.

In the first part of study 2, we found that the evolution in attitudes can be predicted by this course evaluation with explained variances between .03 and .07 (small to medium effect sizes). The use of recognizable and clarifying examples and exercises is often stressed in literature (e.g. Cobb, 1987; Singer & Willett, 1990) and our results suggest that when students describe these factors to be present in their course, they indeed contribute to some extent in improving students' attitudes toward statistics. The extent to which students think the lectures contribute to their learning process, however, does not seem to contribute to predicting the evolution of students' attitudes.

In the second part of study 2 we used our questions about the course to predict the post-scores. We found proportions of explained variance between .02 and .12, or small to medium effects. Other studies using subjective course characteristics in the study of attitudes toward statistics found comparable proportions of explained variance in predicting the attitudes toward statistics after the course (.03 and .06, Roberts & Saxe, 1982; .12, Mvududu, 2003). In comparison with predicting the evolution of attitudes, in this model more course characteristics contribute significantly to the model. Every course characteristic contributes significantly in predicting at least one subscale score.

We can conclude that subjective course characteristics are useful in predicting students attitudes changes and attitudes after the course. Factors often claimed to be important for students' attitudes and learning process, indeed seem to contribute to a positive evolution of students' attitudes. The construction of a more elaborate questionnaire on course characteristics hence seems to be an interesting option to learn more about how students' attitudes toward statistics are influenced by course characteristics.

General Conclusion

This study learned us more about the evolution of attitudes toward statistics over a longer period of time and in relation to course characteristics as reported by the students. The evolution of attitudes toward statistics proved to be relatively stable, although some small but statistically significant differences between administration moments were found. We think it is interesting for further research to look at the evolution of attitudes in different student groups to confirm this stability and to look for factors that interact with the (possible) evolution, like student characteristics and course characteristics. We furthermore found that the course characteristics as described by students contribute in predicting the evolution of attitudes and the attitudes after the course, but only to a fairly limited extent. Further research could concentrate on the relation and interaction between students' evaluation of the course, objective course characteristics and student characteristics. The development of a more elaborate evaluation questionnaire for students would also be interesting, both for research and for educational practice.

We can conclude that, although the changes in students' attitudes toward statistics are rather small, it is possible to predict them using students' evaluation of their course as a predictor. This means that even though attitudes are relatively stable, it is possible to (positively) influence them, although no large changes in a short time are to be expected.

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Epistemological and Didactic Obstacles: The Influence of Teachers' Beliefs on the Conceptual Education of Students

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ABSTRACT: *This investigation addresses the topic of teachers' beliefs about mathematics, mathematics education and their expectations of this discipline as regards their students' learning of two specific topics: fractions and angles. International research in this field has produced many results, the first being a good definition of the concept of beliefs, and has taken the first steps towards understanding the impact of these beliefs on didactic activity. This research is based principally on the classical Theory of Obstacles. Within this framework, our aim is to establish whether or not, beyond objective epistemological obstacles, teachers' beliefs create some misconceptions among students. So didactical obstacles, arising specifically from some beliefs of the teachers, must be added to the epistemological obstacles. This is also clearly reflected in teachers' beliefs regarding the knowledge that their students must acquire.*

Key words: *Epistemological obstacles, Didactic obstacles, Teachers' beliefs, Students' beliefs, Misconceptions, Fractions, Angles.*

THEORETICAL FRAMEWORK OF THE RESEARCH

In view of the complexity of the research, there are two distinct specific theoretical reference outlines:

1. One relating to beliefs and to changes in beliefs, as well as to teachers' expectations regarding their students;

2. A general one relating to misconceptions.

Beliefs, Changes in Beliefs, and Associated Aspects

Changes in teachers' beliefs regarding a subject - mathematical, epistemological, or didactic - always represent an issue which is not easy to confront, because, in some cases, they come into conflict with sensitive personal and professional aspects. However, as the current research shows, they have considerable didactic significance. It is, in fact, universally acknowledged that beliefs are important constituents of the set of that which is known, given that they establish and condition this set, as noted by Schoenfeld (1983) more than twenty years ago.

For an example of the studies conducted in this sector, and for the vast range of specific bibliography analysed therein, see D'Amore and Fandiño Pinilla (2004), a work dedicated to the changes in beliefs in mathematics, epistemology, and didactics on the part of teachers (secondary school) in training programmes.

Since we plan to discuss the theme of beliefs, we consider it to be of some interest to state explicitly that we will utilise the following definitions of the following terms (D'Amore, Fandiño Pinilla, 2004):

«*belief* (conviction): opinion, a set of judgments/expectations, that which one thinks about something;

someone's set of beliefs (A) about something (T) gives the *conception* (K) of A relative to T. If A belongs to a social group (S) and shares that set of beliefs relative to T with the other members of S, then K is the conception of S relative to T. Instead of 'the conception of A relative to T' we often speak of 'the image that A has of T'».

The shift from a personal belief to a shared conviction is extremely specific, given that beliefs are conditioned by complex interactions within social groups. In fact, since we are unable to separate the analysis of individual beliefs from the analysis of beliefs of the group one belongs to (Hoyles, 1992), we must also consider the micro-social aspect, which is very important in this research, as we will demonstrate.

That personal beliefs influence didactic and methodological choices has already been demonstrated in a work by the Research Nucleus of the University of Bologna, dedicated to the theme "area and perimeter" (D'Amore & Fandiño Pinilla, 2005). In this study, the personal beliefs of researchers, teachers, and pupils were placed in close relationship.

As highlighted by the research, it is true that beliefs can have deleterious effects on didactic action, but the opposite can also be true, as the results of our research show. We are also encouraged in this by the following statement "Beliefs can be an obstacle, but also a powerful force which allows the carrying out of changes in teaching." (Tirosh & Graeber, 2003). Moreover, in our work, great epistemological importance is placed on the socio-cultural perspective and on the interpretation of "practice" as constituents of the processes of conceptual construction, in a clear anti-Platonism, or even anti-realism view.

On the interpretation of the socio-cultural perspective, see the D'Amore, Radford and Bagni interview-conversation (2006). On classroom, mathematical activity, see D'Amore and Godino (2006). On the realism – pragmatism dichotomy, see D'Amore and Fandiño Pinilla (2001), D'Amore (2003), D'Amore, Godino (2006). This last work puts forward a historical-critical analysis of the derived perspectives; both anthropological and ontosemiotic.

Misconceptions

One of the first documented appearances of the term "misconception" in mathematics occurred in the USA in 1981, in a text by Wagner (1981) which treated the learning of equations and functions. Again in 1981, a celebrated text by Kieran (1981) discussed equation solution activity. 1982 saw the publication of an article belonging to the algebra learning domain: Clement (1982). In 1983 we have works by Wagner (1983) and Kieran (1983), again on algebra. Then numerous works published in 1985 specify the term "misconception": Schoenfeld (1985), Shaughnessy (1985) and Silver (1985), who use it mainly with regard to *problem solving*, together with beliefs or to explain their interactions.

In Silver (1985, pp. 255-256) it is stated explicitly that there is a strong tie between misconceptions and mistaken beliefs.

Schoenfeld (1985, p. 368) highlights how students can correctly develop some incorrect conceptions, particularly regarding procedures.

As we can well see, in the first half of the 80s intense work on this theme was conducted by Mathematics Education scholars. Fischbein himself, in the 80s and 90s worked in this field, only sometimes, however, using the term explicitly, particularly in relation to the learning of probability (Fischbein et al., 1991; Lecoutre, Fischbein, 1998).

Interesting citations of the term appear in Furinghetti and Paola (1991) and in Bonotto (1992), where it is given as a synonym for "incorrect rule" (p. 420). The term is used several times in Arzarello, Bazzini and Chiappini (1994) with regard to the learning of algebra. In all of these works, the term is interpreted in the negative sense acquired from the literature.

The term "misconception" is also used considerably in Gagatsis (2003), with the same meaning.

Bazzini (1995) maintains: «In the realm of the most recent studies, a fascinating area of investigation is the one relative to the function of analogical reasoning in the process of the restructuring of individual knowledge and of the overcoming of misconceptions (Brown & Clement, 1989)».

Further on, citing Fischbein, she reports: «We should not, however, forget that if different kinds of analogical reasoning on the one hand can encourage the construction of knowledge, on the other they can induce erroneous conclusions at the moment in which particular aspects are emphasized or distorted to the disadvantage of others. If the analogy is a potential generator of hypotheses, it can also cause misconceptions or misunderstandings (Fischbein, 1987, 1989). It often happens that when the subject finds himself in considerable uncertainty, when facing a problem to be solved, he is brought to transform a certain nucleus of information from a well known domain to another less known one by means of a transfer by analogy. It can happen, then, that one can assume analogical correspondences to be valid which, instead, are not plausible for those particular systems. We are speaking of tacit analogies which can insert themselves into the cognitive process and disturb it».

The concept of misconception was not precisely defined at the moment of its entrance into the world of Mathematics Education research, but, as we have seen, it was, and still is, used with its intuitive meaning.

Having taken into account the positions of the different authors and the sometimes rather diverse occurrences of this term, we maintain that the attention given to misconceptions, since their appearance in the world of the sciences (not mathematical), has been very productive because it has forced scholars to no longer identify errors as something absolutely negative, to be avoided at all costs, but as human products caused by situations on the path of evolution. Increasingly often, over the years, a shared meaning of "misconceptions" has been delineated as causes of errors or, more precisely *perceived* causes of errors; causes which are often very justifiable and sometimes even convincing.

It is therefore undeniable that this kind of study has forced us to examine the interpretation of the reality of the subject; an interpretation created on the basis of convictions which have matured partly as a result of learning; therefore, to view misconceptions as the fruit of something known, not as an absolute lack of knowledge. (D'Amore & Sbaragli, 2005).

In recent years, we have also been starting up a first classification of misconceptions, observing their specific features. A first distinction regards those which we have called avoidable and unavoidable misconceptions (Sbaragli, 2005).

Avoidable misconceptions derive *directly from the didactic transposition of knowledge and from didactic engineering,* in that they are, precisely, a direct consequence of the choices of teachers. These misconceptions arise from scholastic practices "undermined" by improper habits proposed by teachers to their pupils.

In fact, it often happens that decisions taken by teachers complicate the learning of mathematical concepts: decisions, sometimes deriving from the proposals of the noosphere (textbooks, programmes, magazines,...), to supply to the pupil, day after day, always and only unambiguous conventional representations which are in this way blindly accepted by the pupil because of the *didactic contract* established in class and the phenomenon of *scholarization* (D'Amore, 1999a).

The continuous and unambiguous demands by the teacher ensure that the student, and even sometimes the teacher himself, confuses the proposed representation with the mathematical concept that he wants to teach: «The student doesn't know that he is learning signs that stand for concepts and that he should instead be learning concepts. If the teacher has never reflected on this point, he will think that the student is learning concepts, while, in reality, they are "learning" only to use signs» (D'Amore, 2003).

The repetitiveness of the representations supplied is not the only cause of avoidable misconceptions, which also often depend on representations that have been badly chosen by the teacher himself (Martini & Sbaragli, 2005; D'Amore, Fandiño Pinilla, Marazzani & Sbaragli, 2008).

The unavoidable misconceptions are those which derive only *indirectly from the choices* made by the teacher, in that they are a consequence of the need of having to say and prove something of a non-definitive nature in order to explain a concept.

Such misconceptions can therefore be attributed to the necessity of having to start from a certain knowledge in order to be able to communicate; an initial knowledge that, in general, will not completely encompass the entire mathematical concept that one wants to propose.

At this point, we can find a connection between the two analysis elements of the knowledge construction process. The ontogenetic and epistemological obstacles seem to be tied to the idea of unavoidable misconceptions, given that they depend both on the maturity of the pupil in his ability to conceive a specific piece of mathematical knowledge (ontogenetic obstacle) and on the concept itself, often epistemologically complex (epistemological obstacle) which is proposed. At the same time, the avoidable misconceptions are tied to the idea of didactic obstacles depending on didactic transposition and on the didactic engineering choices made by the teacher.

It is, therefore, the task of the teacher to pay close attention to misconceptions and to the obstacles which can arise during the teaching-learning process; to be aware that what the student thinks are correct conceptions can in reality be misconceptions and that the cause of these misconceptions can depend on different types of obstacles. So our research focuses strongly on the didactic aspects. Obstacles and misconceptions are therefore strongly connected, particularly if these issues are viewed from the aspect of the problems encountered by the pupil in terms of mathematical conceptualisation.

RESEARCH QUESTIONS AND PRELIMINARY HYPOTHESES

The research questions are listed here below.

Q1. What beliefs do elementary and secondary school teachers hold about some aspects concerning the subjects of fractions and angles, from both the cognitive and the didactic aspects?

Q2. What are the expectations of teachers, at this scholastic level, regarding the performance of their pupils relative to the two mathematical topics in question?

Q3. What are the beliefs of pupils, aged 9 to 14, about these aspects concerning the topics of fractions and angles? How do these beliefs evolve over the years?

Q4. Is there coherence between the expectations of teachers regarding the performance of their pupils and the results effectively achieved during tests?

Q5. Do the beliefs of the teachers referred to in Q1 affect those of the pupils referred to in Q3? In what way? Is there relevance amongst the results obtained between the two samples in question?

Q6. On the basis of the answers obtained in Q1-Q5, can we hypothesise that the epistemological obstacles related to the subjects in question (angle and fraction) are not solely responsible for the potential cognitive failure of students, which can also be attributed to didactic obstacles?

Listed here below are the respective preliminary hypotheses formulated at the beginning of the research and thereafter confirmed or proven wrong by the results of this same research.

H1. Although all teachers will have completed a teacher training programme, we hypothesised that some teachers (often, but not only, primary school teachers), would have demonstrated some uncertainty in the implicit mathematical formulation in our proposals, particularly because of the transition, testified by international research, linking academic Knowledge, acquired in the years of study, to the school practice of didactic transposition, which, in the end, also conditions the knowledge of the teacher. In our opinion, this also implies that the didactic convictions of teachers are "modelled" on the their cognitive expectations regarding their students.

H2. The considerations of H1 condition the expectations of teachers, creating the illusion that their cognitive proposals, if appropriate from an adult perspective, are easily acquired and constructed by students. We therefore assumed that teachers would give excessively positive preliminary opinions on the actual results of their students.

H3. We decided that the intentionally generic question would only be responded to after the tests. However, we hypothesised that the increasing age and the consequent increase in level of skill would not have excessively modified the knowledge acquired in elementary school.

H4. As already said in H2, we hypothesised an excess of positive pre-evaluation by the teachers.

H5. We hypothesised that there would be close coincidence, easily identifiable by comparison, between on the one hand the answers to the questionnaires and the interviews done with pupils, and on one other hand, the interviews conducted at the same time with the relevant teachers, on the other.

H6. We hypothesised that, as noted by the international research in other sectors of mathematics, also in this case the undoubted existence of epistemological obstacles would have been supported by the creation of didactic obstacles, which would be easily noted on the basis of the answers given to the preceding research questions. In short, the distinction between the two categories of obstacles is foundationally correct, but perhaps didactically nonexistent.

RESEARCH METHODOLOGY

The test was designed according to the following methodology. Researcher 'A' gives a class a questionnaire on fractions and angles (see the attached document), with questions that are designed to highlight potential misconceptions. At the same time, researcher 'B' interviews the teacher of the class, outside the classroom, on the same topics as those listed in the questionnaire given to the pupils. When necessary, follow-up interviews are conducted with pupils to clear up possible ambiguities. There is therefore no interference between the class teacher and the work of the pupils.

In academic year 2007/2008, pre-tests were conducted in 4 classes; 2 in the ES (elementary school) and 2 in the MS (middle school), on a total of about eighty pupils, to start calibrating the research topics and the methodological approach to be followed. The results of these initial investigations allowed us to clarify the feasibility of this research and to prepare the final versions of the tests. According to the research plan, therefore in academic year 2008/09, 12 classes in Ticino Canton (Switzerland) were evaluated, 6 ES and 6 MS, giving a total of 105 ES pupils and 115 MS pupils.

The same tools were used for all tests:

1. Two researchers A and B for each visit, with the same functions, but now equipped with an improved grid both for recording the interviews and for their subsequent analysis; tape recorders were used during almost all the interviews.

2. Students were given written tests consisting of both multiple choice questions and questions with semi-open answers; in part, on subjects already known and in part, on new subjects.

3. The interviews were carried out with the teachers at the same time as the questionnaires were given to the students. The technique was that of a clinical interview.

4. Immediately after each class visit, the group met up to analyse the notes taken during the interview itself and to listen to the audio recordings.

INITIAL RESULTS

The section below reports the results of the research, divided between the beliefs of the pupils and those of the teachers for both of the mathematical contents subjected to this analysis: fractions and angles. We have chosen to highlight more the beliefs of the teachers, which allows us to give more exhaustive answers to our research questions.

4.1 General Results of the Pupils

The section below reports the results of the pupils obtained from the questionnaire and from the subsequent interviews.

4.1.1 Fractions

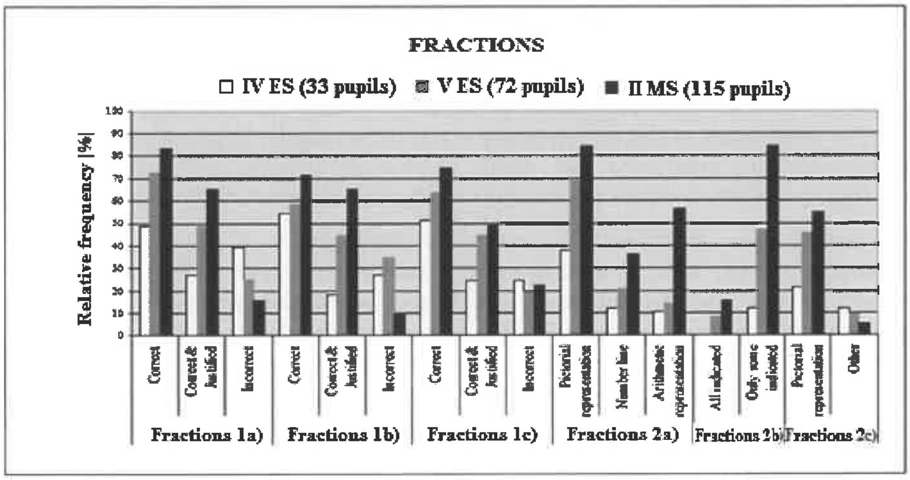

Figure 1. Results of the teachers on the topic fractions.

We will begin with the most predictable results. Globally, more pupils answer the questions correctly than make mistakes. If, however, as also transpires from the expectations of the teachers, we consider only the correct and justified answers, the global result is much less satisfactory. Comparing these results in the various classes (4th and 5th ES, 2nd MS) we note that increased scholastic level is associated with constant progress in all the items.

Amongst those who did not respond correctly, the following error typologies, encountered at both scholastic levels, were highlighted; students:

went astray on the non-congruent parts: “No, because they are not divided into equal parts” (5th ES);

considered only half of the figure: “No, because they are divided in two” (5th ES);

couldn’t conceive that two distinct parts could constitute one single part: “No, because neither of the parts is near or attached” (2nd MS);

didn't know how to arithmetically manage the fractions: "No, because it doesn't represent the ½ part, but 2/4" (5th ES); "Because part A is ¼ and part C is also ¼ and in all it is 1/8" (2nd MS).

It should be noted that there was a significant difference between the answers to questions 1a and 1b, which makes us hypothesise that the form of the figure affects the recognition of the corresponding fraction.

The pictorial register is considered (even by the MS pupils) the most appropriate of the semiotic registers for representing the fraction ¾. This could be ingenuously surprising considering that, in Ticino, it is in this scholastic period that the theme of fractions is addressed again, placing emphasis on the arithmetic register. We also noted some difficulty in recognizing the same meaning for different representations. Moreover, it should be noted that on several occasions the answers turned out to be contradictory. Some pupils may have read question 1d too quickly, and may have failed to consider the *not*. For example, one pupil (and not the only one) who overall responded well to all the fraction questions, wrote the following answers for question 1b), 1c), and 1d):

1b) 1 and 2 because they both make ¾. 3 and 6 because they both are 0.75.

1c) NO.

1d) 1 and 2.

The concept of fraction as a rational number is not greatly focused on in the ES in Ticino; here also, one can note an excessive fear on the part of the teachers.

4.1.2 Angles

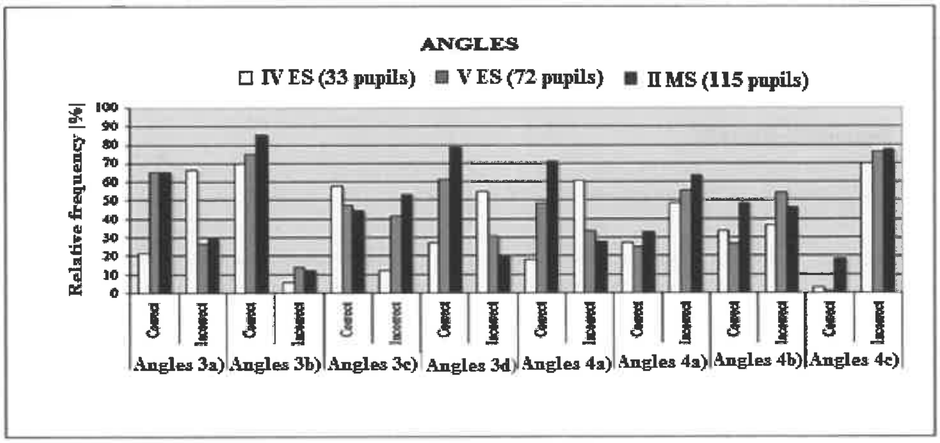

Figure 2. Results of the teachers on the topic angles.

The fact that the concept of angle as part of a plane is not generally tackled in the ES, leads us to believe that several unjustified answers, particularly of the 4th ES pupils, are casual. For example, for item 3c the pupils of the 4th ES were even better than both those of the 5th ES and those of the 2nd MS.

To answer questions 3a, 3d and 3e, a substantial number of the ES pupils used a goniometer (protractor), a strategy used in the classroom by many teachers, and this has made it even more complicated to interpret the results because small inaccuracies or errors in measurements have significantly influenced the percentage of success. This is not true for the pupils of the MS, who knew how to apply their knowledge of vertically opposite angles.

The answers to questions 3b and 3c have highlighted a very common kind of error, which consists of confusing the sign used to indicate the angle (the symbol) with the angle itself (the object): "It isn't 'inside' the angle" (2nd MS).

The lack of conceptual knowledge of the angle profoundly influenced their success in item 4, in which poor quality answers were encountered. At least three different interpretations of the idea of angle are seen in these incorrect answers:

point coinciding with the origin; "Because the angle has only one point, which is B" (2nd MS) or with the vertex of a polygon; in 4a) one pupil answered: "NO: because D is from another part" and justified points C and P in the same way;

union of segments: [D does not belong] "Because the two lines of B do not connect with the [...] point" (2nd MS);

limited surface: "Because it is completely outside, even from the diagram" (2nd MS); the expression "inside angle" probably contributes to reinforcing this last view of angle coinciding with the polygon;

"arc"; confirming that for some pupils the angle is identified with the "arc" which is usually used to represent it, one pupil answered question 4a): "NO: because it has nothing to do with it and it's too far away"; the same thing happened with question 4b).

4.2 Teachers' Results

4.2.1 Elementary school teachers

We report some considerations deriving from the interviews conducted with 6 elementary school teachers; this report is also divided between the subjects "fraction" and "angle", accompanied by conversation extracts reinforcing the considerations.

4.2.1.1 Fractions

First situation - Conceptual aspect

In terms of the conceptual knowledge of the elementary school teachers on this piece of knowledge, the 6 subjects questioned they were not perceived to experience any particular difficulties in answering correctly the questions submitted to the pupils. Only one teacher demonstrated significant indecision in answering question 1c) and this indecision was due to the requirement to obtain only congruent parts amongst themselves when talking about fractions.

We report an excerpt of the interview from which this indecision emerged:[1]

1: You've put me in a difficult situation, for that initial one, [of the rectangle] I wouldn't know, for that one under it, yes, it is half, but of that …

I.: Does all of this coloured part A + C together represent ½ of the total rectangle?

1: Yes, however there is still another half and another half of C which are white.

I.: But what are you looking at in this figure?

1: I'm trying to report the figure here.

I.: But what about these figures: look at the shape, the perimeter, …

1: The problem is that if there are *two different figures*… No, in the end nothing changes if this little rectangle has the same dimensions as this one. But, I don't know.

I.: You always refer to the first rectangle, but is it different if you refer to this rectangle? Does anything change?

1: No, because it's equal.

I.: Ok, then A and C together?

1: They make half.

I.: So, the difficulty depends on the different shapes?

1: Yes, for me yes, if C were also like this (the same shape as A), then it would jump out at me that there are 4 equal parts.

The questions asked were answered well by one teacher, but she emphasised that she considered the concept of fractions differently if talking about arithmetic or about geometry, not realising that the fractions proposed are associated with plane figures in a continuum and therefore refer to areas. So, in the situations proposed to the teachers and to the pupils, the arithmetic aspect turned out to be profoundly connected to the geometric aspect.

2: This is a conversation [questions about the rectangle] which, for example, amongst other things, "If you speak about areas, it definitely works, even if I then say that it has nothing to do with fractions. Let's say that the fact of transporting a part of the figure in order to understand, for example, I

[1] We decided to number the teachers and to indicate the interviewer with I.

don't know, the area of the parallelogram, I transform it into a rectangle, moving a triangle, I don't know if I'm explaining myself well, eh .... Surely, doing activities more... they will certainly get more used to seeing, even if I repeat that this is not important with fractions, I don't want to create confusion with fractions at this point. But, when you do activities with young people, such as I don't know, geometry, I think they use those activities in other environments, it isn't that they don't use it, but without confusing the subjects. I don't want you to think that I'm confusing the areas with the fractions".

First situation - Didactic aspect

As concerns the didactic aspect with reference to the subject of fractions, 5 out of 6 teachers interviewed maintained that it is indispensible to show congruent parts to their pupils when speaking about fractions and they were not able to hypothesise how it would be possible to do otherwise in class, also because they are concerned about giving pupils problems that are too difficult for them; these hypotheses have not been proven in practice, however. These teachers turn out to be closely tied to the traditional practices in use in the Ticino Canton. Although recognising that, in the geometric register, the unit can be partitioned into non-congruent equi-extended parts, in 5 cases out of 6 examples of this kind had never been proposed in class: the teachers stated that they had never considered the problem.

I.: Do you ever give fractions of this kind where the figures are not congruent as in this case?

3: No, absolutely not, never.

I.: When do you tackle examples like these, where the parts are not equal, but represent the same fractions?

3: I don't know when I will do it. I do fractions with DiMat.[2]

I.: Does DiMat cover cases of this kind?

3: No, they are not covered in DiMat, not even in the fifth year. When we present fractions, it is explained that they must be divided into equal parts, because, otherwise, they would never spontaneously divide them up in that way [meaning: in non-congruent parts].

I.: Does DiMat specify that making fractions means dividing into equal parts?

3: Yes.

[2] DiMat (Differenziazione dell'insegnamento della Matematica – Differentiation in the teaching of Mathematics) is a commonly used approach to the teaching of mathematics in Ticino in the 3rd – 4th – 5th elementary school classes. DiMat specifies the use of worksheets, which allow differentiated student learning paths.

I.: And, in your opinion is it right to say that making fractions means dividing into equal parts?

3: As an introduction, of course. Otherwise, they wouldn't understand. They already find it difficult to understand fractions in this way. Just imagine, with this more difficult example, they wouldn't understand. This could be a later example, but definitely not the initial one.

I.: Do you show examples where it is not necessary for the parts to be equal?

4: No, in my opinion they must divide into equal parts.

I.: Do you ever show examples like this?

1: Like this one, no. I've never shown it.

I.: Why not?

1: Well, I've never shown an inside rectangle divided in this way... that is, I have already shown, for example, a rectangle subdivided like this [all equal parts] divided in half, in tenths, but the same figure divided in two, no, I've never done it.

I.: You've never shown examples with differently-shaped parts?

1: Yes, that's right.

I.: Do you use DiMat?

1: Yes.

I.: In DiMat, are there no situations like this?

1: There are a number of situations where they have to... where they have, for example, some squares where they have to colour 1/8, 1/10, or they have to subdivide some chocolate bars and then I have them construct a sheet of paper with a grid, then, for example, there are 4 bars and there are 6 children who have to divide them up. Last year I had a wonderful experience with fifth year pupils, which was given by the 'school board'. So, I've never done it, but I am not so tied to the DiMat. I sometimes prefer doing lessons that are face-to-face, because it isn't an easy subject. I often show them cakes where I divide them into 4 parts and I show them ¼, it is a bit real, like the slices. It's difficult, also in DiMat. At F level they colour in, they understand it well, but already at M it is already difficult, even for the good students,[3] they don't finish it, precisely because it is complex. They know that 1/5 of 25 means divided by 5, however ... it isn't obvious.

Only one teacher stated that he had recently worked (confirmation proven by the results of the class) on situations (graphic register) representing equi-extended, but not

[3] F and M are two levels of the DiMat.

isometric, fractions. When dealing with these situations, the pupils of this class based their work on counting the squares and not on the relationship between the parts and the whole.

Comparison of expectations and class results

Three teachers stated that some children could be mislead by the fact that the shapes of the parts are not congruent and, as we have previously seen, this comes from their didactic practice which does not cover these cases. At different levels, all three of them overestimate the results of their pupils.

I.: In your opinion, will your pupils have problems?

1: Perhaps some, yes, they are not all... I think that only some will have problems.

In particular, teacher C overestimated the results for question 1b) where the correct answers were only 40% versus his forecast of "a few" who would make mistakes.

The second teacher thought that his pupils would answer well in part, but would not know how to explain the correct answer, the subject of fractions is generally rather complex.

5: They will say "yes", but they won't know why. They are generally quite unsure about fractions and they will then give random answers. But ... I don't know how my pupils will answer. It is really difficult for them. Some could answer well, but I don't know if due to real knowledge, the others will answer at random without saying why. Not even the good ones will say why.

Although predicting some difficulties, this teacher:

I.: Where are the greatest difficulties with these questions?

5: As we were saying before, precisely because the parts are not equal, if it had been divided into 4 equal parts, it would have been easier for everyone to understand, but as it is ...

5 completely overestimates the results. The results are, in fact, 17.6% for the first question and 5.9% for the following two. It should be remembered, however, that we're dealing with year four pupils.

The third teacher also overestimated the results, maintaining that most should answer well, while, in fact, only 35% answer 1a) correctly and 45% answer 1c) correctly.

2: I think that they should know it, but I think the fact that there is a triangle here and a rectangle there, could, instead, create a bit of confusion, because it is not always easy when there is a small complication within the situation.

...

2: I don't know if all, I don't know, some have difficulties, a substantial little group, 4-5 pupils who work at their level of calculation and automatism, but of reasoning no, but I'm a bit of a pessimist... the previous teacher, however, also said so, that it is not an easy class.

Instead, one teacher evaluated the abilities of his pupils well. In particular, the fact that revision and further study had recently been done on fractions allowed her to anticipate that most of her pupils would answer the first situation questions satisfactorily. There is, however, an exception to question 1c). Only 55% (versus the expected 70-80%) stated that A and C together represent ½ of the whole figure, with pupils often giving explanations contradictory to those of the first two answers:

1a) Yes, because the two equal parts have been divided up;

1b) Yes, because one half has been divided;

1c) No, because, if not, it would be completely coloured.

One teacher was not off balance in his predictions, because most of the questions concerned subjects barely or not yet tackled.
Another teacher, for questions 1a), 1b) and 1c) predicted a success rate of 60% and broadly speaking reached this level.

Second situation – Conceptual aspect

As regards the second situation related to the subject of fractions, no particular conceptual difficulties were noted regarding recognition of the different representations as equi-meaningful to ¾, except for most of the cases which associate the representation 4 to 3/7, as was predictable and also correct. However, it should be noted that the context certainly made them pay more attention when giving the answers.

3: Yes, they are all definitely ¾.

I.: For you, are they equi-meaningful or not?

3: Yes.

2: 0.75 is 0.75; I would say 3, 5, and 6 say the same thing which is 0.75. And also the first and the second and this one here, left hanging [number 4] is left a bit on its own, although if it could be interesting. But, I see it as 3 over 7 not 3 over 4.

I.: Does each of these representations say the same thing or something different?

2: Perhaps yes ... But, but in the end they say the same thing, but actually I don't like the mathematical expression very much, let's say it expresses the same quantity, that's a way of saying it.

I.: In your opinion, do all of these mean the same thing?

C.: This one no, because it is 3/7, the others are all ¾.

One teacher considers 0.75 as ¾ of 1 and not as an equi-meaningful representation of ¾ in itself.

2: 0.75 represents ¾ of 1.

I.: ¾ of 1?

2: Of 1, because 1:4 makes 0.25 times 3 makes 0.75.

The preference for the representation which better expresses the concept of ¾ is concentrated, more than others, on the first, in one case on the fifth and in one on the third:

I.: Is there one of them which gives the idea of ¾ more than the others?

3: I prefer the first, because I now reason like my pupils.

I.: Why?

3: It's the one that, at first glance, gives more the image of ¾.

2: The first is the one I'm more used to for ¾, also the children.

I.: Is any one of these representations that is better than the others or not?

2: I like number 5, or the first one, even number 6 isn't bad, I like decimals.

1: I don't know, they are so different, perhaps this one [number 3] because it's more original. But, then, I see either 3 or 5 in fraction, otherwise I don't see the fraction.

Second situation – Didactic aspect

From a didactic point of view, the teachers state that they don't explicitly work on all these semiotic representations of the same concept, at most only on some. The impression is that, despite being quite inclined to admit the equi-meaningfulness of the representations (except in the ambiguous case of number 4), in class the teachers feel hampered by their concern about showing excessively complex examples. So, the use of various semiotic registers is willingly deliberately reduced and/or postponed for as long as possible.

1: I don't show this kind of activity in class.

2: I don't explicitly say that ¾ and 0.75 mean the same thing, at the most ¾ and the first drawing.

One teacher stated that he used only two registers: the graphic one (equi-extended areas) and the fractional one. Other registers were not presented to the pupils.

One teacher stated that he had only mentioned "without insisting" the fraction-decimal number passage and that he had not ever used the representation of rational numbers on the numeric straight line. Moreover, he maintained that the representation of the single interval [0.1] constitutes a substantial epistemological obstacle.

6: "The pupils are used to handling the numeric straight line with positive whole numbers. A few simple decimals have already been inserted, but it's quite another thing to deal with the rational interval only [0.1]".

Comparison of expectations and results of the class

In four cases, the forecasts, although negative, were better than the results. For example, for the first representation, regarding which a teacher stated he was certain the pupils would recognise the concept of ¾, 50% recognised it. A similar result was recorded for the fifth representation, which was predicted to be recognized by 50% of the pupils but which was actually recognised by only 20%. Another teacher also predicted a good recognition success rate for the first three representations, but this was not the case, and percentages of only 2.4%, 23.5%, and 17.6% were recorded. The third teacher predicted good recognition success for the first, the second, and the fifth representations, while the results were 20% for the first two and 0% for the fifth. One teacher predicted a success rate of at least 50% for the second and third representations, but the results were much lower.

Different semiotic registers are used with extreme prudence and particularly in the fifth year, so one teacher correctly predicted that the 4th year pupils (this was a multi-level class, with 10 year 4 pupils and 8 year 5 pupils) would limit themselves to discrete object subdivision and to geometric subdivision.

For the fifth year pupils, she evaluated her pupils' abilities well. She predicted that there would be satisfactory answers for representations 1 and 2, but considerable difficulty in recognising the fraction ¾ in the others. As, in fact, happened, she anticipated that many would see the fraction 3/7 and not ¾ in situation 4.

One teacher was not off balance in his predictions, because most of the questions concerned subjects barely or not yet tackled. When he abandoned caution, he made optimistic forecasts ["Everyone should answer this question, except for the usual ones..." questions 1a), 1b), and 1c)]. In fact, the class successfully answered around 50%.

According to the teachers interviewed, the pupils don't realise that the representations are equi-meaningful, since this item of knowledge had not been introduced in class.

I.: In your opinion, will your pupils realise that we are saying the same thing?

2: This is difficult, I don't think so, I'm a bit pessimistic, and I admit it.

I.: Do you think the children will match them as you did?

2: Not all of them, I don't think that all the children, perhaps 0.75 and 75/100 that yes, but match them, no, if only! It is difficult because, in certain things, it isn't so easy to know what they'll answer and what they'll not answer. In this class, you often have to revise for a small group.

Regarding the prediction of the choice that the children will make for the representation which best expresses ¾, everyone agrees that it will be the first:

3: The first in my opinion. The ones that will sidetrack them most will be numbers 4) and 6). Matching ¾ to 0.75 is not a logical combination.

1: Number 1, because it goes over material that they have seen and then there is this one figure and that's all. There is no doubt. Instead, here, you at least wonder, but what is the sack [he's speaking about number 2]? Thinking about DiMat, if each sack is a tenth and the unit is sometimes not very clear inside the sack.

I.: If you had to represent the concept of ¾, which representation would you suggest?

1: I would take a cake and divide it into 4 and I would say that we can eat 3 slices.

I.: So, like number 1?

1: Yes, with them, yes.

4.2.1.2 Angles

First situation – Conceptual aspect

Of the 6 teachers interviewed:

2 answered all the questions correctly;

1 stated a situation of uncertainty in question 3b), about whether point C belonged to the highlighted angle (a question related to whether the side is part of the angle or not);

1 demonstrated not having ever thought of the concept of angle as part of unlimited plane;

1 got the answers to questions 3b) and 3c) wrong, saying that Q did not belong to the angle and revealing in the interview that he didn't have a clear definition of angle.

He also got the answer to question 3e) wrong, saying that the two angles considered were not of the same size.

I.: Does point Q belong to angle 1?

1: No, the vertex and the sides belong to the angle, the point belongs to… that is, no, the size is equal both here and here, that is, I've never done the problem where a point belongs to an angle. We usually do the vertices and the sides of the angle. We don't do the points. Maybe now, we are discovering other things. The point belongs, huh!

3c)

I.: And in this case?

2: I think it should be the same thing, that is, not exactly the same thing, because the position is different.

3e)

I.: Are angles 1 and 2 the same size?

2: No.

1 did not understand the meaning of the term 'point belongs' with reference to an angle, being exclusively tied to the metric aspect of angle, that is, making the size and the angle coincide. However, this teacher proves consistent with himself; when he defines angle, he actually speaks about its size (even if in an incomplete way).

3b)

1: In what sense does it belong to angle 1?

I.: Is Q part of the angle that we have called 1?

1: (silence)

I.: Is Q a point of that angle, or not?

1: Uh, the angle you measure with the size, no? This will be... I don't understand the question. Roughly, you could say that the angle is always equal in size both here and here.

I.: But what is an angle?

1: The size of the angle is determined by the position of the two straight lines, from how they touch.

I.: So is Q part of this angle or not?

1: (silence)

I.: If I have a segment and I ask you if a point belongs to the segment... I mean a point is also a point of the segment besides being a point in itself (it represents the segment and indicates a point).

1: If it is on the segment yes, because it is part of that.

I.: Now, I ask you if Q is part of the angle which we have called 1, or not?

1: In what sense? I don't understand your question. In the sense if in this case there is an right angle [he points out a vertex of the table]...

I.: And if I ask you if this point is part of the right angle [he indicates a point on the table].

1: If to be part of means that it is on this surface, yes, but, however, the angle defined as degree… I don't understand it as point.

I.: How will your pupils answer this question?

1: In my opinion, they will answer yes, because they see it inside. [Next question] And for P, it is clear that it is always inside of the angle, but they will be influenced by the sign of the angle which is up to here and P remains outside even if, in its own right, it is still inside the angle P.

I.: So you would have answered to question c) that P is still inside the angle; it belongs to the angle.

1: Yes, apart from belonging, in any case the angle is this, this is 90 degrees [he indicates the table again]. If I represent it on a sheet of paper and I do this or this [he indicates two different positions to measure] the angle is always the same.

I.: Therefore, this point here belongs as much to this as to this [he indicates points on the table]? Do they belong to the right angle?

1: Huh! Let's say yes [he wants to go on and continues with the next question]. They seem equal to me.

First situation – Didactic aspect

The didactic choices made by the teachers regarding the angle never involve showing possible misleading aspects of the concept of angle arising from graphic situations.

3: Not exactly graphics like this, but we have seen that the angle is not only the one marked, but the whole part between the two half lines.

I.: At the graphic level, do you introduce situations like the ones seen here?

3: No, to tell the truth, one of your students [Giada] did it. She had done a course on the introduction of angles and showed these things that, in any case, the points were inside.

I.: If you think that the representations are misleading in this way, why do you not use them didactically?

3: To tell the truth, I had never thought about it.

I.: Have you ever worked on the misleading aspects of the representations of angles?

5: No, to tell the truth, no, it should be done, I admit it.

I.: Do you show your pupils activities of this kind?

1: No, these I've never done, that is, it actually seems like the same angle, but the surface misleads… It's interesting! This indicates the size which here is smaller than the other; at first glance, for me, the children will be misled by this.

It is clearly noted that, in the classroom, the teachers speak mainly about angle as a measure, rather than from the conceptual and graphic point of view.

I.: How do you introduce the angle?

2: As a part of space, perhaps we can say that it belongs to the space. Honestly, we introduce the angle in daily life. They have to understand that it is definitely a part of space, then, that it has a vertex and sides which they must understand which can be lengthened, but we say that it belongs… I don't know. I don't remember.

I.: How do you introduce the angle?

2: I do it in real life, then we construct angles of different sizes or we give pre-made angles and have them find the size.

I.: So, it is a measurement activity.

2: Good for you, I have to admit that here we should broaden a bit, this yes; I admit that time constraints or deficiencies on our part mean that we should probably look at angles again. They know how to recognize them, but as representations sometimes, perhaps we could perhaps change a bit, sometimes we are very tied to measurements, also.

One teacher, however, stated preference, in class, for a representation to help the pupils remember that it isn't a limited figure (∢). He recognizes, in any case, that as a teacher, he often tends to always use the same representations which can create incorrect intuitive models.

The interview of one teacher showed how geometry tackled in class was reduced to minimal levels and that the concepts, such as the concept of angle, were not studied at all. In the fourth year, the image of angle that the pupils have is the one acquired in life, therefore very unrefined, empirical, and with all the potential misconceptions. In class, they had only classified the angles as acute, right, and obtuse. In year five they spoke of the angle of a polygon as part of a plane lying between the two sides. The teacher has never thought of the angle as an unlimited part of the plane and so has never mentioned it to his pupils. He recognizes that year 4 pupils in particular can have the idea of angle as a circular sector defined by the arc; part of a plane which the teacher usually has the pupils colour in. Moreover, he has never dealt with intersection of lines.

Comparison of expectations and class results

According to the teachers, several pupils will be misled by the questions proposed, as they were not part of their didactic practice. In this situation, there are fewer overestimated cases, while also some underestimated cases. In gêneral, the forecasts are closer to the results.

I.: Will your pupils be misled or not?

3a)

3: In my opinion, yes, I already reason from the point of view that there is a reason for asking the question, but several make mistakes because they don't reason from the point of view that, if they ask you a question, there is... We have seen this argument, but for this question, I think many will make mistakes. Unless they use some means to deal with it, a set square.

3d)

3: It's the same as before, we have the half-lines that form two equal angles and two unequal angles, but the representation is very misleading. In my opinion, those using their acquired knowledge can answer correctly, but others will be misled by the graphic aspect.

I.: Will your pupils have the same problem?

2: I don't know. Maybe. Here, a well defined part is coloured, but it could be coloured here also. The angle measures here, measures here, the size is equal. It belongs, ... huh! We have never placed a point that belongs.

1: We have never seen graphics quite like these, but we have seen that the angle is not only that which is marked, but all of the part included between the two half-lines so they should know it. Some, however, will say that Q yes [belongs] and P doesn't.

4: Yes, I admit it; my pupils will fall into this trap. I should have worked on it more.

3a)

1: In my opinion, they will say that angle 1 is larger.

I.: And is this true, in your opinion?

1: No, they will be misled by the dotted line part, but to me, they seem equal.

3d)

1: If they can understand and make the extension, it is like the earlier problem.

I.: So, in your opinion, your pupils will not be misled by the length of the sides of the angle, as a representation.

1: Some might, yes, but perhaps someone with a ruler could also extend the line, basically the half lines can also be extended.

In view of the small amount of geometry done in class (and the way in which it was presented), one teacher considered making any forecast excessively risky.

One teacher correctly predicts that his pupils would answer well to questions a) and b) (only in terms of Yes and No and not in terms of the justification), while for c) his evaluation is rather negative, "They will fall into the trap" which, however, is not confirmed by the result that wasn't so bad after all (50% correct answers). Many of the pupils' answers lack justifications, particularly for questions b) and c), or give justifications that are completely irrelevant (Does Q belong to angle 1? "Yes, because there would be degrees" or "because it is in the same square"), confirming the teacher's prediction.

Second situation – Conceptual aspect

Of the 6 teachers interviewed: 1 answered the questions correctly; 1 showed a situation of uncertainty about whether point C belonged to the highlighted angle;

When talking about the "internal" angles of a polygon, 1 teacher considered only the part belonging to the polygon, but, after having been alerted, returned to the part of the unlimited plane. It isn't, therefore, really a misconception, but a non-assimilated deeper understanding. So in didactic practice he had probably, perhaps unknowingly, induced this error in his pupils;

1 teacher made mistakes in the answers to questions 4b) and 4c), stating that the points do not belong to the angle, thinking of it as limited, even going against the chosen definition and, therefore, proving inconsistent.

I.: Does point C belong?

3: I say no, but what is the right answer?

I.: And P?

3: P no, P is outside.

I.: Why, in your opinion, is it called an internal angle?

3: Because it's inside the figure.

I.: What is an angle?

3: Part of a plane delimited by the two half-lines.

I.: Do you see half-lines in the polygon?

3: No, but in this case I see the sides.

I.: What do you see as unlimited half-lines? So, is the angle limited or unlimited?

3: No, but here it is limited because it is internal... Perhaps I should take a course, too.

1 teacher proved very uncertain in 4a) and 4c).

4a)

2: The point is in an internal angle this, no, meaning extending the sides of ...? If D belongs to the angle extending the sides, then, yes.

I.: So, for you, point D belongs to the angle.

2: No, no, but here they won't understand.

I.: You told me you're a pessimist.

2: But so much the better, if they surprise me, I've given my soul to these young people.

4b)

I.: And point C, on the other hand?

2: Point C is directly on the side, so it is part of the angle since it is on the side you consider it still inside, therefore it belongs to the side, you're teaching me.

4c)

I.: And point P, on the other hand?

2: If you do this [extends the sides], it is, yes, actually it is still inside, basically, even if here you have to truly see the figure on the plane with the extensions of the sides, otherwise you don't see it.

I.: Is it right to see the extensions of the sides?

2: Perhaps yes, here we should be even more... do activities on this and make the figures less static. I try to do it, but probably we should do it even more.

1 teacher didn't accept question 4). He only accepted that the points on the sides belonged to the angle and was undecided about 4c), still failing to understand the concept of belonging.

4a)

1: No.

I.: Why?

1: Because the angle is formed between BC and BA and this, only in this case [he indicates all the way up to the vertex of the side].

4b)

1: Returning to our earlier conversation, the answer is yes. It is its maximum and minimum part respectively.

4c)

1: Ah, good question. Yes, because if I extend here, it is inside. But then, also D… [he thinks]. I have a bit of a problem. Perhaps yes, I'm not sure, because I don't understand what you mean with this question.

I.: How do you deal with angle at school?

1: With them, I begin with practical things. We start them in the fourth year. The precise use of the goniometer is then consolidated in the fifth year. I begin by saying that a square has a right angle which measures 90 degrees, then they look around the room for right angles and they say a table, a corner of the blackboard, a book, a piece of furniture… I begin with the right angle which is easier and I let them draw it with a set-square on a sheet of paper and they revise and I explain that this is really only 90 degrees; not a degree more, not a degree less, while the difference with the other angles, I say that there are acute angles which I say measure from 0 degree up to 90, but 90 is already too much I say up to 89 degrees, in the fourth year I say this because they don't yet have the concept of decimal number at the beginning of the fourth year. The 90 degree right angle, then the straight angle, they draw a line and take the goniometer and then we look and there is also the round angle.

I.: So, you do a classification of the various kinds of angles.

1: Sometimes, I give them some exercises where they do it roughly without the goniometer and then we use the goniometer which isn't clear.

I.: Don't you show situations like the ones that I showed you?

1: No, I've never thought that I could tackle it like this. I wouldn't do the one with the point, but the rest is interesting, where you go a bit beyond, like with the fraction of the rectangle [he indicates the first question] where you go beyond the classic formula.

Only 1 teacher was uncertain about all the questions.

Second situation – Didactic aspect

No teacher shows his pupils the unlimitedness of an angle in a limited context such as polygons. Un limitedness is considered, not by all the teachers, only when presenting angle as a figure on its own. When polygons are addressed, they are usually presented as part of a plane defined by a line segment not as the intersection of angles. Therefore, for the pupil, the angle of a figure is limited by its sides. In any case, irrespective of the definition of polygon, there is no discussion of the unlimitedness of the angles in polygons. To confirm how, at the didactic level, discussion of angles is sometimes separated from that of plane figures, one teacher states, "Angle was done last year, this year they are working on polygons".

Generally, the teachers proved to be tied mainly to the measurement of space, and little to the conceptual and graphic aspect of angle.

I.: Do you ever show examples like these to your pupils?

5: No, I have to say no, also because we still haven't got to figures with them. We talk about angles. We measure the angles inside figures, but examples like this, no. I have never introduced them, not even in the fifth year.

One teacher stated that he never returns cyclically to topics in order to sort them out later on. For example, the concept of angle as part of an unlimited plane "is done at the beginning of the fourth year", then never touched again.

The same teacher stated that intersecting lines had been dealt with only in the case of perpendicular half-lines. Particularly meaningful: the idea an unlimited part of the plane was not taken up again when using the goniometer. The pupils know that when they use this instrument, they can lengthen the sides, but this is treated as simply a "way of using" the goniometer.

4: This year, I did a bit of revision on the angle as part of a figure lying between two sides. I have never spoken about the unlimited angle. However, regarding the use of the goniometer, I said that sides that are too short can be extended.

Comparison of expectations and class results

In conclusion, according to the teachers interviewed, it is difficult to make forecasts about this situation, since the subject has not been addressed in class. The majority of teachers predicts negative results, which however, in several cases were overestimated.

I.: How will your pupils answer?

3: They will guess, since it's a subject we haven't dealt with.

In particular, one teacher correctly predicts that all the pupils will really struggle (at most 5-10% correct answers) because "for them, point D indicates another angle and point P is outside the figure". In fact, only 15% answered that D belongs to the angle B, 25% for C, and 0% for P.

In three cases, at the end of the interview, the teachers had understood the kind of problem proposed, recognising the potential and stating that it would be a good idea to introduce them in the elementary school.

I.: Would you introduce an activity like this in class?

1: Yes, because it is interesting, because you get used to reasoning in a different way.

4.2.2 Middle school teachers

Here we report some considerations arising from the interviews with the 6 middle school teachers. It is interesting to note that the most unexpected answers were given by a teacher who trained and who has worked in the German part of Switzerland, and who is doing his first experiences at the Ticino school system.

With respect to the *conceptual aspect*, with reference to fractions, the middle school teachers were not perceived to encounter any particular problems in responding correctly to the questionnaire that was given to the pupils. The same cannot be said for angle. Uncertainty was noted in some teachers, and misconceptions in a few cases.

In terms of the *didactic aspect* with reference to the subject of fractions, however, all the MS teachers specified that they also introduce situations with non-congruent parts; stating their didactic value. The semiotic register chosen to represent fractions is usually the pictorial register, as observed even more clearly in the ES. There is not seen to be any particular didactic focus on preparing activities which aim to represent the same concept in different ways.

As regards the subject of *angles* and the didactic choices made by the MS teachers, they never consider the problem generated by possible misunderstandings of the concept of angle deriving from the graphic situation. For angle, the teachers clearly usually present the aspect of its measure, rather than the conceptual aspect of figure. To the question, "How do you introduce angle to your pupils?", most of the teachers answer, "I do it in the real world, then we construct angles of different sizes or we give them angles and have them find the size".

4.2.2.1. Fractions

First situation – Conceptual aspect

The 6 teachers interviewed do not demonstrate any insecurities concerning the conceptual aspects presented in the first situation. They all show that the importance lies in the area, and not in the form. Their concept of fraction includes both the arithmetic and the geometric aspects.

We note that the MS Training Plan in Ticino (Middle School Training Plan, 2004) specifies that the concept of fraction as operator is dealt with in the second year. It is therefore not surprising that all the teachers answered the first three questions with certainty. Teacher 1 explicitly states that "[...] it is clear, you work with the fraction as operator on the area" thus confirming a clear conceptual vision of the subject.

First situation – Didactic aspect

Unlike the results reported for the ES teachers, the MS teachers maintain that pupils must be shown congruent parts, but they are aware of the importance of the non-congruent part approach. Despite this, only half of them have actually introduced didactic situations with non-congruent figures.

Three teachers (teachers 3, 5, and 6) have already dealt with the topic of fractions as operators in the first part of the school year, introducing examples with non-congruent subdivided figures. These teachers maintain that it is important and useful to show situations that are different from the classical one (with congruent parts). Since the pupils have already encountered similar situations, the teachers in question expect good results in this part of the questionnaire (for example, one teacher states "Most of the pupils should answer 'yes' to the three questions, justifying their answers with the fact that the figures have the same area, although the shapes are different. [...] We have dealt with many similar examples with non-congruent figures. The rectangle is therefore the most commonly used figure [...]"). One of these teachers has already addressed the addition of fractions and thinks that the strategy used by the pupils to answer the third question could be that of calculating ¼ + ¼, further complicating the situation. In fact, two students took this road: one concluded correctly that "1/4+1/4=2/4 reduced to the minimum terms is 1/2". The other made a mistake stating that "1/4+1/4=2/8".

One teacher (teacher 4) introduced the theme "fraction as operator" in the first year of middle school, but without presenting examples with figures subdivided into non-congruent parts. "I dealt with the subject last year, working with figures, segments or other sizes, however rarely with figures with different shapes". He thinks that the pupils could struggle with the first and third questions.

Teachers 1 and 2 had not revised the theme of fractions, but, in any case, they expect their pupils to produce positive results, because they maintain that these concepts are already acquired by the pupils in the elementary school.

Comparison of expectations and class results

In general, the teachers expected good results in all three items. Only two of them (4 and 5) predicted difficulties in the third item.

For the first two questions, this prediction proved to be justified for the answer by itself (without a stated reason). In fact, all the classes had a success rate of about 70%. The situation changes if you also take into account the explanations. In this case, the positive results go down, in one class even to 38%. This could be explained by the greater cognitive difficulty inherent in the request to give reasons for the answer.

The third question also produced a basically positive result of over 70%, except in the case of one class - for which the teacher did not predict any difficulty, despite not having worked with non-congruent figures - where we note a significant drop of about 90% in item 1.b to 57% in 1.c. On the other hand, the class results of the two teachers who had predicted difficulties were good (over 70%).

Second situation – Conceptual aspect

No particular problems related to teachers' beliefs emerged in this sphere. All the teachers interviewed stated with certainty that figures 1, 2, 3, 5, 6 represented the fraction ¾.

Figure 4 was seen as 3/7 by five teachers. Only one teacher (number 4) hesitated, but did not go as far as to state that figure 4 does not represent ¾.

The teachers preferred the first representation. Teacher 6 saw this partiality as a legacy from the elementary school. Teachers 1 and 5 stated that they were more used to representations of this kind. One teacher (4) even stated that, "It's clear that it is divided into 4 parts and 3 of them are taken". Teacher 3 also indicated representation 2 as his favourite. Teacher 2 was the only one to differ. In his opinion, the best representation was number 5, "since it is already written as ¾".

Second situation – Didactic aspect

The general tendency is to work with a reduced number of semiotic representations in order to avoid confusion.

The practice in Ticino leads one to prefer the first two representations, which seem very commonly used in the ES. This practice finds justification in the belief that it would be rather inappropriate to change the approach followed in the ES. Teacher 6 indirectly justified such a choice, referring to the Training Plan for mathematics in which representations different from 1 and 2 would never be explicitly indicated. He stated that "representations 5 and 6 are more for the third year of middle school", in the sphere of the subject of rational numbers, implicitly justifying the failure to address it in the second year. In fact, in the new Training Plan for the second year includes the specific objective "Recognising the fraction as a result of division [...]".

It is interesting to note that teacher 2, who comes from outside Ticino and who indicated number 5 as the best representation, is the only one to nourish any perplexities about representation 1, even stating that such an image could evoke "[...] a 2×2 table".

The teachers thought that representation 3 was difficult to understand. Teacher 3 even asserts that, "It is the most difficult". In fact, it was not always understood as a representation of the straight line of numbers. It was sometimes confused with the representation of the length of a segment. For example, in representation 3 teacher 5 did not see the straight line of numbers and therefore did not know how to associate it to ¾ as a number, but rather to ¾ as an operator applied to the length of the segment represented, stating that "[...] to be ¾ in the figure, the part equal to ¾ of the segment must be highlighted".

Regarding representations 5 and 6, it is interesting to note how the practice described previously - which leads to separating the aspects of operator applied only to bi-dimensional geometric figures from the numeric ones - causes teachers 5 and 6 to predict that "[…] representations 5 and 6 will be placed in relationship to each other, but without connecting them to ¾".

Comparison of expectations and class results

All of the teachers, except number 4, thought that the first (and possibly the second) depictions were the most chosen. This prediction was largely confirmed by the data.

Only teacher 6 explicitly expects positive results for representation 3, in the knowledge that he had worked on it. The data confirm the predictions of the teachers. We have a success rate of about 75% for the class of teacher 6 compared to an average of less than 25% for the others.

As expected, representation 4, recognised by all the teachers as unrelated to ¾, was not taken into consideration, not even by the pupils.

3 of them (1, 3, 4) hypothesised that representation 5 could also be one of those identified as ¾, due less to any true understanding of the concept, but rather because of a purely formal aspect. "[…] They could get there, if they think about the use of the "fraction" sign used in geometric formulae; for example, h=A/b=A:b". The remaining 3 teachers predict that representation 5 will not be amongst those indicated, since the topic had not yet been addressed in class. Surprisingly, the data proved the predictions of all 6 teachers to be partially wrong. For those who hypothesised recognition, only slightly more than one quarter of the pupils actually recognised the representation. For those who hypothesised non-recognition, on the other hand, almost half of the pupils recognised it. The data also proved teachers 5 and 6 wrong; they predicted the association of representations 5 and 6, but separate from ¾.

Teacher 2 differs from the others. He thought that the pupils might see a table in the first illustration, and that representations 5 and 6 would clearly be chosen as the best representations of ¾. The data proved him wrong in that only a few pupils answered as he had predicted.

4.4.2.2 Angles

First situation – Conceptual aspect

The conceptual aspects of this situation are clear for 5 teachers. Teacher 2 admits to not knowing the definition of angle as part of a plane. For him, the angle "[…] is the measure of a rotation". However, all the teachers answered all the questions correctly, including teacher 2 after the definition of angle as a geometric figure was made clear to him.

First situation – Didactic aspect

In general, after being introduced in the first year of middle school, the concept of angle as a geometric figure is never used again. It is a sort of inert concept.

Their didactic practice favours operative aspects rather than conceptual aspects. None of the teachers interviewed has ever presented their students with problems concerning the angle as a set of points. All of them dedicate significant time to angle measurement, particularly to the use of the goniometer.

Opposite angles at the vertex, in particular the aspect relating to their congruence, are already addressed in the first year of middle school and the topic is applied to many problems.

Comparison of expectations and class results

All the teachers expected that the pupils "[...] will immediately ask to use the goniometer". They also thought that questions would generally be answered correctly. For them, the pupils "[...] except for a few" will answer correctly "[...] without being distracted by the arcs or by the extension of the coloured surfaces".

The analysis of the answers indicates:

for item a): two out of three pupils answered correctly, in most of the cases without justification; one of the most frequent justifications is the one related to measuring with a goniometer;

for items b) and c): the success rate of item b), more than 80%, coincided with the teachers' expectations; on the other hand, the success rate of item c) was half, dropping to a little more than 40%, clearly proving the teachers wrong; this highlights how the large number of correct answers in b) actually hides a misconception held by the pupils;

for items d) and e): in this case the teachers' predictions proved to be justified; approximately 3 out of 4 pupils answered both items correctly; the difference between the two situations did not seem to bother the pupils.

Second situation –Conceptual aspect

All the teachers answered correctly here as well (a couple with some hesitation), despite expressing amazement about questions of this type and stating more or less explicitly that they were seeing it for the first time; teacher 6 "[...] strange, I'd never thought about it".

Second situation – Didactic aspect

None of the teachers had ever introduced this kind of question; not so much because of an informed didactic choice, but rather because they had never considered the possibility of reflecting on similar aspects. Nevertheless, teacher 1 specified that he insists right from the start that the angle is unlimited.

Comparison of expectations and class results

Teachers 2, 3, and 4 did not manage to predict their pupils' answers. The other three made predictions, but only very prudent ones. Teacher 1 said that "[...] they should answer 'Yes', but there could be difficulties since D is the vertex of another angle" and he added that he often uses the expression "[...] angle in B, which could reinforce the idea that the angle is finished". Teacher 6 expected a majority of correct answers only for item a), while teacher 5 expected a majority of correct answers only for item b). Globally, the results for this second situation, were very low, only approaching 50% for item b). Only the class of teacher 1 obtained results bucking the trend, all between 50% and 60%, in line with his expectations.

ANSWERS TO THE RESEARCH QUESTIONS

A1. Notwithstanding the studies conducted, the most diffuse beliefs amongst the elementary school teachers are similar to those held by their pupils; mainly with regard to the concept of angle, and to a lesser extent, fractions. Didactic habits, reference to one's own experience and that of colleagues, reference to texts or to specific materials used in Ticino, very often reaffirm that the knowledge possessed by teachers greatly influences what they demand of their pupils. One does not see a real didactic transposition understood as a re-elaboration of Knowledge (D'Amore, 1999b). It seems that the teacher limits himself to providing his own knowledge to his pupils. Therefore, the cognitive and didactic points of view are not far apart; there is a tendency to identify the belief about a concept with the way in which it is introduced, with the way with which it is cognitively understood. The situation is slightly different for the middle school teachers, some of whom have a deeper understanding of the concepts in question. However, this culture does not modify the didactic methods, so even in this case, preference is given to executing a minimum didactic transposition, almost identifying the cognitive and didactic aspects.

A2. There is no difference between the two scholastic levels. The rules implicit in the didactic behaviour associated with the teacher's epistemology (Brousseau, 2008) come into play and the teacher maintains that the performance of the pupil should coincide with the repetition of the activities (oral and written) which have marked his teaching action. The expected performances are rather superficial and repetitive. In fact, faced with situations defined as 'new' within our proposal on this theme, there are opposite, but cognitively similar, reactions: a) these proposals cannot enter the didactics because they would change the scholastic programme; b) I had never though about these proposals and I will use them in the future. Both reactions demonstrate that teachers' expectations tend not to enter into Vygotsky's 'zone of proximal development', but to limit themselves to the effective zone.

A3. A double answer is given to the first question of Q3; one general and one specific. The general answer: as with all other mathematics topics, fractions and angles are also intrinsic concepts in the scholastic programme, for which the teacher gives a definition which must be accepted (on the whole identical for both the elementary and the middle schools), the learning of which is not important. The important thing is to tell the teacher what he expects to hear. The specific answer: angle and fraction are subjects which are not encountered in the extra-scholastic reality and which are supplied only with information, explicit or implicit, given in the classroom. The answer to the second question of Q3 is formulated on the basis of the observation-comparison of the responses given in the elementary school and in the middle school. If we compare the two school levels, we do not see any significant differences between students' beliefs regarding angles and fractions. Notwithstanding the age and maturity differences, the mental model formed in the elementary school remains in the middle school and the apparently paradoxical situations proposed in order to initiate discussion are perceived as such at both scholastic levels.

A4. The answer to Q4 is positive; in the sense that the teachers do not seem surprised if a student gives a wrong answer, or in any case an answer not coherent with the definitions. The teachers, adult and educated, seldom fail to see the "deception" in some of our questions. However, when analyzing the test, they tend to recognise the answers of their own pupils.

A5. To answer Q5, it is necessary to make a significant distinction between the two scholastic levels. In the elementary school, as said previously, the knowledge of teachers and the expected knowledge of students are virtually identical. In the middle school, as also said previously, the teachers possess a greater mathematical conceptual education, but they do not submit their own knowledge to a true didactic transposition. Nevertheless, although this knowledge nourishes different conceptual beliefs, it does not greatly alter the expectations of the teachers at the two scholastic levels. Their expectations of these mathematical subjects are rather similar. So for different reasons, the beliefs of the teachers influence the beliefs matured by the students; misconceptions included.

A6. The history of mathematical thought demonstrates, without a shadow of a doubt, that angle and fraction are concepts for which one can speak about epistemological obstacle. (D'Amore, 1985; D'Amore, Marazzani, 2008; Fandiño Pinilla, 2005, chap. 2). However, the analysis conducted shows that, precisely because of the choices (which for the elementary school teachers are obligatory, while for the middle school teachers they are due to their beliefs about the abilities and possibilities of their pupils), the epistemological obstacles are superimposed by didactic obstacles. As we have demonstrated, these didactic obstacles are tied to the beliefs of teachers regarding the two mathematical concepts and the learning of their pupils.

CONCLUSIONS

The results demonstrate that, in some cases, the cognitive and didactic points of view of the teachers are not in any way distinct from each other. There is a tendency to identify the belief about a concept with the way in which it is proposed and with the way in which it is cognitively expected. The situation is slightly different for the middle school teachers, some of whom have a deeper understanding of the concepts in question. However, this culture does not always modify the didactic methods for angles, so even in this case one ends up by executing an ineffective didactic transposition.

The beliefs of the teachers, both cognitive and didactic, precisely define the classroom activities and also influence their interpretation of their role; what to teach, how and why.

The preliminary beliefs of the teacher also run the relationship in the classroom. The teaching of correct mathematics is not important, and therefore neither is hypothesising correct learning process. It is more important to obtain the answers that were expected right from the beginning of the activity.

The questionnaires proposed by us include questions chosen purposely in order to highlight the fact that even students who seem to have conceptualised a concept well, in fact limit themselves to good contractual performance: they answer the expected questions, probably because of the "Topaz" effect (D'Amore, Fandiño Pinilla, 2008) or similar.

The most striking example involves the angle; even having admitted and repeated in response to an explicit question that the angle is a certain unlimited part of a plane, in fact, the addition of the adjective "internal" referring to polygons profoundly modifies conceptualisation, leading to situations expressing explicit contradictions.

The results therefore demonstrated that the conceptualisation of these two concepts, fractions and angles, *did not always happen*, particularly as regards the angle. There was no real construction of knowledge; not only from a conceptual aspect, but not even from a semiotic aspect, because the different semiotic representations of the angle modify their conceptualization.

Amongst the many possible causes, our research clearly shows the influence exerted by teachers beliefs, which, as has been seen, determine both the beliefs of the students, and in some cases their failure to develop significantly over time, and also the teachers' fear of proposing sufficiently varied and rich situations, adding didactic obstacles (avoidable to (objective) epistemological obstacles.

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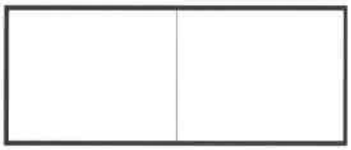
APPENDIX

Pupils' questionnaire

FRACTIONS of a continuous unit

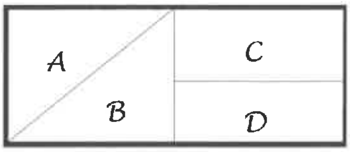
Here is a rectangle divided into two parts.



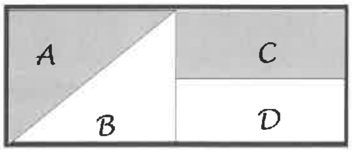
FRACTIONS of a continuous unit (continuation)

Now, each of the two parts is divided into two equal parts.

So, there are four parts: A, B, C, D.



Two of these parts have been coloured in.



Does part A represent $\frac{1}{4}$ of the first rectangle? Yes □ No □ Why?

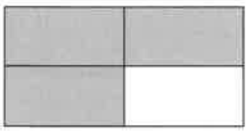
Does part C represent $\frac{1}{4}$ of the first rectangle? Yes □ No □ Why?

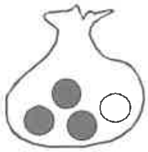
Does all of the coloured part represent $\frac{1}{2}$ of the first rectangle? Yes □ No □ Why?

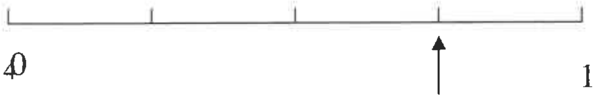
Representation of FRACTIONS

Look at the following figures, diagrams, and writing.

a) Beside each representation, write what it is trying to communicate.

1. 

2. 

3. 

4. 3 : 4

5. 0,75

b) Of these preceding five representations, which of them say the same thing? Why?

____

____

c) In your opinion, does one of them 'better' represent the fraction $\frac{3}{4}$? Yes □ No □

If yes, which and why?

____

____

d) Are there any that do not represent the fraction $\frac{3}{4}$? Yes □ No □

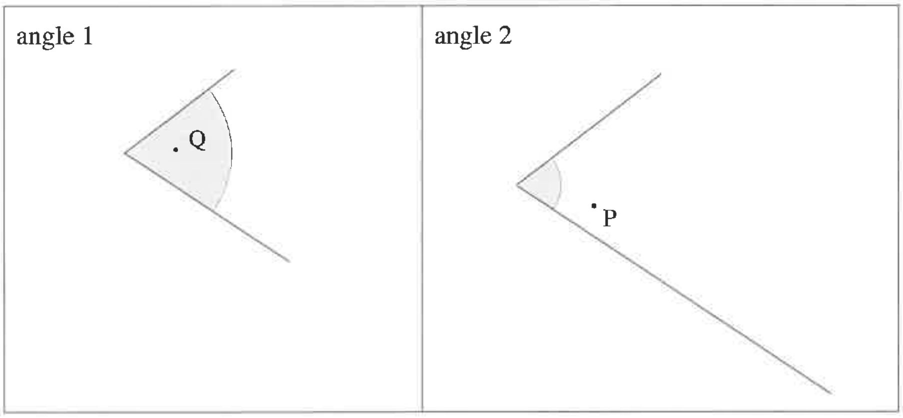
If yes, which and why?

____

____

ANGLES

Look at the two angles represented below, which we have called angle 1 and angle 2:



Answer the following questions:

a) Between the two angles 1 and 2, is there one that is a larger size than the other?

Yes ☐ No ☐ Why?

____

____

b) Does point Q belong to angle 1? Yes ☐ No ☐ Why?

____

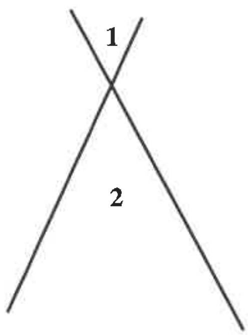
____

c) Does point P belong or not to angle 2? Yes ☐ No ☐ Why?

____

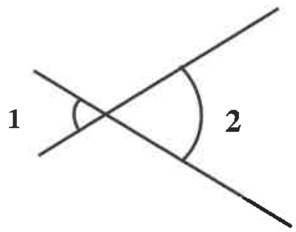
____

d) Two straight lines intersect forming the acute angles 1 and 2.



Is the size of one of these larger? Yes ☐ No ☐ Why?

e) Two straight lines intersect forming the acute angles 1 and 2.

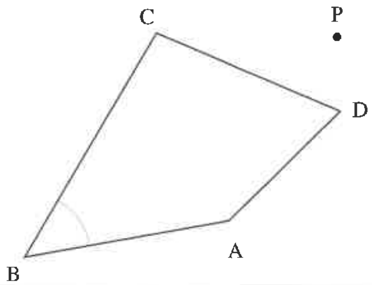


Is the size of one of these larger? Yes ☐ No ☐ Why?

#

INTERNAL ANGLES

In the quadrilateral ABCD, an internal angle has been highlighted.



Answer the following questions:

a) Does point D belong to the highlighted angle? Yes ☐ No ☐ Why?

b) Does point C belong to the highlighted angle? Yes ☐ No ☐ Why?

c) Does point P belong to the highlighted angle? Yes ☐ No ☐ Why?

Questions of the type posed during the teacher interview

Fractions

Have you seen the test that we have proposed? In your opinion, what will your more able pupils answer? Why?

And the ones that are less able? Why?

How do you usually introduce the concept of fractions?

Have you ever proposed examples of this kind? Why? Which ones do you usually show?

Does the shape of the highlighted parts affect the answer? From what point of view?

With reference to the representation of fractions, since it deals with different semiotic representations of the same concept, they should be equi-meaningful with each other. However, certainly for the children it will not be like that. Do you use all of them?

Which do you maintain are the most advantageous?

In your opinion, which one or ones will the children choose?

Do you have in mind others that you would propose to your pupils?

Angles

Have you seen the test that we have proposed? In your opinion, how will your more able pupils answer? Why?

And the less able? Why?

How do you usually introduce the concept of angle?

And how do you handle it after that?

Do you propose situations of this kind?

Yes / No, why?

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Problem-Solving Activity Ancillary to the Concept of Area

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ABSTRACT: *This paper concerns the results of the second stage of a two tier program designed to enhance students' technique usage in area measurement. The first stage involves 11 year old students; certain techniques were didactically introduced with the dual purpose of cementing the concept of area and area preservation, and of giving the students tools for explicit area measurement (either exact or estimates). The second stage deals with the development of the same techniques, but the focus is not now primarily on the direct enhancement of the central concept (area) but on the re-assessing, re-examining and adapting of the techniques themselves. The paper reports on a case study concerning two 13-year old students' output analyzed from this latter context. Their work in particular shows several ways that they could refine the 'technique' of decomposition of plane figures.*

Key words: *Problem-solving, Development of techniques, Area.*

PROLOGUE

Consider the following task:

Given a triangle T with base a and height h, divide the triangle into three sub-regions such that the sub-regions can be re-positioned to form a rectangle, one side of which is a. Deduce the area of T in terms of a and h.

The result is well known, but the method to obtain it might seem unorthodox. Lying this apart, simply suppose that you are asked about what domain is the task situated. From the perspective of the result, one might say area determination. From the perspective of the method, perhaps the re-arrangement of sub-regions forms the best candidate; in practical terms the concept of area takes a nominal role as long as some basic principles of area preservation are well understood. From the perspective of motivation, we come back to area determination; why should we be interested in the manipulation of the sub-regions unless it yields what we want? This example illustrates problem-solving activity that does not contribute to the enhancement of a concept that sponsors it, but at the same time it retains ties to that concept. This kind of situation we shall call 'problem-solving activity ancillary to the concept'; we will expand on this idea, with its educational significance, in the next section.

It is also possible that students in doing the task above and others of the same ilk eventually weaken the background reliance on the concept of area and the problem solving raises new conceptualizations that are not directly related to area. In particular, the students' work reported in this paper (which includes a task that is closely related to the one stated above) come close to the notion of dissection, a relatively sophisticated way to match up pairs of congruent sub-regions of two plane figures.

The paper, then, concerns a kind of tug-of-war between problem-solving that respects a certain concept (area) but at the same time is suggesting a tentative development towards another sphere of conceptual thought.

INTRODUCTION

This paper primarily examines 13-year old students' ability to work with success on area measurements. This is done in terms of a particular perspective of problem solving that has a more global setting, explained below.

Problem solving has been taken by the literature in two main guises. Either one regards problem solving as result oriented, where the focus is how the solution enhances the understanding of some concept, fact or principle, or as strategy-based, where the stress is on how the solution itself was obtained. This distinction is reflected in Schroeder and Lester (1989) which refers to 'teaching via problem solving' and 'teaching about problem solving'. (Another category is mentioned also, 'teaching for problem solving'; it concerns the application and the transfer of the knowledge and experience obtained from previous problem-solving activity, but it would seem reasonable to regard such activities to be subsumed in teaching about problem solving, as executive control issues remain predominate factors). For 'teaching about problem solving' there has been a long tradition established by authors like Polya (1973) and Schoenfeld (1985) for which notions such as heuristics and executive control take central roles. This means that solvers are encouraged to monitor their own progression when solving problems, and also they are taught a number of strategies (heuristics) from which they can use in carrying out a problem - solving plan. Heuristics tend to have the form of general advice in aspects of problem solving. Some typical examples of such strategies are: looking for patterns, solving a simpler problem and working backward. This tradition does not usually stress a constant mathematical theme. 'Teaching via problem solving' in contrast tends to keep to a succession of tasks consistently relevant to a particular concept. These tasks tend to be non-standard, so a contribution might be expected to occur in the students' cognition concerning the concept at hand through the deliberation required to resolve the tasks. However, sometimes this expectation of a contribution to the students' cognition is void. If the solver already possesses the requisite conceptual background, what can happen is that the task reduces to how to maneuver the already assimilated methodology associated to the concept. The problem-solving aspect, though, does have a context imposed by the concept, so it is constrained to act in a particular sphere; there will always be a connection to the concept, but not necessarily a direct dependence on it. This reflects a situation when 'teaching via problem solving' evolves into 'teaching about problem solving' but retaining a consistent conceptual backdrop.

Quite often, when teaching a new concept, a collection of techniques is introduced didactically with an original intention to promote the understanding of the concept and its inherent properties. (By the term technique we mean something between heuristics and algorithmic procedures. It constitutes a kind of mathematical knowledge which is characterized by: i) a standard sequence of steps, which is its procedural part, ii) usually a standard form for its output and iii) lack of a predetermined approach for achieving the above mentioned steps. One has to encounter several tasks with consonant structure to register a technique, Mamona-Downs & Downs, 2004). After some time, it can happen that the usage of the techniques no longer enhances the conceptualization; instead, the student employs her/his mastery of the concept to organize the methods known to guide strategies in order to tackle given tasks. This concerns a transition; from being regarded as content – specific, the techniques become, one might say, 'concretized heuristic' in character. We shall term the problem-solving aspect involved here 'problem solving ancillary to a concept' (Mamona-Downs & Papadopoulos, 2006). (If the concept is not specified or is assumed, we sometimes shorten this to ancillary problem solving and use the acronym A.P.S.) The term is meant to convey that the problem - solving activity is dependent on the presence of the concept, but is incidental to the actual apprehension of the concept itself. It 'serves' without 'affecting'. The focus is on how to 'make' the techniques to produce results that lie in the conceptual field, rather than extending it. There are several compelling reasons to study A.P.S. It marks a process where problem solving evolves from an orientation towards the mere result to the appreciation of the strategy taken. The existence of a collection of techniques associated with a concept provides a unified base that is rarely found in the tradition of 'teaching about problem solving', so the demands in some of the more exacting factors such as mature metacognition and executive control might be alleviated. In this way, A.P.S. could form a channel to investigate the development of students' ability in problem solving over time. A particular feature of A.P.S. is that it possesses a ready-to-hand well-connected knowledge basis; it is of interest whether, and how, students can take advantage of this source. Finally, the influence of the concept that sponsored the A.P.S. should not be disregarded. Even though we assume that the allied conceptualization is no longer refined, it still acts as an 'anchor' to maintain the integrity of the problem solving activity. Allied to this, it is highly likely that the appreciation of the concept and its properties can play a part in guiding the mental processing of the techniques in order to negotiate a task. Hence, despite we can discuss A.P.S. in terms of a general educational construct, it can be also treated thematically according to the concept involved, as it is done in this paper; we shall consider problem solving ancillary to the concept of area.

Fundamental properties of area and certain actions respecting area preservation can be intuited with some reliability fairly early in the teaching of the topic. The techniques which were introduced to first enhance students' understanding of area determination soon became 'objects' of examination in themselves. How can the techniques be combined, adapted, and take roles in the solving process? Such considerations lie very much in the domain of A.P.S. as discussed above.

The previous suggests a two stage teaching program; first, to introduce didactically a collection of techniques through which properties of area are explored and established; second to challenge the students to overcome the limitations of the techniques (in the form that they know them) to tackle new problems related to area determination. The primary target for the first stage concerns concept enhancement, for the second technique empowerment. Most papers dealing with the learning of area at early years (meaning primary school and first year of secondary school levels) combine elements of both stages without making much of a distinction (Baturo & Nason, 1996; Clements & Stephan, 1998). However, we feel that explicitly introducing the A.P.S. aspect will give a base to study the elaboration of technique usage further that it might be otherwise. In particular, the principles in task design will differ between the two stages.

We implemented such a two stage teaching program on area; the first stage involved 11 year - old students, the second stage took place a year later using some of the same students. Results of the first stage can be found in Papadopoulos (2008; 2010). This paper will report mainly on the results of the second stage, with the specific aim to record and assess the new aspects in technique usage occurring in the A.P.S. environment. Thus, we are seeking to address the following research question: In the context of Problem Solving Ancillary to the Concept of area, to what extent would students be able to re-assess, re-examine and adapt the techniques relevant to the specific concept? Would new spheres of conceptualization occur?

Such questions require an access of the students' ways of thinking more than that mere written evidence would yield. To address them, we conducted a fieldwork within the second teaching program. Both methods of 'talk.aloud' whilst working and individual interviewing were applied. As the issues concerned are quite subtle, requiring close contact with subjects, we gauged that a case study involving only two students was the most useful channel to employ in our exposition.

The rest of the paper is organized as follows. The next section will give some background material, which cover associated literature and students' past experience. After, a section discusses the tasks used and a précis of the students' responses to each. Two sections follow that discuss the results in more detail. The first reports them thematically according to various different perspectives concerning problem solving, and represents the whole process of obtaining their solutions. The second assimilates and refines the same material in such a way that more pointed observations are made directly towards the explicit aims of the paper above; it is more focused on the final positions taken by the students. The final chapter summarizes the content of the paper, with comments on its significance in mathematics education.

BACKGROUND MATERIAL

In the introduction, the issue of problem solving ancillary to a concept, in particular area, was described. In this section will discuss related literature, and students' past experience.

Selected Literature About Students' Skills in Problem Solving

A.P.S. is a perspective of problem solving that is tied to a particular conceptual knowledge base. The resultant restriction does not affect many general problem-solving principles found in the literature. It is relevant to mention some problem-solving issues that will be important in our analysis of a fieldwork conducted in the spirit of A.P.S.
There are many studies concerning students' behavior in the problem-solving agenda. The majority of them are directed to either high-school or university students. The ones that deal with primary school students (e.g., Ball & Bass, 2003; Francisco & Maher, 2005; Lampert, 1990; Yackel & Cobb, 1996) largely emphasise students' behavior vis-à-vis resolving obstacles, as well as specific problem-solving elements like conjecturing, verification and decision-making. (These elements were evident in the work of the participants of our research.) A précis of the research findings on these lines follows.

In the problem solving - tradition, notions such as heuristics play central roles; they constitute general mathematical problem-solving strategies. But when a series of tasks is given with a common conceptual backdrop, the solver rather seeks for more concrete methods relevant to the concept that could be used as a source of knowledge applicable to associated problem solving. Francisco and Maher (2005) worked in this direction, with their task design model based on giving students related tasks concerning targeted mathematical concepts with comparable levels of difficulty and similar problem-solving structure. What they found in their longitudinal study (K-12 up to college) was that these strands enhanced the students' ability to overcome cognitive obstacles in problem solving. On the contrary, English (1997) reports on 6th graders' abilities to recognize commonalties in problem structure to use them when they have to solve parallel problems. Her research showed that what we can expect from such young students is a tendency to focus on the surface features of problems rather than the underlying structural elements, something that obviously limits students' problem solving abilities in general and the problem solving ancillary to a concept in particular.

Conjecturing and verifying are vital parts in problem solving. Many students either assume that a conjecture offered in the class is true as soon as it is proposed, or it is accepted on the evidence of it 'working' on a few cases (Ball & Bass, 2003; Ball, Hoyles, Jahnke & Movshovitz, 2002). Lampert (1990) working with 5th graders found that whether the students accept a conjecture as mathematically 'sound' depends on whether the assertion makes sense in an informal, intuitive way. On the other hand, verification processes include a second thinking through the steps of a solution, a questioning whether the intuitive premises were founded, comparing a solution with those of similar tasks met before, making checks by taking particular cases that can be separately treated, or forming alternative approaches and seeing how they agree in their results. Verification often comes in the form of a visual representation to strengthen the cognitive basis of stricter argumentation. Further, students can verify their solutions based on their experience or on an intuitive empirical approach (Eizenberg & Zaslavsky, 2004).

Regarding the students' willingness to apply verification processes, research shows that students usually do not verify the accuracy of their final answers (Pugalee, 2004; Schoenfeld, 1985). Stillman and Galbraith (1998), working with secondary school students, found that while almost all of their students believed it was necessary to check the final results, only a few of them claimed they always did so, and a large proportion of them said that they did not verify their solutions or did so only rarely. A measure that can be taken to redress such behavior is to give students tasks whose results might be surprising, prompting doubts in their argumentation that will promote verification processes (Hadas & Hershkowitz, 1998, 1999).

Decision-making is inherent in problem solving. On a prosaic level, one could say that decision-making is required to select an appropriate approach. However this usually gives an artificial picture of the solving process. In most cases the solver starts with a provisional or tentative direction and sees what it leads to (see Selden, Selden, Hauk & Mason, 2000). Reflection on what has been achieved vis-à-vis the aims prompts one of the following; an abortion of the approach and adoption of another, an adaptation of the original approach to re-align it, or a suggestion how to proceed further. Here decision-making is not solely making a choice from a selection of possibilities, but it also involves the processes of generating the possibilities that are made available. Decision-making in this sense is metacognitive in character, and is commonly called 'executive control' in the literature. In general, executive control requires mature deliberation in projecting the potential of the present line of thought, married with an anticipation of how this might fit in with the system suggested by the task. (The theme of anticipation is taken further within the framework of 'transformational reasoning' as expounded by Simon, 1996). Schoenfeld (1985) has indicted that many undergraduate mathematics students have very poor executive control skills. On the other hand, some quite young students do seem to have some ability to make deliberate decisions that lead to effective changes in approach, see e.g., Schoenfeld (1992).

The Students' Past Experience in Working with Areas

The study concerns a research program with two stages. In the first stage certain 'innovative' methods and teaching materials were put forward aimed at enhancing the concept of area for 11- year old students. This had a dual purpose of researching young students' developmental understanding of area, and of arguing whether the pedagogical practices introduced should be represented in class and in mainstream textbooks. There are previous studies in the same ilk (for example, Baturo & Nasons, 1996; Clements &

Stephan, 1998). We noticed, though, that much of the students' solving did not really contribute to conceptual enrichment of area. This motivated us to undertake the second stage of our program, introducing the notion of A.P.S. as explained in the introduction. The paper concentrates on this second stage; however it remains important to state the previous background that the participating students had vis-à-vis the concept of area. This will be briefly described here.

Through their regular classes in mathematics, these students were taught some basic concepts of geometry, concerning the recognition of certain 'basic' shapes (mostly limited to triangles, quadrilaterals, and circles) and some of their properties. They were also familiar with the role of formulae as devices for calculating the area of these shapes.

In parallel to the normal teaching in primary school, the students had taken part in the first stage of our research project. This stage aimed to explore and enhance students' comprehension of the concept of area, with an emphasis on problem solving techniques for the estimation or the exact determination (if possible) of the area of irregular plane figures. By irregular here, we mean that the students do not have direct means to obtain the area without identifying or bringing in other regions. In some cases the figure could be divided into other shapes that were simpler and familiar to the students so it was easier for them to calculate the area. But in other cases these figures could not be split into other simpler ones (e.g. the outline of a lake or an island) and consequently the students had to develop different methods to calculate their area. The students' participation in this phase of the project gave them on the one hand sufficient content knowledge that was prerequisite for problem solving ancillary to the concept of area and on the other hand experience in the usage of various techniques enabling them to calculate area of some irregular shapes. The techniques that will be referred to in the paper are: A) *The usage of grids*. We refer either to a grid explicitly given or suggested indirectly by an array of dots provided in the presentation of the task, or to a construction of a grid made by the solver where there was nothing in the task's environment prompting such an action. B) *The subdivision of a shape into sub-shapes*. The students split the original shape into sub-shapes that are known to them and they calculate the area in two steps. First they calculate the area of each sub-shape and second they add to find the area of the initial shape. C) *The subdivision of an area unit (usually a square) into sub-units*. Since irregular shapes are not usually covered fully with whole measurement units, the students subdivide the units in order to deal with the remaining uncovered part of the shape. D) *Length measuring tools that allowed calculating area especially for the separate parts of a decomposed region*. We refer to the usage of the array of the dots provided in the tasks as a device to measure bases and heights so as to apply known formulas. E) *The cut and paste method*. The students 'remove' a sub-region of the figure on the boundary and attach it on the boundary elsewhere to create a new shape of different form but of the same area. This action is not necessarily enacted physically, but is enacted by imagining a suitable transformation in space. It can be applied several times in the same task.

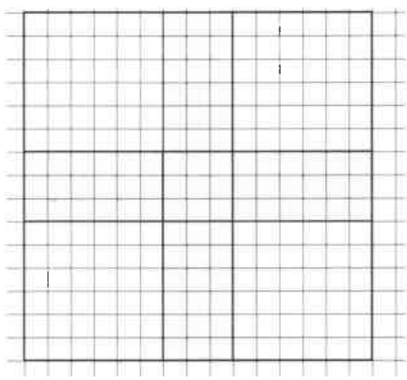
THE DESIGN OF THE FIELDWORK AND SUMMARY OF THE STUDENTS' RESPONSE

This section sets out details about the design of the fieldwork and a simple outline of the students' behavior for each task as a prelude to the deeper analysis presented in the next section.

Description of the Tasks and Methodology

This paper presents a case study with two students participating, chosen for their lively involvement in early stages of the project. We select three of the tasks concerning area determination or comparison posed to them in the second stage of our research project and we describe the students' problem-solving behavior. The tasks were non-routine in the sense that "*they are not typically covered in school geometry courses... The non-standard nature of the problems ensures that the students will not be able to solve them by simply recalling and applying familiar solution patterns*" (Schoenfeld, 1985, p.2). The students in the earlier stages of our project already had experience in coping with non-routine tasks that to some degree fostered their geometrical intuition and their level of mathematical reasoning. However, the tasks at this last stage were designed to be non-routine at a higher 'rank'. The difference could be characterized by noting that at the earlier stages, it was mostly the task environment that was acted on in order to apply the known techniques, whereas in the last stage even the techniques themselves could become a focus of examination and may be adapted. The latter situation captures the essence of A.P.S. as explained in the introduction.

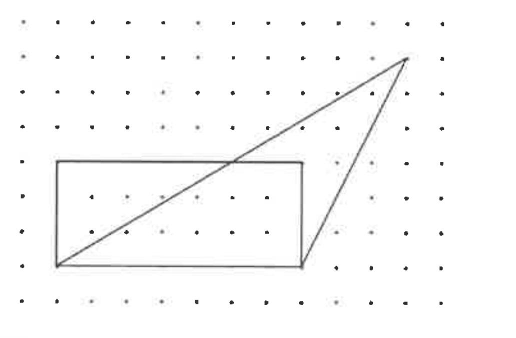
Below are the statements of the tasks as they were presented to the students (including the diagrams). For each, we say in brief what motivated their design.



You place four square -cards of length 9 onto a square region of length 15. In the picture, the shaded region represents where more than one card are covering the big square. What is the area of this shaded region? Suppose that a colleague of yours tries to calculate the area in the following way: "The area of the big square is 15 * 15, and the area of each card is 9 * 9; there are four cards. So, the area of the shaded region is 4 * 9 * 9 -15 * 15". Is the student right?

Figure 1. Presentation of the 1st task.

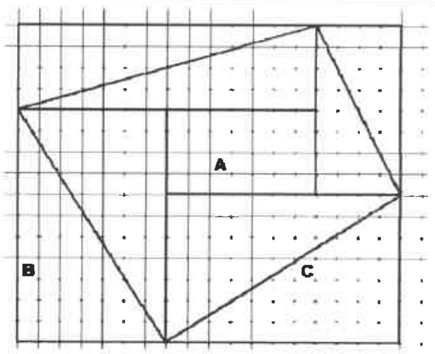
The first question of Task 1 could be accomplished by a straightforward counting of unit squares. (As such it can be thought to be within the ambit of stage 1). It was only included as a check for the numerical answer for the second question, which forced the students to extend their existing techniques in order to cope with areas involving overlapping figures.



Apply two cut and paste actions to obtain the rectangle from the triangle. Is the area of the rectangle bigger, smaller or the same as the area of the triangle?

Figure 2. Presentation of the 2nd task.

Task 2 requires a double application of a technique where for the realisation of the second application there has to be an anticipation of how to make the first. How the students would achieve this? (The last part of the task has a minor role, as a check that the students understood that the actions preserve area).



Find the area of C (*quadrilateral*) in terms of A (*rectangle in the interior of C*) and B (*exterior rectangle*)

Figure 3. Presentation of the 3rd task.

In task 3 the treatment of region A of C has to be processed in a different way from the other parts of C, causing a complexity in its resolvement and a resultant challenge to review the existing forms of the available techniques.

The students' work was conducted by pencil and paper. (The first stage of the project was done in both environments of the computer laboratory and paper-pencil; this change is partially explained by the stress on concept attainment then, and the stress on problem solving now). The duration of the session was 2 hours. The students worked individually and were asked to vocalize their thoughts while performing the task. This was conducted on the lines of protocol analysis as set out by Simon and Ericsson (1984). (For verbalization during problem solving activities see also Schoenfeld, 1985). Protocol analysis gathered in non-intervention problem-solving sessions is especially appropriate for documenting the presence or absence of executive decisions in problem solving, and demonstrating the consequences of those executive decisions (Schoenfeld, 1992). Protocol analysis in character minimises the interference of the interviewers (the authors), but it was desirable to use more direct questioning concerning the motivation of the students' working. In order to do this, we interviewed the students a few days after the session. Both sets of data were tape-recorded, transcribed and translated from Greek into English for the purpose of this paper. The researchers examined the data to identify material that was of special interest to A.P.S. From this material, certain categories were made according to how the students were working with the known techniques. The titles of the categories are phrased without reference to the concept of area, so they could be regarded as a whole to suggest a provisional and partial characterisation of different kinds of interactions that occur in A.P.S. generally. Nonetheless, the main intention of introducing these categories was to organise the presentation of the results concerning problem solving ancillary to the concept of area in particular.

Précis of the Students' Solving Behavior

Here, we shall give a short summary of the two students' solving behavior over the session. This summary is not intended to constitute a statement of the 'results', but to act rather as an orienting source when we analyze the data in detail.

1st Task

Both of the students (named Katerina and Nikos) noticed the overlapping of the regions in the middle of the big square. Katerina was confused by the fact that some parts of the shaded region were covered twice whereas the central part was covered four times. Despite the fact that she was sure that the student's expression in the task's description was wrong she did not manage to find the correct expression and finally she gave up.

Nikos realized that something was lacking in the problem's arithmetical expression. He understood that this was related with the overlapping regions in the center. He 'interpreted' this expression and made the suitable adjustment to it necessary to reach the correct result.

2^nd Task

The students divided the part of the triangle that was exterior to the rectangle into two triangles (a right and an obtuse one). They transferred these triangles into the remaining part in the interior of the rectangle and arranged them so as to fit exactly and fulfill the region, but without being fully able to justify their propositions.

3^rd Task

Using the geometrical properties of the diagonals in the four 'exterior' rectangles they tried to decompose shape C. They recognized the triangles had area half of the rectangles. But this leaves out part A. From this point the two students followed somewhat different approaches: Katerina 'doubled' C. So she 'had' now the whole exterior rectangles (since each rectangle is composed of the two congruent triangles formed by taking a diagonal) plus part A taken two times. It meant that C doubled was equal to B plus A, i.e., 2*C = A + B.

Nikos dealt with the situation concerning the triangles in B by noting that they gave a total contribution (B-A) / 2. In order to obtain C he added A so he finally reached the expression C = (B+A) / 2.

DETAILED RESULTS

In this subsection we characterize from a problem-solving perspective some of the ways in which the students employed the source of knowledge afforded by the previously assimilated techniques of area determination. They are listed below:

How Does a Student Select a Particular Technique? How Does (s)he Choose to Abandon One?

We will present how the students applied the known techniques in order to solve the specific problems and when possible clarify the motivation for selecting or abandoning a certain technique. The available techniques were: the usage of a (provided) grid (technique A), the subdivision of a shape into sub-shapes (technique B), the subdivision of an area unit (usually a square) into sub-units (technique C), the usage of length measuring tools for calculating the area (via known formulae) especially for the separate parts of a decomposed region (technique D) and finally the cut-and-paste method (technique E).

In the first task the students easily calculated the area of the shaded region considering the big square as a synthesis of specific sub-shapes. Both were familiar with the idea that they could split the initial shape into sub-shapes knowing that its area equals with the sum of the areas of the sub-shapes that constitute it (tech. B). So, they calculated the area of the big square and subtracted from this the sum of the areas of the four white squares to find the area of the shaded region.

In the second task Katerina initially constructed the grid squares lying completely either in the given triangle, or in the rectangle (tech. A). An influence for this is that over the last two years, there was an accumulated experience where the task environment included an array of dots, hence the construction of the grid squares was a familiar strategy for her. The array of the dots acted as a cue to argue in terms of completing and counting squares (Mamona-Downs, 2002). The next step (also according to past experience) was to divide the partial square units into sub-units (tech. C). This expanded the original approach rather than being a rebuttal. When we asked her in the interview to explain her decision, she justified:

K.I.T2.1.: As I saw it, I immediately remembered the tasks we dealt with in primary school.

But very soon she rejects this approach:

K.4.5.: What I have done is useless.

Her criterion was that this method did not meet with the task specification of 'two movements':

K.I.T2.2.: The instructions are talking about two movements. If I were to use grid I would have to move the square units one by one and consequently I would realize more than two movements in order to solve the problem.

Note that in taking account of the instructions, she envisioned 'moving' squares of the grid, whereas the previous image of a grid would be static. Finally she divided the triangle into three parts to form a decomposition (tech. B) (see Figure 4).

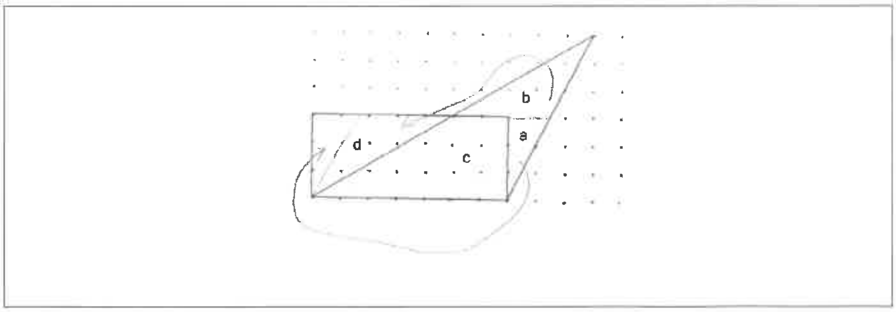


Figure 4. Katerina' s separation of the triangle.

She decided to transfer the 'a' and 'b' parts by cut and paste to 'fill' the remaining part of the rectangle (tech. E). However she had to think about how to apply this method since she had to choose specific parts for obtaining the transformation. She explained her choice verbally:

K.4.10: I do not have to move the 'c' part because it is already in the interior of the rectangle.

K.4.18: I will transfer the a-part so as to fit with the right angle in the d-part.

K.4.20: And then I will move the b-part in the remaining space of the d-part

This could be regarded as an act of control, based on exploiting perceived structural similarities (Mamona-Downs & Downs, 2005). She had a problem in that she thought she needed a length measuring tool to check that the triangle 'a' in its new position ensured the 'completion' of the rectangle.

K.I.T2.3: I counted the dots (*technique D).*

K.I.T2.4: I knew that the right angle fits perfectly in the upper left corner so I counted the distance between the edges

was her response during the interview.
Nikos also split the original triangle into three parts (tech.B) (AEDB, BDW, EWX), two of which were triangles (Figure 5):

N.4.17: I could cut the part of the triangle that is out of the rectangle and put it inside.

N.4.39: I will create the BDW triangle and then I will cut also the other one above it.

He felt that he needed to estimate their area arithmetically and he turned to technique D; he used the dots trying to estimate the heights and bases of both triangles based on his past experience in order to apply known formulae. Only when he encountered difficulties in doing this, he changed his approach; by cut and paste (tech. E) he transferred these two parts into their new position.

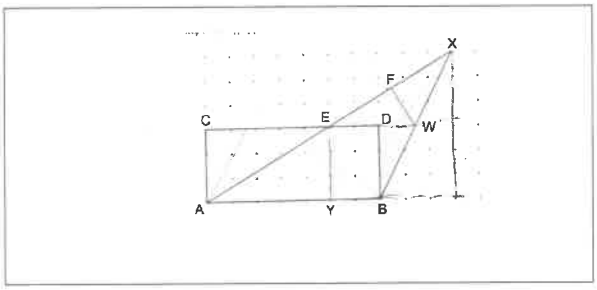


Figure 5. Nikos's partition of the triangle.

We saw that some times students' first reaction was to focus on the surface features of the task as happens with the dots of the array provided in the second task. This is in accordance with English (1997) who ascertained similar behavior from even younger students. On the other hand, the students' final achievements in this study show that they can often work displaying deep understanding of structural aspects, as evidenced in the following items.

How Does a Student Act on the Given Task Environment to Allow an Application?

After the students decided about the technique needed for solving the problems they had in some cases to act on the task's environment in order to apply this technique. In Task 2 for example, they decided to split the part of the triangle that was left over out of the rectangle. However, they had to choose a specific way for splitting it. Both of them preferred to draw the DW line segment (c.f. Figure 5). The criterion was the right angle that would be created in one of the two triangles.

For Katerina:

K.4.18: I will transfer the a-part so as to fit with the right angle in the d-part.

For Nikos:

N.4.114: I will cut the DBW triangle and I will adjust it to the ACE angle.

Additionally both of them drew the line segment in the 'd' part to signify the result of the first transference (see Figures 4 and 5).

In other cases the students did not merely 'act' on the task environment in a direct way but resorted to making representations that allowed them to realize actions in this milieu, which are then transferred to the original situation. In particular, Nikos in Task 3 constructed symbolic and pictorial representations of the task's environment which suggested the inclusion of C in B and the inclusion of A in C (see Figure 6). (This is what Simon and Ericsson (1984) call an 'isomorphism').

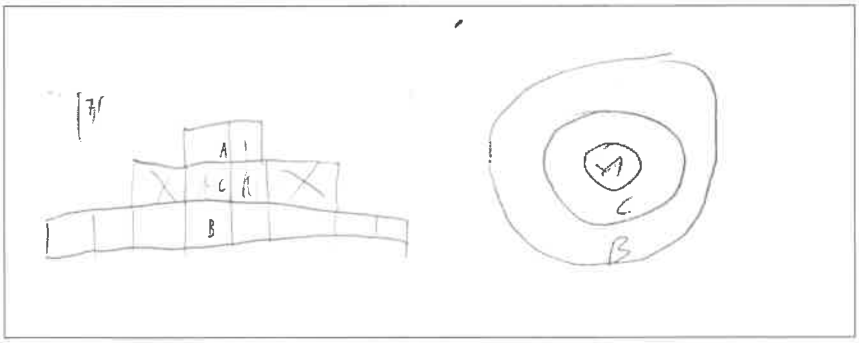


Figure 6. Nikos's isomorphisms.

In this new task's environment Nikos represented A, B and C with concentric circles, with bars on his paper, and finally he materialized a third representation (physically constructed this time) using a ruler (B), a clip (A) and a rubber (C). In this latter environment Nikos's action of removing the A rectangle ('clip') facilitated him to apply efficiently the technique B of decomposition:

N.5.129: It means now that firstly I have to take the clip off (*that is the 'A' shape)*

N.5.130: What I found is (B-A)/2. So now I must add the clip to take the C

N.5.131: Therefore the expression will be (B-A)/2+ A

Student's Ability to Adapt Known Techniques

In their effort to apply known techniques for determining the area of more irregular plane figures, students sometimes adapted these techniques so as to facilitate solution approaches. As a first example, we mention the use of technique-B in the first task. Students were aware of calculating the area of an irregular shape adding the areas of the sub-shapes that constitute it. However in this task we did not have a direct application of this. They chose to find the area of one of the initial shape's parts. So they adapted the technique on the principle that the wanted equals the whole less the remainder:

K.3.1: I will find (*the area of*) the whole square.

K.3.3: Then I will estimate separately (*the area of*) each one of the four white squares

K.3.4: I will multiply the one of them times four.

K.3.5: Then I will subtract this from the initial big square.

In the second task students decided to apply repeatedly the cut-and paste technique in order to transform the triangle into the rectangle (as was suggested in the task statement). According to their past experience in applying this technique, a single application meant that they had to transfer one piece of the shape and put it in another place so as to share one side with the initial shape. However, in order to transfer the first region (the right triangle), the students did not follow exactly the cut and paste protocol. The shape was put into the frame of the rectangle, but <u>not</u> such that one side was shared with the figure that it had been cut from. So they adapted the previous methodology in a way that made it more flexible.

The Effective (or Otherwise) Interaction Between Different Techniques

Some of the techniques associated with the problem solving ancillary to the concept of area stressed numerical measurement, others actions preserving area. This gave students the possibility for interplay between geometric and arithmetical approaches during their attempts to a problem relevant to area. The context of the tasks, which was mainly geometric, influenced how the students started. Because of this they preferred initially to involve techniques that were mainly geometric in character. However, whenever they got 'stuck' they tended to turn to another technique (among those available) usually based on an arithmetical approach. Doing this allowed them to work with geometric and arithmetical aspects in tandem.

In the second task Nikos, working geometrically, constructed the BDW and EWX triangles with the technique of subdividing a shape into sub-shapes in his mind. He conjectured that these two triangles taken together had the same area with a third one (AEC). He tried to verify it but his inability to continue working in the geometric environment made him to turn towards technique-D (length measuring tools for calculating area) that gives an emphasis to numbers. So he measured approximate bases and heights and applied the known formula for the calculation of the area of a triangle. This instance is of particular interest because this interplay did not work so effectively. The two outcomes were not consistent. Nikos accredited this to inaccurate measurements and repeated the arithmetical approach that again did not confirm his conjecture. Thus, he came back to the geometrical approach with the cut and paste technique:

N.4.114: I will cut the DBW triangle and I will adjust it to the ACE angle.

N.4.115: Then, I will cut the EWX triangle and I will put it so as the side EX to be adjacent to the AE side and the W vertex to look towards the C point

In the interview later, as an act of post-reflection, Nikos in his own words related the conflict that he saw in using numerical aspects and the aspect of retaining regions that can be transformed:

N.4.122: I think I delayed reaching the solution because I dealt from the very beginning with the *formulas (tech. D)* and did not consider it as a single shape.

N.4.123: I did not try to find the relationship between the shape I was asked to make through the *transformation (tech. E)* and the one that already existed.

N.4.124: And this is why I started purposeless *measurements (tech. D)* which eventually did not help me in any way.

In the third task both students made a geometrical start. From the very beginning they realized that the C part is constituted from sub-shapes (tech. B) and that the right triangles in the interior of the quadrilateral C were half of the rectangles containing them, using the general property that for any rectangle, a diagonal divides it into two triangles of equal area. But neither managed to continue geometrically; their next step was to calculate arithmetically the area of these triangles (i.e., turned towards tech. D). This was done in accord to what they had achieved before.

Generally both students persisted in their confidence in technique D (length measuring tools) even though in the end it did not offer them a solution path. It took considerable time for them to move away from this line of thought. Katerina insisted on using numbers to find the expression asked from the problem until a shift of attention, (Mason, 1989) prompted her to consider the regions A, B and C forgetting for a while the numbers. Similarly, Nikos at a certain point realized that the intense usage of technique D was maybe a factor that could hamper him to solve the problem:

N.5.84: I need an alternative way to work. I do not want to be confused because of numbers

This 'demand' for something else except numbers made him to 'see' the shapes apart from their arithmetic values. This changed Nikos's perspective because then he started to explore the situation using the previously mentioned representations (see Figure 6).

The Laying Down of the Foundations of New Techniques

According to their previous experience from the research project the students already were in a position to divide a complex shape into sub-shapes in order to find its area (technique B). It is indicative of their awareness that "*...if two figures P and Q do not overlap, then a(P∪Q) = a(P) + a(Q), with a(P) calling the area of the figure P...*" (Hartshorne, 2000, p.205). However, if the situation is that we want to calculate the area of a figure that is completely contained in another whose area is known, in some cases it is more convenient to calculate indirectly by considering the area 'between' the two. As related before, both students utilized with ease this circumstance in the first part of Task 1. In Task 3, the same tactic was involved, but in a far more complex context; both students struggled with this task but eventually attained the 'correct' answer. One student in particular had recourse to mentally remove a sub-region (A) and consider the area of the rest of the regions, before replacing A. Now although it is not a completely new technique this tactic is certainly a new perspective. The students do not follow the protocol of technique B. They do not divide a shape in order to find its area and they do not simply have to think in terms of addition of partial areas. The new idea is that now the students subtract: the wanted area equals the whole minus the remainder (see above extracts of the protocols K.3.1-K.3.5).

In Task 2, the students are asked to apply two cut and paste actions to obtain one region from another. The students' behavior shows some deviation from this direction. As previously noted, the students fit the two pieces of the first region into the 'frame' of the second but not respecting the idea of pasting. Further, one student in particular shows some evidence of evolution from thinking in terms of explicit transformations of sub-regions to comparing the areas of conjectured congruent parts of decompositions of the two regions (that does not require the image of 'movement'):

N.4.58: I must divide the part of the triangle that is outside of the rectangle into two parts that is in the triangles DBW and EWX and then I must paste both of them inside the rectangle

N.4.81: Reasonably the EWX and DWB triangles must have together area equal with the area of the ACE triangle

N.4.82: So, I have to calculate the area of these shapes so as to be able to compare the area of the two triangles with the third one.

These points of behavior suggest that the students' image in the end is approaching the more general notion of dissection, as described in Hartshorne, 2000:

> Given two figures, to find, if possible, an efficient decomposition of the first as a nonoverlapping union of smaller figures, not necessarily triangles, that can be reassembled into the second. A dissection exists if and only if the two figures are equidecomposable. In this case we also say that one figure can be dissected into the other, or that they are equivalent by dissection. (p.213)

The Double Role of the Techniques. The Students Apply the Known Techniques Not Only to Calculate / Estimate Areas but to Verify their Solutions Also

Testing and verifying are heuristics whose importance is attested by leading researchers on problem solving, such as Polya and Schoenfeld. Here we briefly illustrate how students used the methods that were available in their 'tool bag' not only for achieving the solution of a problem but also for purposes of checking by considering alternative solutions, or obtaining supportive estimates.

In Task 1 Nikos made an initial calculation of the shaded area based on the usage of formulae and on technique B of decomposition of the big square to sub-shapes. The fact that he had a feeling of uncertainty due to the novel nature of the task made him seek an alternative way to estimate the area and he resorted to an arithmetical framework because he felt comfortable with numbers. Thus he applied the technique-A (usage of the grid) to verify his previous result.

For Task 1, Nikos stated:

N.3.17: I am not quite sure whether the way I thought for the calculation of the shaded area was a correct one. This is why I will count the squares (*of the grid*) that are inside the cross (*shaded area*)

N.3.18: And I will compare this outcome with my previous result

N.3.19: So, 1,2,3...., 54 + 18 =72, 72 + 9 = 81 (*he counts squares in the grid*)

In the second task both Katerina and Nikos solved the problem by the cut and paste method (technique E). But before the transference of the 'a' and 'b' parts to their new position was effected, Katerina made an effort to verify arithmetically that the two triangles 'a' and 'b' have together the same area as the 'd' part. She tried to estimate the area of each sub-shape based on the square units that could be formed from the array of dots provided (technique A). The same happened with Nikos but instead of square units he used the dots as a tool for length measurement for estimating sides or bases and heights (trying to calculate the area of the triangles BDW, EWX, ACE), hence invoking technique D. This action in itself is an important act of control; the matching of the pieces was not so transparent that it could be left not argued.

DISCUSSION

Ancillary problem solving is focused on a set of techniques initially introduced to help students to appreciate some properties of a concept (in our case area of plane figures). Finally the apprehension of the concept becomes stable and the usage of the techniques ends up solely as a problem solving activity. This actually occurred to some degree in the first stage of our research project, where the techniques were mostly taken separately. This paper deals with the second stage, a year later, involving two students who attended the first. The aim was to examine how they were able to combine and extend the techniques. In the eventuality, the students provided a rich output in this regard. Below is a list that covers some of the most important observations.

(1) Both students take the same approach for the first part of task 1. The researchers expected a direct method by simply counting the unit squares comprising the wanted figure (technique A). However, the students argued this way: the area of the wanted figure equals the area of the 'big' square minus 4 times the area of the 'smaller' squares. This approach is indirect in the sense that the asked area is completely determined by the areas of other figures that lie outside it. It is a combination of a decomposition (technique B) and a 'principle' (picked up probably from other mathematical themes met) that states 'the wanted equals the whole minus the remainder'. The latter depends on the idea of decomposition because it provides a context for which the notion of the 'remainder' becomes meaningful.

(2) The second part of Task 1 forced the students to encounter intersections of plane figures, as another way to solve the original problem. One student obtained a full explanation, the other specified the central square of 3x3 units as the crucial feature to consider but was unable to sort out in her mind how to deal with it. In the task statement a putative student suggests that the total area of the cards minus the area of the 'big' square should provide the area of the shaded 'cross' figure. This argument is flawed because thinking only about the intersection of the cards ignores the fact that the middle square is covered by four cards rather than two. This means that the 'real' students had to make an adjustment, i.e., the right answer is $4 * 9 * 9 - 15 * 15 - 2 *$ (area of middle square). The consequent switch between a figure formed by overlapping regions and taking account how many times each part been covered is quite subtle for the students of the age we are dealing with. Using intersections to calculate areas could be regarded as an extension to the technique B of decomposition, but the arithmetic modeling is far more complex compared to a simple decomposition. This is suggested by the student who failed to correct the number, though was able to identify the 'middle' square as being the crux.

(3) For Task 2, both students started with approaches depending on the array available and only later followed the cut and paste as suggested in the task formulation. Probably, the existence of the array influenced them in doing this, but it must be said that their initial approaches were not well thought through. The students were eloquent about why their approaches were not working, but less so about why they had first taken

them. Also their first attempts did not seem to have much linkage to their eventual approaches. (For example, after some working one student resorted to the task statement that indicates that the solution should be approached by the cut and paste method, but the resultant switch is made in a rather artificial way.)

(4) Both students finally solved Task 2 roughly in the same way. The students were accustomed with tasks concerning the cut and paste technique multiple times, but never (as in this task) where the cuts were dependent on the previous ones. The difference here is that in one case one simply recognizes instances where the same shape juts out and cuts in the region at separate places, whereas in the second case a construction (i.e., the line segment DW shown in Figure 5) has to be realized to make available the second part to be transferred into the rectangle. In this way the triangle AXB is divided into two pieces; the strategy involves both decomposition (technique B) and the cut and paste (technique E) applied two times. This (new) composition of two techniques seemed to be quite natural to both of the students for this particular task.

(5) Task 2 does depend on the array; for example a way to approach the task (without the cut and paste technique) is to note that both the triangle and the rectangle have the same base, whilst the triangle has twice the height. The students knew how to calculate the areas of the two and so they could see that they are equal. Even though there was a direction in the wording of the tasks to apply the cut-and-paste technique, both students started by disregarding this direction. So, in that sense they were free to use the above mentioned approach based on the comparison of bases and heights. However, they did not do this. Perhaps when one is asked to compare the areas of two regions, it is more natural to show their areas equal by thinking in terms of exchanging sub-regions between the two figures to form one from the other. This conjecture would suggest cognitively that given two plane regions, how they are 'shaped' vis-à-vis to each other is more meaningful than comparing two numbers that give their areas separately. However, the equality of area of two figures is not necessarily expressed through space transfers of parts of figures; one student showed evidence that he eventually did not transfer but associated a piece in the triangle with a congruent one in the rectangle. This is analogous to a broader perspective beyond the cut and paste that was explained in the previous section, i.e., dissection.

(6) The third task was somehow unusual for the students in that it asks for a relation between the areas of several figures, where explicit numbers do not appear (instead the letters A, B, C are used). The diagram given in the task's presentation contains a grid that could encourage the students to regard the squares in the grid as units of area in order to approach the problem. Both students started on this basis, but eventually they appreciated that the task is more general than the specific diagram suggests, and they ceased to use grid squares as area units. Another aspect of the usage of the notation of the letters A, B, C is their dual roles as the figures themselves and representing area. Both students showed cases of this kind of abuse of notation (as, it must be admitted, occurs in the statement of the task and indeed within the text of this paper). However, there is much abuse of symbolism even by professional mathematicians that is freely acknowledged. 'Loose' symbolism can be very helpful to aid mental associations, but it

is so only if its different meanings are closely controlled. In the case of APS concerning a measure (area) and its sponsoring object (figure), there is little danger of confusion if both are symbolized in the same way. This is evident in how the students vocalized their solutions for the third task.

(7) The diagram provided in Task 3 at first seems to suggest a simple decomposition of the rectangle B. But the task involves a complication in how this decomposition is to be handled. Both students notice the four rectangles with diagonals already provided. They access their previous knowledge that says given a triangle determined by a rectangle and one of its diagonals, that triangle's area is half the area of the rectangle. Hence the total area of the four triangles inside C is half of the total area of the rectangles on the sides of rectangle B. But this argument does not seem to account for the central rectangle A in the quadrilateral C. Both students spent a considerable time trying to 'fit in' A in their approaches, whilst keeping to the original idea. Both obtained satisfactory solutions, but different in character. One student 'doubled' C in order to obtain B, but whilst doing this, A was taken twice; hence it had to be taken away once resulting in the formula B= 2*C-A, which can be re-expressed as C = (B + A) / 2. This method is impressive in two ways. First we see here the facility of switching between area and region (as mentioned before). Indeed, the idea of 'doubling' does not have any real meaning except if the student is simultaneously drawing on areas (just numbers) and plane regions in a mental combination. The fact that the student took some time in formulating this strategy does not detract from the close association she attributes between area and region. It is a case of having to explore the task environment until the plan became apparent. In the analysis of the data, there are several stages identified in this process of exploration. This reflects the basic character of A.P.S. because the conceptual aspect of area being a measure of a region is always comprehended, and it is only in the context of problem solving that these stages can be materialized. Second, and related to the first point, the student makes an action on the wanted quadrilateral C. This creates another principal object B to center the argument, but the functioning of the action must be held in mind at the same time. The most obvious indication of this is that the student first obtains an equation for B in terms of A and C instead of what was directly asked for, C in terms of A and B.

(8) Perhaps the most natural approach to solve Task 3 is to notice that the sub-region (call it D) of B comprising all triangles inside B (Figure 3) is complementary to A vis-à-vis B, so has area B-A. At the same time, of the eight equivalent triangles forming D, only four lie inside C, so the part of C comprising these 4 triangles has area (B-A)/2. What remains apart in C is A, so C = (B+A)/2. The second student eventually obtains a correct answer that parallels this argument. However, there are interesting cognitive aspects in how he actually achieves this in variance of the strategy stated above. The main point is why should he identify the region D at all? After all, he has only processed the fact that the total area of the 4 rectangles lying on the sides of B is twice the total area of the triangles in C. A numeral association has been made based on separate rectangles, but there is no evidence that the student registered this in terms of the overall region (i.e., D). Because D was not registered as a region where the same

mathematical behavior occurs, the student had to recapture D in another way. How he does this is to make several representations that seem to highlight the fact that A was included in C and C was in B (beyond this the representations no longer respect the more detailed structure of the task environment). Probably what influenced him in featuring A, B, C in these representations is the fact that the task demands a relation between their areas. Eventually he proposed a representation that explicitly involves three physical objects; this situation encouraged him simply to remove the object representing the region A, and later to replace it (to preserve area) once the remaining structure is evaluated. But this structure dealing with the triangles has already been considered; the real significance in the act of 'removing' A is that he identifies the region D. From this point, the student solved the task almost immediately.

So, one approach is to identify the region D first, and realize its relation with region A second, whilst the student first isolates A and then (after some time) realizes D as its complement in B. Both lines of thought challenge the students' previous usage of decomposition (technique B). The student's experience was to 'divide' a given region into shapes whose area could be calculated separately and then added. They were not familiar with transforming a given decomposition (in our case, with 9 'pieces') into another by merging several adjoining pieces to obtain fewer (i.e., two; D and A). Effectively, the student was able to reach the same reduced decomposition by considering D as the complement of A in B. But this was achieved through a representation which encourages the idea of a removing and replacing of A; this mental operation differs cognitively to the notion of decomposition. This example shows nicely that the very basis of a technique (as found in A.P.S.) could be altered to obtain a new one, and how new cognitive perspectives, not quite aligned to how the technique was first conceived, can help in achieving the change.

THE CONCLUSIONS

This paper was mostly concerned about what we called 'problem solving ancillary to the concept of area of plane figures'. What first motivated us was the observation that certain techniques introduced to a primary - level class with the purpose of enhancing the concept of area (and for us to research the effect) soon did not serve to that end; the relation of the concept and the techniques became so secure that the tasks given reduced to the handling of the techniques themselves. Once this point was reached, we deemed that it was in place to design tasks that prompted and stressed the students' command of their usage of the techniques. From our own study, the effect of doing this led to a broad range of student activity. For instance, we observed cases of students altering the task environment in some way to allow an application, adapting the technique to allow a solution of a particular task, generating a new technique from an old one, combining two known techniques in tandem. Of course, by involving only two students (and relatively gifted ones too), we can only regard the content of this paper as a case study. Thus we cannot say much about how a larger population of 13-year-olds would cope with the tasks. Nonetheless we contend that the two students, who were generally successful in obtaining correct answers but usually had to overcome some difficulties on the route, exposed modes of thinking that were essential to achieve these tasks and others of the same ilk.

We summarize the results as reflected in the previous section. From the first stage of the project, five techniques were introduced to the students. These are: usage of square grids; subdivision of a shape into sub-shapes (decomposition); subdivision of an area unit; usage of length measuring tools; cut -and-paste method. There are inherent and rather obvious associations between them, and indeed, the students realized these at the first level of the study. What we report here is more intricate problem solving based on the techniques, but the basic associations will still play a potent role.

The notion of decomposition in fact dominates the results. The primary image is that the area of the whole equals the total area of its 'parts'. Both students though, seemed at ease in applying the principle 'the wanted = the whole - the remainder' in the context of calculating areas via decomposition. Somewhat related, but much more difficult to process, is a case where a critical region in problem solving terms is identified by considering its complement (Task 3). It was effected by one student by imagining physical movements of regions. This has some similarity to the cut and paste method. In Task 2, an application of a multiple cut-and-paste required one figure to be decomposed. Further, one student indicated that he was not explicitly 'moving pieces' in the task; rather, he was forming a bijection between the decomposed parts on both figures involved such that the bijection preserves congruence. This suggested a more abstract method called dissection. In another direction, the notion of decomposition is that of a set of regions that do not intersect and fills a 'larger' region. Suppose the case where the regions do intersect. One student coped well in adapting to this changed situation (forced by the design of Task 1). The identification of a simpler structure within one more complex is in general of great importance in problem solving. For this reason, it is interesting that both students did not directly isolate a particular 'sub-decomposition' in Task 3 that would facilitate the solution. On the other hand, on the same task, one student made an impressive mental action on a decomposition of one region to obtain one for another region (where areas in fact are not preserved); this puts the first decomposition cognitively on the status of an object that can be handled as a whole.

So, from the previous paragraph, we see that from just two students, a single technique that originally was 'read' from one simple angle, was branched out in many ways.

The students involved were only one year older than when they participated in the first stage of the project, so their sophistication and creative output appeared to develop swiftly over the interim. The design of the tasks was made to prompt the students to think in certain ways and consider certain issues concerning area determination for which they did not have previous experience, but many of the approaches the students produced were not anticipated by the researchers. A main factor in the richness of the output of the students lies in that they could always choose to think in terms of regions or the corresponding numbers representing their areas. Despite the intimacy between the two, sometimes there seemed to be an advantage in introducing areas early, other times to argue first in terms of maneuvering regions and convert to areas later. This dialogue between geometric and algebraic thinking in this context is quite intricate, and we plan to analyze it further in another paper.

Our results reported above tend to focus on the final outcome of the students' solving attempts as analyzed in the section titled 'Discussion'. Of course there were stages and aborted starts, which were more fully described in the previous section titled 'Detailed results'. The latter revealed a wider usage of all the five techniques, in contrast with the stress put only on decomposition as above. The technique (A) of employing square grids or square arrays in particular was widespread; the actual counting of grid squares was often evoked, but later abandoned as this method led only to approximations rather than exact results. A more practical use was to determine lengths of line segments, but again problems were caused when the line segments were not horizontal or vertical.

The preliminary stages of working, taken with the final resolution, reflect also 'role setting' states in problem solving, such as accessing an appropriate heuristic, forming conjectures, verification processes and assessing, exploring and executive control. It would be interesting to relate the special perspective A.P.S. with these more general aspects of problem solving, for which there is abundant existing literature. (In particular, we noted in this study the need that the students felt to verify their solutions.) In the trust that this avenue will attract future research, we included a short section earlier in this paper to give a sketch of this literature, with a bias towards research dealing with primary and middle schoolers.

This work addresses technique elaboration associated to a single concept, i.e., area. For this we introduced a framework, called ancillary problem solving, suggesting that we can address other concepts in the same sort of spirit. We envisage that the strength of the framework lies mainly in promoting technique adaptations that are explicitly associated to the concept in hand. Having said this, there are grounds to study students' behavior concerning A.P.S. from a more general scope. Indeed within this paper, the titles of the items laid down in the section describing the results might serve well as a good start.

In conclusion, A.P.S. brings out a more specific and consistent platform to problem-solving activities than is usually adopted in the literature. In particular, the coherence of the techniques in A.P.S. due to the common reference to the associated concept constitutes a platform for a thoroughly tight knowledge basis. This gives students an unusual support to their reasoning, and this study indicates that at least some of them can take advantage of this to develop quite impressive strategy -making. Further A.P.S. creates a bridge that crosses two educational traditions concerning conceptualization and problem solving that typically are treated as separate agendas. The formation of frameworks that transverse different educational fields is essential for the sake of the integrity of the whole discipline. It is a strength of A.P.S. in how it forges such a link, without compromising the need of detailed examination of students' behavior towards particular mathematical topics or issues.

We believe that the aspect of problem solving raised here deserves further attention from educators. There are several avenues for future research to take, but here we only mention one. Much exposition of problem solving can seem erratic if seen from a mathematical theoretical viewpoint. A.P.S. retains a constant conceptual basis, even if it stays at the background, and so some theoretical elements are respected. However, in theories concepts lead to others. In our study, the students showed that their thinking was approaching the concept of dissection. The context of A.P.S. was instrumental in bringing this up. On the other hand, the same context might well not be the most suitable for actually developing an explicit expression of the new concept. Further research could resolve this issue.

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Dimensional Analogy and Different Coordinate Systems, an Onto-semiotic Approach

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ABSTRACT: *Dimensional analogy as a technique and different coordinate systems, apart from their intrinsic mathematical interest, are used in many types of applications in the sciences, engineering and art. As part of the process of the construction of an epistemic network for the subject, the identification of objects, and their dualities, that emerge from this mathematical activity was carried out. The transformation of expressions to content through semiotic functions, and the identification of chains of signifiers and meanings, can be accomplished because of the rich layering and complexity of these mathematical concepts. Questions related to the geometrical aspects of multidimensionality and different coordinate systems, as well as to algebraic methods of transformation were presented to multivariate calculus students, with a questionnaire and formal protocol. Written work and group interviews were analyzed. It was found that dimensional analogy is used by students to create understanding, even when training in specific techniques has not been formalized.*

Key words: *Multivariable calculus, Dimensional analogy, Coordinate systems, Onto-semiotic approach, Semiotic functions, Epistemic network.*

INTRODUCTION

The present work is an attempt to relate dimensional analogy as a technique, to the study of multivariate functions and change of coordinate systems, which is usually initiated in a multivariate calculus context. There are few antecedents in terms of research on this topic. Thomas Banchoff (1996) has developed computer aided visualization techniques to enhance the understanding of multidimensionality. His pioneering work includes techniques for visualizing the fourth dimension, and higher dimensionality is an important part of his studies. Hillel (2000) analyzes the problem of change of basis in Linear Algebra, which is related to the notion of change of coordinate systems. Moore-Russo and colleagues study the concept of *dimensional reasoning*, which involves a person's ability to fluidly move between working in one, two and three dimensions. Dimensional reasoning can be geometric or algebraic, and can be used to understand how the change between dimensions impacts measurement, and how this relates to the number of coordinates of points, the possible variables in an equation, etc. (Moore-Russo, Bateman, S. & Chiu, 2009).

The following section develops and illustrates the concept of dimensional analogy, and presents the type of issue that is addressed in this study.

DIMENSIONAL ANALOGY AND DIFFERENT COORDINATE SYSTEMS

In the familiar 3-dimensional space that we live in, there are three pairs of cardinal directions: up/down (altitude), north/south (latitude), and east/west (longitude), usually labelled x, y, and z respectively. The possibility of spaces with dimensions higher than 3 was presented by Bernhard Riemann in his 1854 habilitation dissertation *Über die Hypothesen welche der Geometrie zu Grunde liegen* (On the Hypotheses that Underlie the Foundation of Geometry) at the University of Göttingen. This work was a landmark in the formal academic activity of geometry in higher dimensions, including, in particular, four dimensions. A space of four spatial dimensions has an additional pair of cardinal directions, usually labelled w.

Often, when a reference is made to the fourth dimension, it is the temporal interpretation which is meant. In this case, the four coordinates are understood to represent 3 dimensions of space plus 1 dimension of time. This 4D-space is used in Einsteins' theories (special and general relativity), modelled by the Minkowski space and notations (Walter, 1999).

To understand the nature of four-dimensional space, a device called *dimensional analogy* is commonly employed. Dimensional analogy is the study of how $(n-1)$ dimensions relate to n dimensions, and then inferring how n dimensions would relate to $(n+1)$ dimensions. Emphasis is made on the fact that dimensional analogy, as used in this study, is a mathematical concept, and should not be confused with analogy as a cognitive mechanism or as a linguistic construct.

The relationship between spatial dimension as a geometric concept, and the algebraic representation of dimension as lists of coordinates, has been recorded in textbooks as well as books and articles of general mathematical circulation, both pure and applied (Rucker, 1977; Banchoff, 1996; Pocock & Rosebush, 2002; Tucker, 2004; Doren & Lasenby, 2007). The abstract relations are present from the beginning levels of algebra, but the present study privileges the multivariate calculus course where, as expressed in Montiel, Wilhelmi, Vidakovic and Elstak (2009):

> It is in the multivariate calculus course where students, many for the first time, are expected to deal with space on a geometric and algebraic level after years of single variable functions and the Cartesian plane. They must define multivariable and vector functions, deal with hyperspace,... find that certain geometrical axioms for the plane do not hold over (lines cannot only

> intersect or be parallel, they can also be skew), and work with functions in different coordinate systems.

As can be appreciated in an interchange on the Unidata website, listed in the references, the need to reach common understandings, conventions and representations in applications ranging from earth system science, flow problems in physics and engineering (Brykina, Rusakov, & Shcherbak, 1991), robotics (Weber, Elshow, Triesch, & Wermter, 2007), to computer graphics (Solomon, 2004) is a 'hot topic'. Methods for solving three-dimensional flow problems with the aid of two dimensional solutions require the analytical and geometrical notions that are, in most educational systems, first introduced in a multivariate calculus course. For example, it is necessary to understand the concept of dimension and use dimensional analogy to understand Green's theorem, part of the standard content of the multivariate calculus course. Research on the epistemology and didactics in general of multivariate calculus is virtually non-existent but it is in this subject that, for the first time, students must learn operations that are dimension-specific (such as the cross product) and make generalizations in terms of dimensions and their algebraic and geometric representations, which require flexible mathematical thinking.

The issue of transiting between different coordinate systems, as well as the notion of dimension in its algebraic and geometric representations, are significant within undergraduate mathematics. Deep demands are made in both conceptual and application fields with respect to understanding and competence.

> The move into more advanced algebra (such as vectors in three and higher dimensions) involves such things as the vector product which violates the commutative law of multiplication, or the idea of four or more dimensions, which overstretches and even severs the visual link between equations and imaginable geometry (Tall, 1995).

As an example, when dealing with functions in rectangular coordinates that can be graphed as surfaces in R^3, the vertical line test still applies. In a similar fashion, when dealing with functions whose domain is some subset of the polar plane and range some subset of R ($f(r,\theta) = z$), the vertical line test also applies. The polar plane (or some subset of it) now forms the domain, and r is not necessarily a function of θ, but z does depend on the multivariable domain and, geometrically, there is only one value for each pair. On top of all these new elements that are usually introduced in a one semester course, this last example is presented in the context of *cylindrical coordinates*. The cylindrical coordinates refer to a three dimensional domain and a function whose graph can only be imagined in hyperspace. However, in some of the standard exercises that students must confront, they are asked to find the three dimensional volume of that domain when given a triple integral, with an 'r' inside that does not represent the function $f(r,\theta,z)$ but is the determinant of the Jacobian matrix. The Jacobian itself (the determinant) is not usually introduced until after the specific cases of polar, cylindrical and spherical coordinates are studied.

It is common to think of dimensional analogy as a method which permits the visualization of the fourth dimension (Weeks, 1985; Banchoff, 1996). To those interested in understanding the nature of multiple spatial dimensions, by taking a step back to the second dimension and trying to understand certain physical aspects (like the classical Flatland[1]), and then looking at the third dimension, an understanding of what the fourth spatial dimension would mean can be developed. For example, there are techniques that use generalizations, such as looking at the boundaries of one, two and three dimensional objects. On the other hand, there exist two ways of looking at dimension, that is, *extrinsically* or *intrinsically*. Briefly, the "extrinsic" point of view considers curves and surfaces, in particular lines and planes, as lying in a Euclidean space of higher dimension (for example a plane in an ambient space of three dimensions). According to the "intrinsic" point of view one cannot speak of moving 'outside' the geometric object because it is considered as self-contained. The inhabitants of Flatland consider their plane from an intrinsic point of view. For them, there is no space, no third-dimension.

The mathematical activity with lists of coordinates beyond the triple is usually introduced in the linear algebra context and, although mentioned in the multivariate calculus course, it is primarily the triple that displaces the 'ordered pair' (although, when working with gradients, the ordered pair returns, often in a way that may not be clear to the student). However, in triple integration, as was mentioned, the functions have a three dimensional domain (and one dimensional codomain), which means that the list indicating their geometrical representation would have four components. When dealing with functions whose three dimensional domains are expressed in the cylindrical or spherical coordinate system, it is not indicated, in any of the texts that were consulted, what that fourth component might look like. These are the types of mathematical issues that this study attempts to analyze when referring to dimensional analogy and different coordinate systems.

CONCEPTUAL FRAMEWORK

Mathematical Objects

A mathematical object, in this study, will be considered anything that can be used, suggested or pointed to when doing, communicating or learning mathematics. The onto-semiotic approach (Godino, Batanero & Roa, 2005; Font, Godino & D'Amore, 2007) considers six primary entities which are:

1) *Language* (terms, expressions, notations, graphics);

2) *Situations* (problems, extra or intra-mathematical applications, exercises, etc.);

[1] The original term "Flatland" is used in a science fiction novella by Abbott (1884) that has several new editions; for example, the annotated reference work by Stewart (2001). There are literary works, feature and short films, chapters in TV series, role-playing and video games, etc., related to the Flatland topic. See, for example, http://en.wikipedia.org/wiki/Flatland.

3) *Definitions* or descriptions of mathematical notions (number, point, straight line, mean, function, etc.);

4) *Propositions*, properties or attributes, which usually are given as statements;

5) *Procedures* or subjects' actions when solving mathematical tasks (operations, algorithms, techniques, procedures);

6) *Arguments* used to validate and explain the propositions or to contrast (justify or refute) subjects' actions.

The 'ecology' of the primary entities is not uniquely determined nor objectively established in the cognitive and instructional processes. Mathematical knowledge 'lives' in institutions and within a social context and it 'manifests' itself through concrete practices. In other words, the proposed ontology of mathematical meaning is based on *anthropological* (Chevallard, 1985) and *socio-cultural* (Radford, 1997) principals, as well as *cognitive* aspects (Tall, 1991).

As important as the mathematical objects are: 1) the agents that move them and the meaning (straightforward or not) that is assigned to them; 2) the concrete appearance of these objects and the reference to ideal entities; and 3) their contextual and relational function with other mathematical objects. For these reasons, in the onto-semiotic framework, the following dual dimensions are also considered when analyzing mathematical objects (Godino et al., 2005, p.5):

1) Personal / institutional;

2) Ostensive / non-ostensive;

3) Intensive / Extensive;

4) Unitary / systemic;

5) Expression / content.

These dual dimensions demonstrate how the primary entities must not be understood in an isolated manner, but according to their function and their relation in a contextualized mathematical activity. Furthermore, the primary entities and the dualities offer a 'photographical' way of seeing the didactical systems, that is, they permit the elaboration of models that capture a changing and dynamic reality. In fact, they are indicators for the identification of the basic processes of mathematical activity. For example, generalization is a mathematical process in which an intensive (type) is determined from a class of extensive (examples) that share a common structure or function. On the other hand, particularization is a process by which an intensive, or general case, is deemed pertinent in justifying or solving a concrete case, determining an extensive that explains or solves the problem.

The determination of the mathematical objects involved and their function in the mathematical activity is important, as is the determination of the generic processes inherent in the activity. Based on Godino, Batanero and Font (2009), Font and Rubio

(2008) and Font, Godino and Contreras (2008), we summarize this onto-semiotics of mathematical knowledge in figure 1.

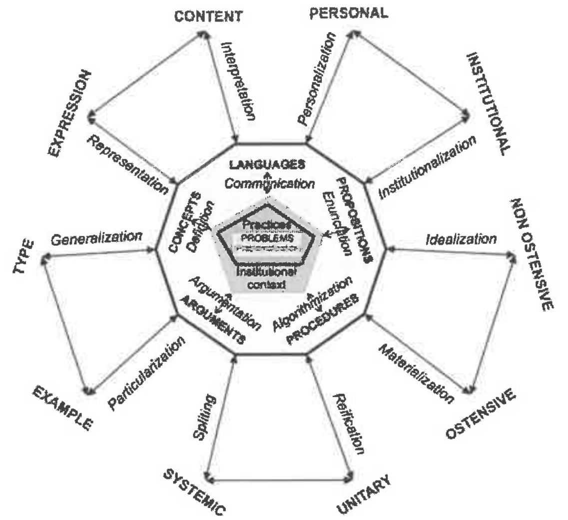


Figure 1. Objects and processes.

The analysis and solutions detonate the use of different concepts, procedures, propositions and previous arguments, and open the possibility that new ones emerge. The activation of these emerging objects is brought about by the processes of definition, of creation of techniques (algorithmic or not), the determination of propositions and argumentation. All these processes are only possible through the use of language in different registers, that is, the use of languages that make the codification and transference of knowledge and meanings of the mathematical objects involved possible. For this reason, the problem-situation is placed in the center of the onto-semiotic analysis (figure 1).

Systems of Practices, Emerging Objects and Epistemic Networks

According to the onto-semiotic approach (Godino & Batanero, 1994; Wilhelmi, Godino & Lacasta, 2007a), it is necessary to determine the meanings (plural) associated with mathematical objects in different contexts and organize them (the meanings) as a complex and coherent whole. The *operative and discursive systems of practices*, and their subsystems, understood as depending on the institutional and personal contents that are associated to a mathematical object, and the objects that emerge within these systems, form epistemic and cognitive networks. This means that if the systems of practices are institutional, the emerging mathematical objects are considered to be institutional objects, and if the systems of practices correspond to an individual, then the objects are personal, according to the duality specified above. Also, following this duality, the objects that emerge can be ostensive (such as symbols and graphs) or non-ostensive, that is, conceptual or mental. The contextualized and functional use of these objects as elemental entities cannot be divorced from their essentially relational nature that, at the end, justifies their adaptation, whether in particular (intensive) or general (extensive) processes.

Because of this, it can be assured that there is a correspondence between the systems of practices and the expression of the mathematical objects (in a contextual and functional way).

Whereas the *meaning* of a mathematical notion represents the structured complex of a system of practices in a context, the *holistic meaning* of a mathematical notion represents the expression of the different (partial) meanings associated with the notion as one system. The holistic meaning comes from the coordination of the partial meanings associated with a mathematical notion (Wilhelmi, Godino & Lacasta, 2007b). *Flexible mathematical thinking* (FMT) is what permits the passage between different partial meanings, and the coordination or partitioning of the different meanings when necessary.

The communication of mathematical knowledge is often based on the premise that knowledge is 'accumulative', and that it is not necessary to establish explicit mechanisms to integrate previous knowledge with the newly obtained. When this occurs, it is supposed that FMT is acquired in an 'automatic' way, just because two partial meanings are known. The mathematical edifice is constructed over axioms (intuitively accepted or formally stated), from which properties are deduced, leading to successive levels of 'expansion' of the mathematical theory. However, often this axiomatic-deductive conception is translated to mathematics teaching, where it is supposed that mathematical knowledge acquisition follows this same dynamic of 'accumulative and linear' expansion. This leads professors to refer, systematically, to the previous levels where the notions were introduced, and the processes and meanings of the mathematical objects, that they 'only are going to use', were developed. In terms of both the knowledge that is being introduced, as well as the previous notions, there is no revision of the field of application, nor of the efficiency of the cost that the use of these procedures and arguments supposes. This way, no mathematical connections between the definitions belonging to the situations are established, and linguistic elements (notations, expressions, graphs) are introduced without distinguishing the new conventions from those that belong to the uses of language previously introduced. The mathematical discourse is centered on the identification and justification of a collection of propositions relatives to the emerging objects, accepting that the semantic relations with the previous objects are transparent, leaving to the student the task of establishing the relevant mathematical connections.

Finally, in order to capture the semiotic complexity inherent in the communication of mathematics, it is important to identify the different objects that make up the mathematical practice in specific content areas and contexts. The resulting network is called a *socio-epistemic configuration*, and it captures the interplay of the objects and relations in a particular mathematical setting. This way, the personal meanings that are constructed when individuals carry out mathematical activity can be described by cognitive configurations. Evaluation of the learning and mathematical behavior of these individuals lies in the analysis of the relation between the socio-epistemic and cognitive configurations.

Semiotic Functions and Representation

From the pioneer work of Janvier (1987) and Douady (1987) to the more current proposals (Goldin, 1998; Duval, 2002) it has been clear (theoretically based and contrasted experimentally) that the notion of representation is central to mathematics education. The onto-semiotic approach places great value on the relation between mathematical objects by means of the *semiotic function* (Godino & Batanero, 1997), as a relation between an expression and a content established by 'someone', according to certain rules of correspondence. Not only language, but the other types of objects such as concepts, situations, actions, properties, or arguments, can be expressions or content of semiotic functions. Font and colleagues (2007) pointed out that to understand representation in terms of semiotic functions has the advantage of not segregating the object from its representation. Indeed, given an object and a representation, in general it is not possible to identify a unique semiotic function between them, and even the representation can constitute the content in another context. For example, Alson (1989, 1991) shows how a Cartesian graph can be given an algebraic structure before the introduction and analytic development of the theory of functions. This fact determines an objectification process of a representation (in its broadest sense) that is prototypical in mathematics.

When talking about semiotic functions, the dependence relations can be either *representational* (one object is put in the place of another), *instrumental* (an object is used as an instrument by another) or *structural* (two or more objects conform a system out of which new objects emerge) (Godino, Batanero & Font, 2007). An example of a representative semiotic function (as opposed to structural or instrumental) could be, for the purposes of this study, a solid presented geometrically as the expression, and the formulation of a double integral 'setup' as the content. An instrumental semiotic function could have as expression the double integral, and as content the numerical answer, while the structural (or 'cooperative') semiotic function could take some region together with a double integral in terms of 'x' and 'y' as the expression, and the set up of a double integral in terms of 'u' and 'v' over a simpler region (using the Jacobian) as the content. It should also be clear that the expression in one semiotic function could be the content in another.

Although there is not sufficient space in this article to pursue the aspect of gesture theory in mathematical semiosis (Goldin-Meadow, Nusbaum, Kelly & Wagner, 2001;

Marrongelle 2007; Sabena, 2008), there will be occasions in which reference will be made to deictic[2] signs in the sense of Font and colleagues (2008):

> An elementary meaning is produced when a semiotic act (interpretation/understanding) relates an expression to a specific content within some specific spatial and temporal circumstances: It is the content that the emitter of an expression refers to, or the content that the receiver interprets. In other words, what one means, or what the other understands. Examples of this use of the word 'meaning' are the deictic signs, rigid designations (Eco, 1979) where the content is indicated by gestures, indications or proper names. The content of the semiotic function is a precise object, which may be determined without ambiguity in the spatial and temporal circumstances fixed (p.7).

RESEARCH QUESTIONS

1) In the transition from single to multivariate calculus, does dimensional analogy emerge as the result of actions and argumentations of students?
2) Is the use of dimensional analogy associated with a particular type of semiotic function?

3) What primary entities and semiotic functions can be identified and classified, as students relate different coordinate reference systems to their previous study of calculus in the 2-dimensional Cartesian coordinate system?

CONTEXT, METHODOLOGY AND INSTRUMENT

The context of the present study is multivariate calculus (calculus III) as the final course of a three course calculus sequence, taught at a large public research university in the southern United States. Seven students were interviewed once in two groups, the first consisting of four students and the second of three. The small group interviews were chosen as the ideal method, as they were small enough to permit each student to explain their written answers and respond to the protocol questions, yet they also provided the possibility of group interaction. The students were chosen on the basis of their performance on exams, in-class and extra-class participation. Three of the students were excellent, three were good and one was average, but all of them had shown a special interest in the subject. Three were mathematics majors, three were chemistry majors and one was a double major, mathematics-economics. The interviews were video-recorded. Each interview was approximately an hour long. The students were first given a questionnaire, which is included in this text, on which they wrote down their responses, and they were then asked to explain them. For each question, the students were chosen in a different order, but it was inevitable that who spoke first would influence, in some

[2] According to the Merriam-Webster online dictionary, http://www.merriam-webster.com/dictionary/, deictic means: showing or pointing out directly <the words *this*, *that*, and *those* have a *deictic* function>

way, the others. They were asked to explain verbally on an individual basis, but group discussion was encouraged when it presented itself. It should be noted that these students participated in the interviews after taking their final exam, so they had completed the course. Two of the authors of this article were present, as interviewers, with each group. They used a common protocol to ask specific 'probing' questions. These protocol questions will be made known below, when the rationale behind the actual interview questions is explained. As final grades for the course had still not been submitted, another of the authors, who was the professor of the course, did not participate in the interviews, so that the students would not feel under any pressure in terms of their grades. The students were assured that their professor would not have access to the video-recordings and their written work until after the final grades had been submitted.

Table 1 contains the questions presented to the students; the expected (institutional) answers, and a sample of some of the actual student answers are found in the appendix.

The first question was included to detect the students' geometrical transition to 3D-space where, in the rectangular context, much emphasis was placed at the beginning of the course on the coordinate planes and the octants. These answers were important to begin to detect the process of dimensional analogy. The interview protocol included the question of why the equations represented planes, and not just points or lines. Although we have been unable to discover any literature on the subject, through informal discussions and comparisons it has been noted that the average student has difficulty with associating the algebraic equation, say, $y = a$, with a plane parallel to the xz-plane, or the actual xz-plane if $a = 0$. The protocol also indicated that, if the sketch was correct, to ask how the angle θ 'turns into' a plane.

The cylindrical or spherical coordinate systems are named as such because of the equations $r = r_0$ and $\rho = \rho_0$ (where r is the radius of a circle on the polar plane, and ρ is the radius of a sphere). However, $r = r_0$ is not a function in the usual set up of cylindrical coordinates ($f(r,\theta) = z$), although it is a function in the set up of polar coordinates, where the independent variable is θ and the dependent variable is r (Montiel, Vidakovic & Kabael, 2008; Montiel et al., 2009). However, $\rho = \rho_0$ is a function within the algebraic representation $f(\theta,\phi) = \rho$ associated with the spherical system (Leathrum, 2002). That is, $r = r_0$ is a relation in the cylindrical system, but not a function, as r is one of the two independent variables in the domain (like $x = x_0$ in single variable functions in rectangular coordinates). On the other hand, $\rho = \rho_0$ is a function, as ρ is the dependent variable. Students are expected to complete exercises about these geometric representations in all the textbooks that were reviewed (Larson, Hostetler & Edwards, 2005; Stewart, 2004; Varberg & Purcell, 2006; Salas, Hille & Etgen, 2007) but simultaneously, in these same sections, they are expected to setup and solve integrals that sometimes will represent volume, and sometimes hypervolume in 4D-space, using the concept of function. The textbook used at the university where the study took place is Salas, Hille and Etgen (2007). It should be mentioned that, as is usual in the calculus textbooks written in the United States, θ represents the *azimuth*, or longitude, and ϕ represents the *zenith*, or colatitude. An important discussion on the issue of conventions in spherical coordinates, and the contradictions found in their presentations between mathematics, on the one hand, and science and engineering on the other, can be found in Dray and Manogue (2002). The two main aspects of this study, dimensional analogy and change of coordinates, could be detected and analyzed by question 1, as will be made clear later on.

Table 1
Questionnaire

Questions
Question 1. In rectangular coordinates the coordinate surfaces: $x = x_0$, $y = y_0$, $z = z_0$ are three planes. *(a)* Draw them. Why are they planes and not just points or lines? *(b)* In cylindrical coordinates, what are the three surfaces described by the equations: $r = r_0$, $\theta = \theta_0$, $z = z_0$? Sketch. *(c)* In spherical coordinates, what are the three surfaces described by the equations: $\rho = \rho_0$, $\theta = \theta_0$, $\phi = \phi_0$ Sketch.
Question 2. What is the name of the following surface expressed as the polar function: $z = f(r, \theta) = r$. Sketch the surface. Find the volume of the solid by triple integration (use cylindrical coordinates) when $0 \le r \le 2$. Does your answer coincide with the formula for the volume of this solid (if you happen to remember)?
Question 3. a) Set up the following integral changing from the rectangular coordinate system to the polar coordinate system: $\int_0^1 \int_0^{\sqrt{1-x^2}} dydx$. Why can we calculate area with this double integral, just like we did with the single integral? Why do you need the "extra" r when you change the coordinate system? How do you calculate this? b) If Ω is the parallelogram bounded by the lines $x + y = 0, x + y = 1, 2x - y = 0, 2x - y = 3$ And we have the double integral $\iint_\Omega (x+y)^2 dxdy$, then we can set $u = x + y$, $v = 2x - y$ And change the region from a parallelogram to a rectangle. (i) Why would we want to do this? (ii) Make a sketch of the two regions. (iii) How would we do this? (iv) Could you calculate the area of the region (parallelogram) without calculus? How could you do this with and without calculus?

The second question asked for the names of the quadric surface, but expressed algebraically as $f(r,\theta) = z$ instead of as the 'formula' that is taught when learning to identify quadric surfaces. Then the students were asked to find the volume of the solid by triple integration which, once again, connects volume to a triple - not a double – integral. The question was raised, as specified in the protocol, as to how the process of finding that volume could be carried out with double integral, using polar coordinates. This was done with the intention of detecting the use of dimensional analogy on the part of the students, in the context of the cylindrical and polar coordinate systems. As the surface is the upper nappe of a cone, it was also asked if the answer by integration coincides with the formula for the volume of that solid. To be able to respond to this question depends, of course, on if the solid was identified correctly, and if the formula for the volume of a cone is remembered.

These first two questions play a methodological role in a long-term research project; that is, they constitute a mutual validation of the results obtained in this phase, and in two previous phases (Montiel et al., 2008, 2009). The groups of students are homogeneous in the three investigations, and the instructional process is comparable (the same institution, the same intended meaning through the same instructor). For this reason, the discussion of the results is based not only in the onto-semiotic analysis (the study of the socio-epistemic configurations), but in the comparison and analysis of the data among these similar groups.

The third question presented a problem in different parts, designed to capture complementary aspects of dimensional analogy and change of coordinate systems. First, the students were asked to set up a double integral in polar coordinates, given the set up in rectangular coordinates. They were asked why the double integral is being used to find area, with the intention of detecting dimensional analogy. They were also asked, with respect to the change of coordinate system, why they need the Jacobian. Then the students were presented with a region in the form of a parallelogram, given by algebraic equations that represent intersecting lines. They were told that, through making a change of coordinate system, they could set up the double integral over a rectangle. First, they were asked why they would want to do this. This answer was the key in detecting if it was clear to the students why the change to different coordinate systems is significant, and if it meant more to them than just carrying out the mechanics of an exercise. Then they were asked to sketch the two regions and carry out the process of integration. Finally they were asked if they could calculate the area of the first region (the parallelogram) without calculus. This last part had a double purpose. On the one hand, it is a way of determining, as in the first part of this question, if it is clear why the methods of calculus can actually economize and 'simplify' tedious geometric calculations. On the other hand, the idea of finding area with a double integral once again comes into play, and it is here where the dimensional analogies can be observed.

ANALYSIS USING THE ONTO-SEMIOTIC APPROACH

While analyzing dimensional analogy and different coordinate systems within the onto-semiotic approach, it is important to remember what was mentioned in the conceptual framework about the classification of the mathematical objects and the primary entities. There are aspects that characterize each of these entities, but by no means can there be a sharp separation between them. The three questions will be analyzed; as there are seven students and two groups, S1, S2, S3 and S4 will represent the participants in the first group, and S5, S6, and S7 the participants in the second interview session. Usually the two sessions will not be differentiated as emphasis will be placed on the questions themselves and the mathematical content. There are also written answers which will be referred to at times. Because of the natural limitations of space in an article such as this, emphasis will be placed on the participations that are directly related to dimensional analogy and the change of coordinate systems.

Each question will be accompanied by a table that represents the socio-epistemic configuration relevant to its content and context. In the lecture sessions, the professor (one of the authors) had a well-defined system of objects and meanings that were meant to be developed within the context of the institutional mathematical practices. As in Font et al. (2008), "the students' *cognitive configuration* shows some agreement with the socio-epistemic configuration, but also some differences" (p.167). The questions can be consulted in Figure 1.

In question 1, the socio-epistemic configuration of primary objects is structured in terms of the given situation (table 1), that is, the transition between algebraic and geometric representations of surfaces immersed in 3D, in different coordinate systems.

Question 1 had a marked geometrical emphasis, as students were asked to sketch what they understood by the algebraic equations representing surfaces immersed in 3D space, in the rectangular, cylindrical and spherical coordinate systems. The instructions were written in a combination of mathematical English (Wells, 2003) and symbols. The results of these instructions were sketches, a graphical language object. As institutional objects these sketches are considered language entities:

-*Verbal*: rectangular, cylindrical, spherical coordinates; dimension (3D);

-*Graphical*: graphs of the surfaces;

-*Symbolic*: $x = x_0$, $y = y_0$, $z = z_0$; $r = r_0$, $\theta = \theta_0$, $z = z_0$; $\rho = \rho_0$, $\theta = \theta_0$, $\phi = \phi_0$

We show in table 2 some of the previous and emergent concepts, procedures, propositions and arguments to Question 1.

Table 2
Previous and Emergent Concepts, Procedures, Propositions and Arguments to Question 1

Objects	Previous	Emergent
Concepts-Definitions	-Cartesian coordinate system in 2D; -Definitions of functions and relations in a single variable; -Graphs of single variable functions and relations;	-Different coordinate systems in 3D; -Graphs of planes; -Definition of functions and relations in a multivariable context; -Graphs of functions and relations in different coordinate systems in 3D; -Extrinsic and intrinsic points of view
Procedures	-Graphing lines and curves in the plane; -Evaluating and setting up single variable functions;	-Graphing planes and surfaces in 3D space; -Evaluating and setting up multivariable functions in different coordinate systems;
Propositions	In rectangular coordinates: From an intrinsical point of view: -In 0D space, $x = x_0$ represents a point; in 1D space $x = x_0$, $y = y_0$ represent lines;	From an intrinsical point of view -In 2D space, $x = x_0$, $y = y_0$, $z = z_0$; $r = r_0$, $\theta = \theta_0$, $z = z_0$; $\rho = \rho_0$, $\theta = \theta_0$, $\phi = \phi_0$ represent surfaces.
	From an extrinsical point of view: -In 1D space, $x = x_0$ represents a point on a line; in 2D space $x = x_0$, $y = y_0$ represent lines on a plane; $r = r_0$, $\theta = \theta_0$ represent curves on the polar plane.	From an extrinsical point of view: -In 3D space, $x = x_0$, $y = y_0$, $z = z_0$; $r = r_0$, $\theta = \theta_0$, $z = z_0$; $\rho = \rho_0$, $\theta = \theta_0$, $\phi = \phi_0$ represent 2D surfaces immersed in 3D.
Arguments	Dimensions as degrees of freedom (intrinsic): a point has 0 degrees of freedom, a line has 1 degree of freedom.	Dimensions as coordinates (extrinsic in this case): a point has 1 coordinate, a line or curve in general has 2 coordinates, a plane or surface in general has 3 coordinates.

Only S1 (see Appendix) showed a complete grasp of the (institutional) meaning in his written work. The other students (see Appendix) were able to translate the equations in rectangular coordinates, but their sketches of the surfaces in the cylindrical and spherical coordinate systems were, to different degrees, inaccurate.

When comparing with the students' sketches from the previous study (Montiel et al, 2009) the representation of $\theta = \theta_0$ seemed to be the least precise, in both a) and b). It was reported in the previous study that they all had mentioned a vertical plane; one of the students admitted that "I just did all this by memorization, I have no good explanation". However, another student had also made the sketches "by the definitions", but was able to give content (meaning) to these expressions by means of a semiotic function in which spoken mathematical English (language) played the role of signified, asserting that "as θ is fixed, r and z can be anything". Another student also was able to give content to his expression, using a *non-mathematical metaphor* (Pimm, 1987), "θ is stuck at one position, like when you cut a pie". None of the students in that study actually remembered that in spherical coordinates, the equation $\phi = \phi_0$ must be broken down to five different cases, with restrictions on ϕ, whereas in the present study all the students remembered that there was more than one case. This could be due to the fact that the professor (one of the authors) emphasized these different cases with the students of the present study more than what had been done with the students of the previous study.

As a reply to the question of why the equations in rectangular coordinates represented planes and not lines, S3 said "For each plane there's only a restriction in one dimension, so that dimension is throughout the whole plane. Each other dimension can be anything, that's how you get just one infinite plane". The deictic signs that accompanied the verbal expression can be used to classify the relations that were established as a representational semiotic function, going from the equations, as the expression, to the sketches and gestures as content (in particular, the communication of the 'infinite plane' was clearly done by spatial gestures). In this same vein, an interchange with the two interviewers (I1 and I2) and S5, when asked why the equations represented planes, will be presented.

S5: I don't think I know how to verbalize why they're planes and not points.

I1: But why do you see $x = x_0$ as parallel to the yz-plane?

I2: For the record, you're looking at the room (*the student points to the edges and corners of the room in which the interview was conducted*).

S5: I'm looking at z, y and x.

I2: You're pointing, what exactly are you pointing at?

S5: The corners, I'm pointing at the corners to get my head around it in space.

On the other hand, S7 used dimensional analogy to explain:

S7: When we have 2 dimensional, say $x = 2$. We fix x at 2 and y could be anything. Now we have 3D, another variable which is z. Instead of say,

$x = 3$ and y going on forever, it would be z going up and down forever as well.

In the context of the cylindrical coordinate system, as was mentioned, only S1 drew all three expected answers. However, S3, once hearing and observing S1's geometrical description and representation of the three surfaces, realized he had misunderstood the question, but "it was asking the same thing as the previous one" (the three surfaces in rectangular coordinates), and offered his interpretation:

S3: $r = r_0$ would be a cylinder of infinite height, $\theta = \theta_0$, would be a slice of the cylinder and $z = z_0$ is an infinite plane, no, not a plane, it's an infinite disk at z_0.

I1: What is the difference between an infinite disk as opposed to a plane?

S3: It's just the coordinate system that you use, it's not rectangular coordinates any more, it's polar coordinates.

I2: But why think of it as a disk? I could think of it as an oval or a rectangular thing.

S3: It has no restriction on θ, it turns into a plane with an angle of 2π, the radius has no restriction, so it goes on and on.

S1: It has to do with how you build the plane in your head. If you take one r value at a time, it's just concentric disks. It's just the shape of the space.

As was established in the conceptual framework, the objects that have emerged, such as language, situations, procedures, definitions, reflect their duality. In particular, the duality ostensive (symbols, graphs, gestures) or non-ostensive, that is, conceptual or mental, can be detected in the chosen fragments related to question 1, through the semiotic functions with their expression and content.

Question 2

In question 2, the socio-epistemic configuration of primary objects is structured in terms of the given situation (table 1), that is, using a triple integral to find the volume of a cone. More precisely, this is a problem of recognizing (i) that a triple integral can be used to find volume; (ii) that the formula (function) represents a component of the domain if it is to be set up as a triple integral; (iii) that the geometric representation of the function, given in rectangular coordinates, is a quadric surface. (iv) how to change the algebraic representation to the cylindrical coordinate system.

In this case, the language considerate entities are:

-*Verbal*: triple integral, volume, cylindrical coordinates; solid (as a domain)

-*Graphical*: graph of the cone

-*Symbolic*: $z = \sqrt{x^2 + y^2}$; $\int_0^{2\pi}\int_0^{2}\int_0^{2-r} rdzdrd\theta$; $\frac{1}{2}\int_0^{2\pi}\int_0^{2}\int_0^{r} rdzdrd\theta$

We show in table 3 some of the previous and emergent concepts, procedures, propositions and arguments to Question 2.

Table 3
Previous and Emergent Concepts, Procedures, Propositions and Arguments to Question 2

Objects	Previous	Emergent
Concepts-Definitions	-Definition of the single integral using Riemann sums -Single integral as representing geometrically the area under a curve as well as the volume of a solid of revolution;	-Definition of double and triple integrals using Riemann sums. -Double and triple integrals and their geometric representations (area, volume, hyperspace). -Change of coordinate system. -A solid as a domain. -Points of view.
Procedures	Setting up and calculating single integrals.	Setting up and calculating double and triple integrals in different coordinate systems.
Propositions	Theorems about properties of single integrals	Theorems about properties of double and triple integrals
Arguments	$\sqrt{x^2 + y^2} = r$ in the cylindrical coordinate system	-The geometric representation is a cone; -A cone is $\frac{1}{3}$ of a cylinder, which points to one possible set up of the integral; -A triple integral can represent the volume of the domain of the function $f(r, \theta, z) = 1$.

In this second question the students were asked to set up a triple integral to find volume. The students had been motivated geometrically to understand double integration as a means of calculating volume and the triple integral, in a spatial sense, as the geometric notion of hypervolume under some hypersurface in 4D-space.

The semiotic function that describes the mathematical activity is compound. First, the expression is a statement in mathematical English with symbolic language embedded in it, and the content is an integral set up which captures the meaning of the expression. Then this content (the triple integral) turns into the expression, whose meaning is a number that represents the volume asked for. This question stimulated dimensional analogy:

S7: To set up the triple integral, usually I understand it more as a double integral, where the function $f(r,\theta)$ would lie inside the double integral, and then I could visualize.

I1: If you were to use a double integral, how would you set it up?

S7: I would put r inside, it would be $\int_0^{2\pi}\int_0^2 r(r)drd\theta$. It's the compensation from the polar coordinates. This is the function, and then we just worry about what's happening on the xy-plane (sic). I understand it much more than to have it bounded above and below by another z-plane. I can't imagine what it would look like with a triple integral.

I1: If you do the double and triple should you get the same answer?

S7: Yes, definitely.

The intensive/extensive and ostensive/non-ostensive dualities are apparent in this interchange, as the student makes his dimensional analogy. The extensive, with its ostensive representation in symbolic language, led to language objects that referred to a type (intensive), in this case the double integral as representing the volume under the surface. However, this analogy, which permits a graphical representation, was not explicit enough to make S7 realize he had to divide the volume of the surface under the cone by $\frac{1}{2}$ to get the actual volume of the cone (see Appendix).

S4 also used dimensional analogy by 'going up' a dimension:

S4: When you have an actual function in double integrals, that is the volume.

I1: Actual function?

S4: With triple integration, if we had a function it would be a hypervolume, but we don't have a function so it is just a volume.

However, as can be seen in the appendix, S4 did not recognize the quadric surface and set up the integral with the radius of the cone as a constant. This reflects that she did not grasp the meaning (institutional), as an extensive, for setting up the triple integral to find volume, even though there is an 'actual' function ($z=\sqrt{x^2+y^2}$).

Question 3

In question 3, the socio-epistemic configuration of primary objects is also structured in terms of the given situation (table 1). This is a problem of changing the coordinate system to find a domain which can be easily represented in the limits of integration.

In this case, the language considerate entities are:

-*Verbal*: double integral, change of coordinates; region; Jacobian matrix; Jacobian (determinant).

-*Graphical*: graphs of the regions (the original and the transformed).

-*Symbolic*:

$$- \ (x(u,v), y(u,v)); \quad \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{matrix} \qquad \text{-Area of } \Omega = \iint_{\Gamma} |J(u,v)|\, dudv$$

We show in table 4 some of the previous and emergent concepts, procedures, propositions and arguments to Question 3.

Table 4

Previous and Emergent Concepts, Procedures, Propositions and Arguments to Question 3

Objects	Previous	Emergent
Concepts-Definitions	-The limits of integration over a region; -Type 1 region, type 2 region; -Change from rectangular to cylindrical and spherical coordinate systems and the "compensation".	-Formal definition of the "compensation" as the Jacobian; -General change of coordinate system (not restricted to cylindrical and spherical systems).
Procedures	Setting up and calculating multiple integrals in polar, cylindrical.	-Setting up and calculating multiple integrals using a change of coordinates that will make the region (domain) more tractable. -Calculating the Jacobian ("compensation" for this change of form of the region).
Propositions	When changing from rectangular to polar or cylindrical coordinate, the integrand must include the variable "r". When changing to spherical coordinate, it must include function $\rho^2 \sin^2 \phi$	The area of a region Ω, set up as a double integral, is equal, under certain conditions of continuity and differentiability of the change of variable function $(u,v) \mapsto (x(u,v), y(u,v)$ to the area of a region Γ, set up as a double integral with the Jacobian as the integrand.
Arguments	Geometrical arguments	Intuitional and geometrical verification. The formal proof is not given at this level

Question 3a) was similar to question 2; however, here the two aspects of this study, dimensional analogy and different coordinate systems, were explicitly contained in the same question related to a problem that was easily set up by all but one of the students (whose mistake was minor). In this case, it was possible to establish the semiotic functions relating the question of why a double integral to calculate the area, to the *r* of the Jacobian. Once again, the extensive/intensive and ostensive/non-ostensive dualities permeated the language objects (equations, graphs, spoken and written mathematical English, gestures). For example:

S3: The *r* is the compensation of the coordinate system. The *r*, I can't remember how to calculate, but in rectangular coordinates each rectangle fits nicely into each other so they take up all the space [*there was much gesturing*], but in polar coordinates they are small circles, so when they're next to each other there's some space that will be lost unless we compensate the lost area.

The appendix should be consulted, in this case, to see the drawing of the 'packed' circles, instead of the concentric circles of the polar plane. Whereas the institutional meaning of the Jacobian factor is transmitted as a compensation for the *distortion* of the area, S3's personal meaning was that the Jacobian was to compensate *lost* area. Of course, this discrepancy of institutional and personal meaning cannot be detected if the specific question is not asked, as the procedure and problem-situation objects will reflect perfect institutional meaning.

On the other hand, S1, when responding to the question of why a double integral was used for area, explained:

S1: In reality we're finding volume with height 1, and the numbers are the same. Then the extra *r* is not really extra, we're in another coordinate system, so the constant function is not the same. In rectangular coordinates, the constant function is '1', in polar the constant function is *r*, and we find the Jacobian matrix. We set $x = r\cos\theta,\ y = r\sin\theta$ and do the partials with respect to r and θ, take the determinant and it gives us the compensation.

I2: Do you know what is compensated?

S1: The circles.

In this case, the definition object provided the guideline, as it should, which permitted S1 to express the apparent anomaly of using double integrals to calculate area and volume, with a coherent argument object. Of course, as he continues and expresses that the constant function '1' in rectangular coordinates turns into the constant function 'r', his personal meaning contradicts the definition of constant function. Once again, this contradiction would not be detected without the specific interaction and language objects that it provoked, because the procedural and problem-situation objects do not present any contradiction with the institutional meaning.

For reasons of space, at this point the analysis of the students' responses will conclude and the closing section will be presented. It is evident that what was shown is only part of the material and, although it is representative, there is much more data in this set that can be analyzed using the onto-semiotic approach to study dimensional analogy and different coordinate systems in the context of the multivariate calculus course. This will be done in the future.

SYNTHESIS, CONCLUSIONS AND PROSPECTIVE

As previous research, within any framework, on these mathematical concepts, and on multivariate functions in analysis in general, is practically non-existent, we are in the process of developing a much more sophisticated description of an epistemic network for this subject (Montiel et al., 2008; Montiel et al, 2009). The transformation of expressions to content through semiotic functions, and the identification of chains of signifiers and meanings, can be accomplished because of the rich layering and complexity of these mathematical concepts.

Dimensional analogy as a technique and different coordinate systems, apart from their intrinsic mathematical interest, are used in many types of applications in the sciences, engineering and art. The generic notion of representation is central in the cognitive and instructional processes involved in communicating these notions. The focus on changes of registers and on individual processes of objectification, conceptualization and meaning contributes to a coherent view of mathematical knowledge and the means of its construction and communication.

Based on the onto-semiotic approach, it can be added that it is also important to emphasize the anthropological and socio-cultural character of this knowledge, indicating the tensions between the personal and institutional meanings. The primary entities and their dualities, together with the semiotic functions, allow the description of this personal-institutional tension, related to the notion of meaning and mathematical objects that are relevant, in this case, to dimension, dimensional analogy and different coordinate systems.

"The notion of meaning, in spite of its complexity, is essential in the foundation and orientation of mathematics education research" (Godino et al., 2005). The onto-semiotic complexity that was identified is an empirical indicator that should guide the search for ways to improve and control the didactic systems related to the learning and teaching of notions, methods and meanings associated with dimension and different coordinate systems in the multivariate context. Based on the previous section, a brief response to the research questions shows the following.

Dimensional analogy is used by the students, especially when dealing with the primary entities of language, situations and concepts, as can be witnessed in the analysis of question 2. However, this concept needs to be formalized and consciously incorporated as a technique in the communication of this mathematical subject, and others similar to it. The representational semiotic function was identified in several instances, where the expression consisted of the notion itself (of dimension), and the content was the actual analogy, manifested through gesticulation, graphical representations, double and triple integral signs, among other forms.

These *personal meanings*, that is, the relation between the students and the mathematical object "Dimensional analogy", show a difference with the intended *institutional meaning*. The institutional meaning assumes that the student is capable of using *argumentation* and of integrating dimensional analogy to *procedures*. The student is expected to go beyond a mere metaphoric use based on gestures and graphical language, without an explicit foundation that would make it a reusable technique (*proposition*).

In the case of the different coordinate systems, definitions and procedures played a larger role. This is to be expected, as the change of coordinates was introduced as a procedure, while dimensional analogy plays more of a metaphoric role. Instrumental semiotic functions and structural (cooperative) semiotic functions were more noticeable for this same reason. The majority of students expressed that it was easier to use the calculus techniques and change the reference system, than to use purely geometric techniques, although their level of institutional success was inadequate.

The notion of fossilization of a metaphor is important here. This fossilization means "there is no consciousness of the use of the metaphor". If we say "your teeth **are** perls", we have no doubt that it is a metaphor; however, when we say "a circle **is** an equation $(x-a)^2 + (y-b)^2 = r^2$", the metaphor is "camouflaged" in a "representation", "identification", "semiotic register", etc., which is evidence of culturally rooted knowledge and is considered transparent.

Font (2007) shows one of the most productive metaphors in the history of mathematics: the interpretation of curves on the plane as a set of points that are the solution of an equation, and comment on the process that converts this metaphor in a subcategorization (particular-general). Just as is shown by this example of plane analytic geometry, the fossilization is a complex process that can require a lot of time. It is a process that implies that an object (the circumference in this case) that "lived" in a specific research area of mathematics (synthetic geometry) begins to "live" in another research area as well (analytic geometry).

On the other hand, given that we can consider curves in the plane as contained in three-dimensional space (thanks to the container metaphor), "dimensional analogy" can be applied, and the characterization of curves in the plan can be transformed into curves and surfaces in space (Figure 2).

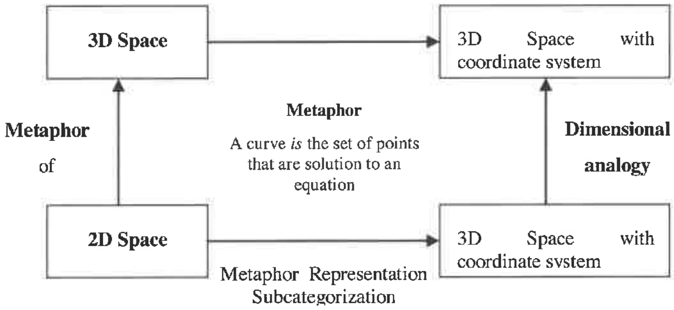


Figure 2. Dimensional analogy and metaphors.

To begin with, the plane is an ideal, two dimensional geometric entity contained in three dimensional space which, in turn, contains infinite points and lines, and can be determined in different ways, for example, by three non collinear points. In this case the duality particular-general (*A* is *B*) is acting, that is, *A* (a concrete plane) is an element of the class *B* (an ideal geometric entity of two dimensions, contained in space and that contains infinite points and lines).

When actually sketching the change of region as a result of using a different coordinate system, the visual result is seen as a language object. The process of change, that starts with one "graph" and results in another, can be seen as a semiotic function in three steps. First, the expression is the algebraic representation of the original region, and the content is the language object, that is, the sketch. Then, there is the transition from the original coordinate system to the new one, which is carried out on a non ostensive level and requires symbolic language. Finally, the new region is expressed symbolically, with a new sketch embodying the content. As was seen in the previous section, there are varying degrees of success when integrating the entire process and the satisfaction of certain institutional requirements does not necessarily translate into personal meaning that will guarantee success in the future.

It is essential to organize what must be known in order to do mathematics. This knowledge includes, and even privileges, mathematical concepts, and it is the search for meaning and knowledge representation that has stimulated the development of the mathematical ontology. The communication and understanding of the mathematical concepts of dimension and different coordinate systems involve so many subsystems, that it is very important not to form descriptions that are simplistic or reductionist. The onto-semiotic approach gives us a framework to analyze, as mathematical objects, all that is involved in the communication of mathematical ideas as well, drawing on a wealth of instruments developed in the study of semiotics.

This process presents a cognitive component as well as institutional. In fact, in this case, the notion of "emergence" does not talk about new knowledge "that emerges" from a mental process "in real time". An epistemic configuration (previous) leads to another (emergent), and this emergence is the mechanism that is triggered when knowledge about a concept is called upon to carry out one of the tasks, or solve a problem.

To understand the meaning of the question 1 ("in rectangular coordinates the coordinate surfaces: $x = x_0$, $y = y_0$, $z = z_0$ are three planes"), the network of semiotic functions in table 1 must be activated. Of similar way for the cases $y = y_0$ and $z = z_0$.

Expression B				**Content B**
Expression A	→	*Content A*	→	Plane
$x = x_0$		Coordinate surfaces		

Figure 3. Network of semiotic functions.

This network of semiotic functions is the result of the fossilization of the metaphor "the planes are a set of solutions of an equation of type Ax + By + Cz + D = 0". This metaphor forms part of the family of metaphors that lead to the interpretation of curves and surfaces in space as "the set of points that are solution to a system of equations".

When the set of solutions of an equation is studied in the context of synthetic geometry, another complex sequence of semiotic functions are activated, whose beginning and end can be represented by Figure 4 and 5.

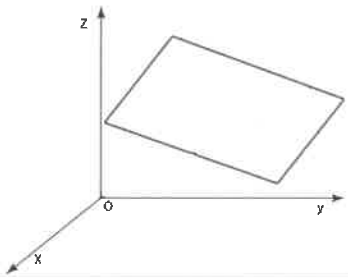
Expression B				**Content B**
Expression A		*Content A*		
	→	Plane	→	Ax + By + Cz + D = 0

Figure 4. Network of semiotic functions.

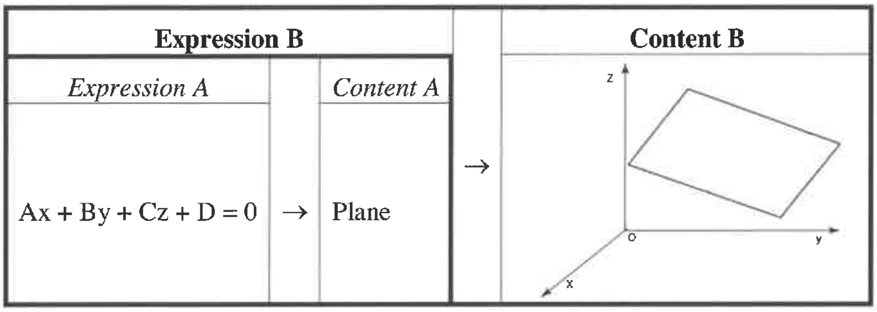


Figure 5. Network of semiotic functions.

Then, depending on the context, the plane figure can be seen as a geometric representation of the equation, or the equation can be seen as an algebraic symbolization of the plane figure. This leads to believe that there is a non ostensive mathematical "plane" that can be represented by the figure of a plane in "synthetic geometry" and by an equation in "analytic geometry". For this reason, the question of part *a, "Draw them",* and problem 1 can be understood as a problem of converting between different representations (algebraic and geometric) of surfaces contained in space (Figure 6, of similar way for the cases $y = y_0$ and $z = z_0$.).

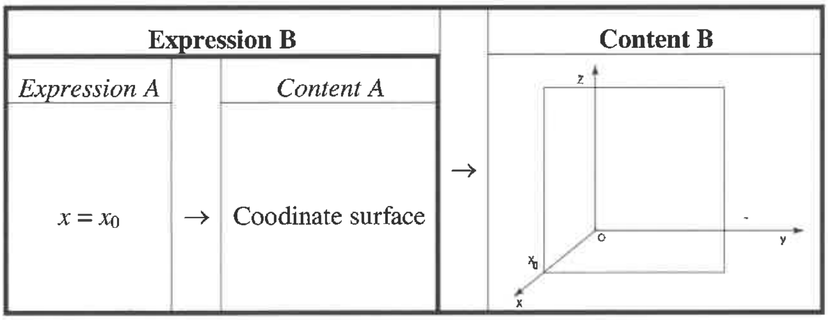


Figure 6. Network of semiotic functions.

In the same way the linking metaphors project epistemic configurations on epistemic configurations, what is called dimensional analogy also projects epistemic configurations (that are considered previous) on epistemic configurations that are considered emerging. In the context of this projection of configurations, it makes sense to formulate the rest of the parts of question 1. The networks of semiotic functions, expressed in Figures 7 and 8 for question 2, are also determined in this manner.

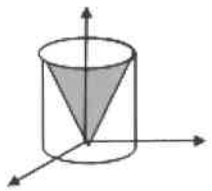
Expression B				**Content B**
Expression A		*Content A*		
$z=\sqrt{x^2+y^2}$ *or* $z^2=x^2+y^2$	→	*Cone (Quadric Surface)*	→	

Figure 7. Network of semiotic functions.

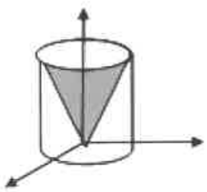
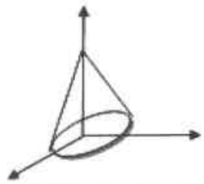
Expression B				**Content B**
Expression A		*Content A*		
 or 	→	Volume	→	$\frac{1}{2}\int_0^{2\pi}\int_0^{2}\int_0^{r} r\,dz\,dr\,d\theta=\frac{8}{3}\pi$

Figure 8. Networks of semiotic functions.

The experimental evidence has shown that these networks of semiotic functions evoke certain notions about dimensionality, without a link to the actions (*procedures*) and justifications (*arguments*) needed to perform as institutionally expected. The responsibility for the formulation of reusable techniques and properties (*propositions*) corresponds to the particular professor, which creates a bridge between whatever personal meaning is really created, and the intended standard institutional meaning. In general, the goal is to create a basis for knowledge on the teaching and learning of multivariate content.

As was mentioned at the beginning of this section, this is a work in progress. Added to the development of the epistemic networks for topics in the multivariate context, there are plans to analyze the particular subject of change of coordinate systems as it is studied in the context of linear algebra, both introductory and advanced.

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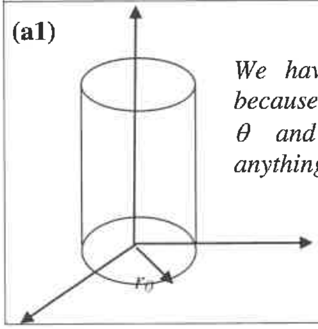
APPENDIX

1. Expected Answers of the Question 1

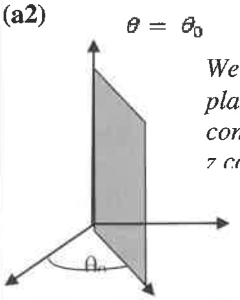
In rectangular coordinates: a1) $x = x_0$ is a plane parallel to zy-plane that it ***contain***s $(x_0, 0, 0)$ point; b) $y = y_0$ is a plane parallel to xz-plane that it ***contain***s $(y_0, 0, 0)$ point; c) $z = z_0$ is a plane parallel to xy-plane that it ***contain***s $(z_0, 0, 0)$ point. One of the variables is fixed as a constant; a plane is generated as there are two other variables that are free.

(a1)



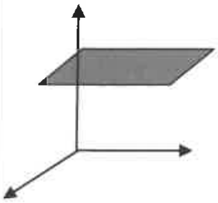
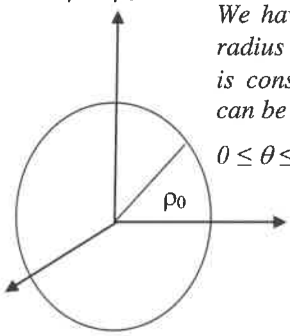
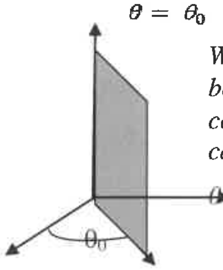
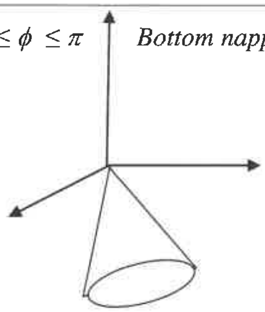
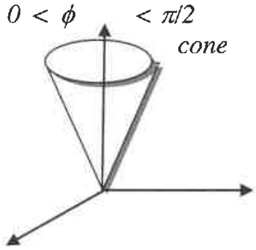
We have a cylinder because r is constant, θ and z can be anything.

(a2)



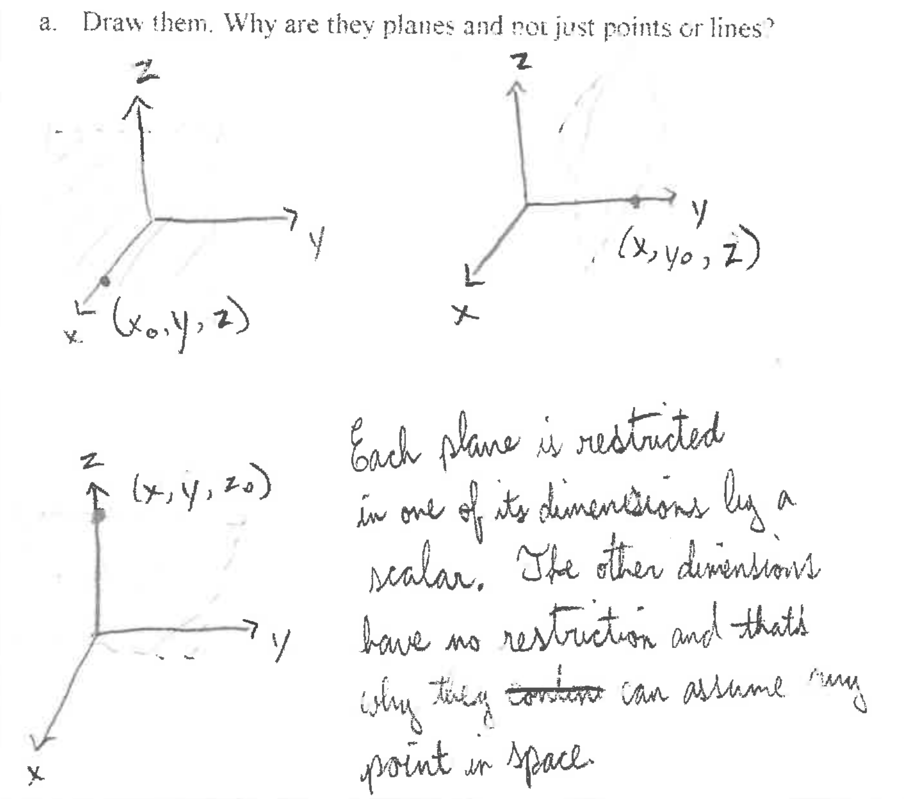
We have a vertical plane because θ is constant while r and z can be anything.

1. Expected Answers of the Question 1 (continuation)

(a3) $z=z_0$  *We have a horizontal plane because z is constant, θ and r can be anything and*	$\rho = \rho_0$  *We have a sphere of radius* ρ_0*, because* ρ *is constant,* θ *and* ϕ *can be anything* $0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi$ **(b1)**
(b2) $\theta = \theta_0$  *We have a plane because* θ *is constant,* ρ *and* ϕ *can be anything.* $0 \le \phi \le \pi$	**(b3)** $\phi = \phi_0$, we have 5 cases: (i) $\phi = 0$: Top half of vertical axes. (ii) $\phi = \pi$: Bottom half of vertical axes. (iii) $\phi = \pi/2$: xy – plane. And the other two cases are:
(iv) $\pi/2 \le \phi \le \pi$ *Bottom nappe of cone* 	(v) $0 < \phi < \pi/2$ *Top nappe of cone* 

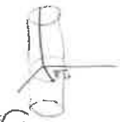
2. Actual Student Answers of the Question 1

S3



2. Actual Student Answers of the Question 1 (continuation)

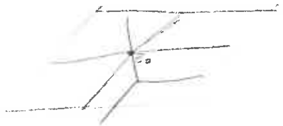
b. In cylindrical coordinates, what are the three surfaces described by the equations: $r = r_0, \theta = \theta_0, z = z_0$? Sketch.



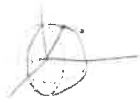
$r = r_0$ is a cylinder with radius r_0

$\theta = \theta_0$ is a plane that contains the z-axis.

$z = z_0$ is a plane parallel to the xy plane



$\rho = \rho_0$ is a sphere with radius $= \rho_0$



$\theta = \theta_0$ is a plane that contains the z-axis

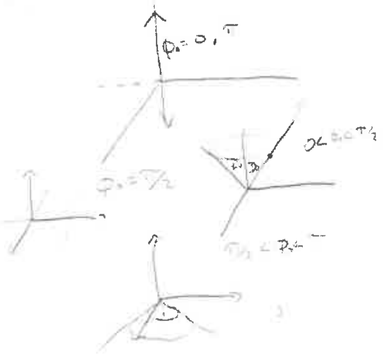
$\phi = \phi_0$ is a . . .

$\phi_0 = 0, \pi$, the z axis

$0 < \phi_0 < \pi/2$, upper nipple of a cone.

$\phi_0 = \pi/2$, the xy-plane

$\pi/2 < \phi_0 < \pi$ bottom nipple of a cone

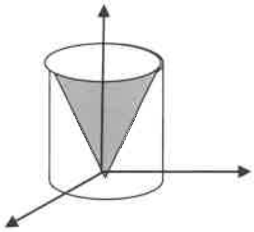


3. Expected Answers of the Question 2

(a) $z = \sqrt{x^2 + y^2}$ *and* $z^2 = x^2 + y^2$ *is the equation of a cone (Quadric Surface). In this case: top nappe*

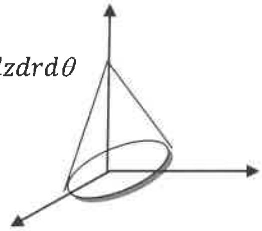
Two ways to set up:

(i) *As the formula for volume of a cone is* $\frac{1}{3}\pi r^2 h$, it is $\frac{1}{3}$ of a cylinder's volume.



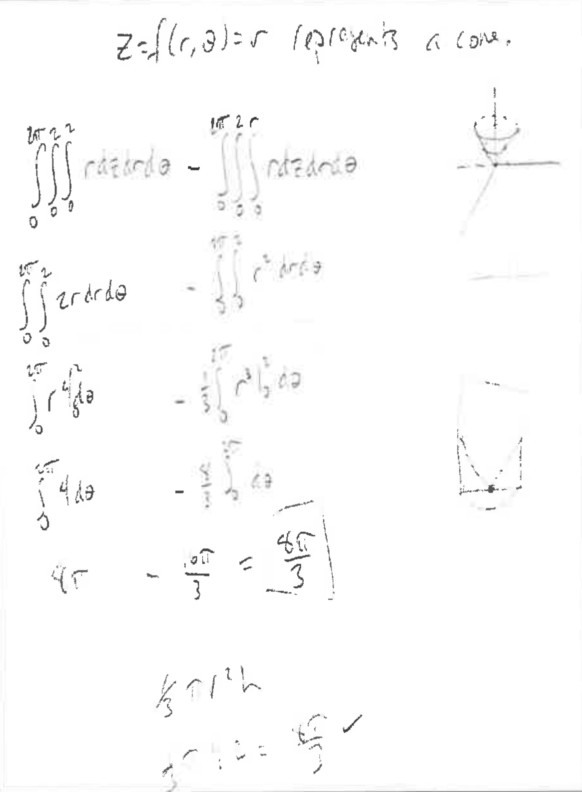
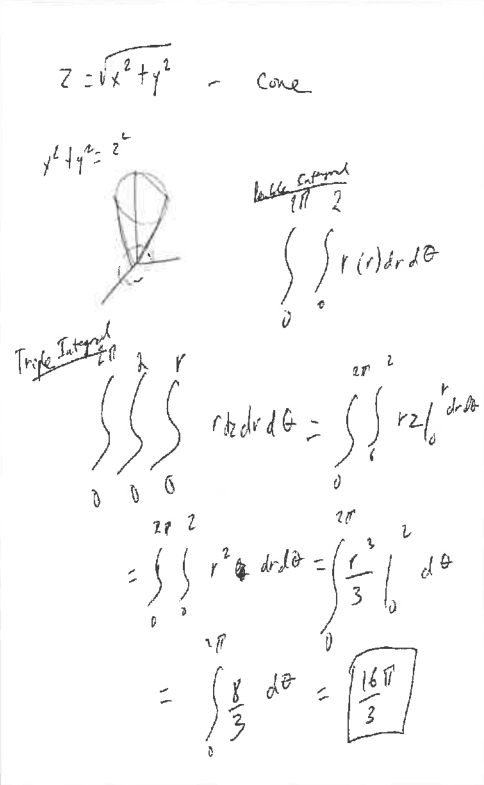
Then the volume is: $\frac{1}{2}\int_0^{2\pi}\int_0^2\int_0^r rdzdrd\theta$

(ii) *"Upside Down"*: $\int_0^{2\pi}\int_0^2\int_0^{2-r} rdzdrd\theta$



Either way we get $\frac{8}{3}\pi$

4. Actual Student Answers of the Question 2

S1	
	

5. Expected Answers of the Question 3

- Because there is no explicit function as an integrand.

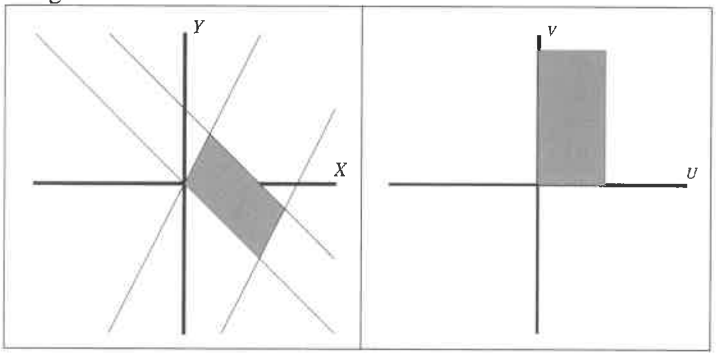
- To compensate the change of region.

- Calculation by: $\int_0^{\pi/2}\int_0^1 r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}$

b)

(ii) To make the integration process easier.

(v) Two regions



(vi) How would we do this?

Let $u = x + y, v = 2x - y$; them:

$u + v = 3x \Rightarrow x = (u + v)/3$

$v - 2u = -3y \Rightarrow y = (2u - v)/3$

And:

$$|J| = \left\| \begin{matrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{-1}{3} \end{matrix} \right\| = \frac{1}{3} \Rightarrow \int_0^1\int_0^3 \frac{1}{3} v^2 dv du = \frac{1}{3}$$

(vii) With calculus: $\frac{1}{3}\int_0^1\int_0^3 dv du = 1$

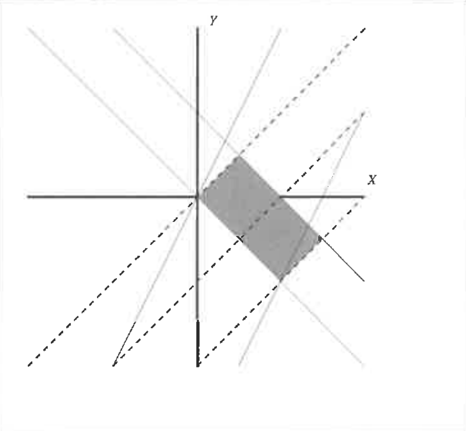
Without calculus: with classical geometry

5. Expected Answers of the Question 3 (continuation)

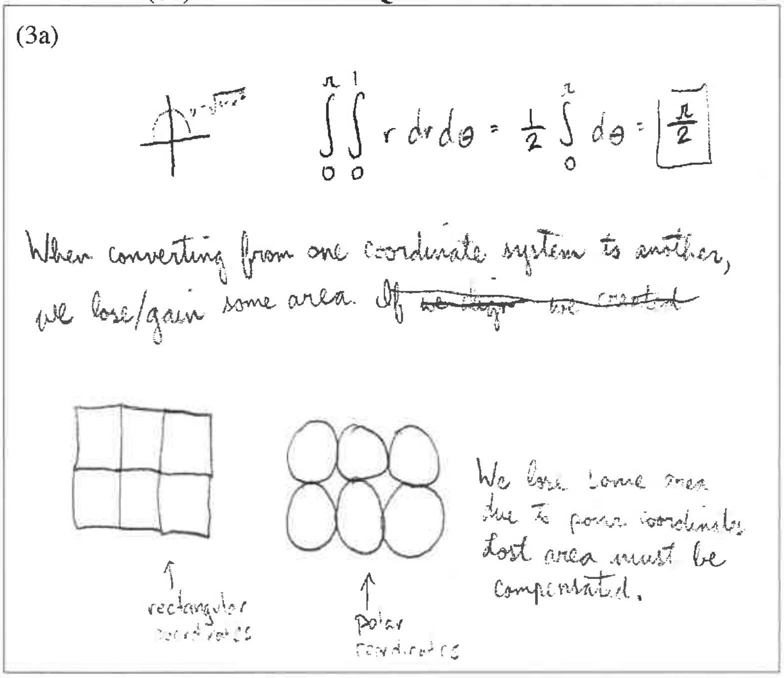
The area of the romboid (b-ii) is equal to that of the rectangle formed by the lines:

$y = -x, y = 1 - x, y = x, y = x - 2$

If we add the line $y = x - 1$, it is easy to justify that the rectangle is subdivided into to equal squares, whose areas are ½; for example, if we observe that the squares can be subdivided in two isosceles triangles, and that the sum of the areas is equal to the isosceles triangle with vertices: (0,0), (0,1) y (1,0). Thus, we can conclude that the area is 1 (= 2 × 1/2).



6. Actual Student 3 (S3) Answers of the Question 3



6. Actual Student 3 (S3) Answers of the Question 3 (continuation)

b) If Ω is the parallelogram bounded by the lines
$x+y=0, \quad x+y=1, \quad 2x-y=0, \quad 2x-y=3$

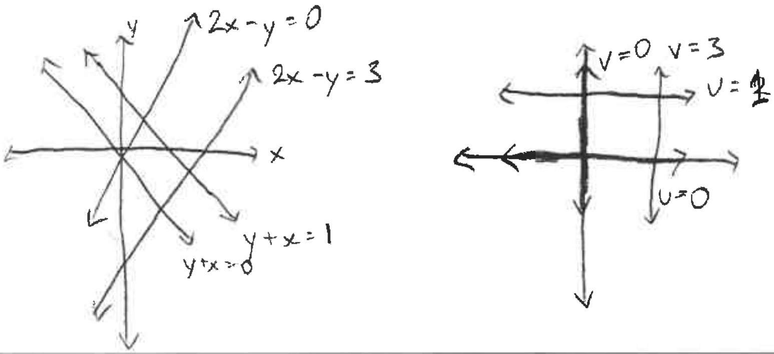
And we have the double integral $\iint (x+y)^2 dxdy$

then we can set $u=x+y$, $v=2x-y$ and change the region from a parallelogram to a rectangle.

(i) Why would we want to do this?

Both the limits of integration and the fuction would be "easier" to integrate.

(ii) Make a sketch of the two regions.



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Teaching Mathematics to Visually Impaired Pupils

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ABSTRACT: *It is almost a commonplace to state that the visually impaired people use different methods than people who can see to receive information about features of the objects and spatial localization. In this paper we present research that was realized in order to better know teaching and learning mathematics of visually impaired students. We try to describe the actual situation of teaching mathematics to non-sighted students in Slovakia. In the second part of the paper we analyse the perception of space and its object by non-sighted pupils within realized experiment. We compared it with sighted pupils and the paper ends with some recommendation related to the teaching spatial geometry of both groups of pupils in view of the results of this study.*

Key words: *Didactical system, Integration, Mathematics education of visually impaired pupils, Perceiving of space and its objects.*

INTRODUCTION

The vision is central to our biological and socio-cultural being. The faculty of vision is our most important source of information about the world. The largest part of the cerebrum is involved in vision and in the visual control of movement, the perception and the elaboration of words, and the form and colour of objects (Adams & Victor, 1993). As for the socio-cultural aspect, it is almost a commonplace to state that we live in a world where information is transmitted mostly in visual wrappings, and technologies support and encourage communication, which is essentially visual.

Nowadays, we notice use of mathematics in lot of disciplines, the serious mathematical grounding is necessary not only for prospective mathematicians, but it begins to be popular also at humane sciences as sociology, psychology, linguistics or philology. We are also witnesses to rapid expansion of information technologies that require new technicians all the time, whose education is based on mathematics as well. Mathematics, as a human and cultural creation dealing with objects and entities quite different from the physical phenomena (like planets or blood cells), relies heavily (possibly much more than mathematicians would be willing to admit) on visualization in its different forms and at different levels, far beyond the obviously visual field of geometry, and spatial visualization (Arcavi, 1999).

If we follow the mentioned facts, we cannot wonder about the attendance of visually impaired people who would like to engage in study of mathematics. Thus, it is needed to create acceptable conditions for studying and deal with problems, which visually impaired people are encountering, as far as everyone should have the same right of education, regardless of disability.

The changes in society, inflow of liberty and humanism, caused that the integration of handicapped people (Slovakia, 1993) have became an actual problem and one can partly speak about it as fashion trend that is carrying its advantage and limitations.

These facts, as well as author's 5 years long experience with working with visually impaired pupils have inspired us to pay more attention to study of mathematics of visually impaired people.

In this paper we describe the second part of the 3years long study. At the very beginning we have specified following questions:

1. What is the attitude of visually impaired people towards mathematics?

2. What is their ability to solve mathematical problems?

3. What approaches and strategies do they use by solving mathematical problems? Are they different in comparison with sighted people?

4. What is the actual situation of teaching mathematics to integrated visually impaired students in Slovakia?

5. How do the visually impaired students perceive the space and how they are able to manage school's spatial geometry?

In the first part of the study we have answered the first three questions (Kohanová, 2005). Briefly said, we find out that visually impaired people are able to solve mathematical problems, although their approach and way of solution is a bit different than approach of the sighted persons. With the regard to the algebra and arithmetic we discovered they mostly prefer arithmetic. They don't use variable very often comparing to the sighted students who do so many times. Nevertheless, we see analogies among other used strategies. In the field of geometry we find out that the visually impaired people use personal geometrical instruments and strong imagination, all object (solids and plane figures) are first touched and then stored. Geometry is for them kind of adaptation to the environment. So we think this adaptation is dynamic in sense that they continually change the system of operations of environment that explores. Since every environment is a new environment s/he has to store all information (tactile, auditory, olphactive, etc.) and so make mental images. All that has inspired us to study more in the field of space geometry, to see how non-sighted people are adapted to various environments, what are their personal tools, since geometry can provide a more complete appreciation of the world.

Before giving the answers for the last two questions it is needed to mention that we have studied the human eye and its behaviour; the personality of visually impaired child, its development; using of other senses and the communication with visually impaired child during the education. We have also devoted to the history of reading codes for the visually impaired and afterwards we focused on problem of Braille notation, in the concrete of problems concerning mathematics. Particularly we dealt with Slovak norm and its limitations for notation of mathematics. Hence, we studied requirements for the suitable notation and semantics of mathematical languages in the sense of possible universal Braille notation creation for mathematics. Finally, we have mapped teaching of mathematics to visually impaired students on each level.

ACTUAL SITUATION OF TEACHING MATHEMATICS TO VISUALLY IMPAIRED STUDENTS

Primary Level

Most of the Slovak visually impaired children (non-sighted and partially sighted) attend special primary schools. Teaching of mathematics on the elementary level means first of all helping children to use and organize their experiences, which they gain from actions and interactions with the world around them. In the opinion of some authors (Csocsan, 2002) the main goal of mathematical education is to develop an awareness of numbers and coping with different relations and dimensions. The most frequent mathematical problems of non-sighted pupils are as follows:

1. Generalizing – finding the similarities in different activities in everyday life;

2. Translating activities and actions into mathematical language;

3. Lack of the flexibility in problem solving and in calculations;

4. Translating and transferring three-dimensional objects into two-dimensional iconic forms [Example: The non-sighted pupil cannot understand a geometrical drawing of a cube from a perspective view because of her/his lack of visual experiences. S/he also has difficulties in enlarging and minimizing two-dimensional forms.].

Secondary Level

There are special high schools for visually impaired students, but mostly oriented on music, some handicrafts, etc. If a student wants to come into contact with mathematics then s/he needs to attend "normal" high school. As we know, mathematics is a subject, which is important for studying not only natural sciences but it also begins to be popular at humane sciences. The direct consequence of this mathematical requirement almost everywhere causes that also more and more visually impaired students today start their education in mainstream schools, which is place, where they can study mathematics.

Communication with visually impaired student

The integration of visually impaired student among sighted ones in the common schools causes that in these specific conditions we find different (new) relations in the classroom. One has to distinguish between communication between teacher and sighted students at the lesson and between teacher and non-sighted student, in addition, between non-sighted and sighted students. The first case is more dominant it runs in various forms. Oral communication is irrecoverable at education; however, this fact does not valid in mathematics. In addition, mathematics requires exactness, definiteness, totality and comprehensibility of presentation. It is very arduous only by oral communication (e.g. when modifying expression or by geometrical construction) and so it is supported by graphical way - text or picture. This connection is typical for mathematics; because of insufficient style of expression some students rather prefer notation or picture. If we are talking about graphical communication in the frame of communication between teacher and non-sighted student, we mean communication supported by for example relief's picture, typhlographic images and plane or space models (construction kit, cubes, skewers, paper). The other case is communication supported by notebook; the non-sighted student takes notes or computes in electronic form, mainly linearized.

Didactical system in specific condition

Following the Theory of didactical situation introduced by Brousseau (1997)[1], when acquiring new knowledge ("connaissance") in the frame of **didactical situation**[2], the teaching of a new notion consists of setting up its situations and carrying out interactions in which the learner can take part. It is itself, an interaction. This interaction is also largely specific to the knowledge being taught but it takes a form of a-didactical situation[3], necessarily different from the non-didactical forms in which knowledge ("savoir") is used. This result changes the entire approach to mathematics education and the education of teachers.

Consequently, we can represent the *didactical system* (triangle) as a system of relationships (didactical contract) between three subsystems: educator (E → teacher: TE, tutor: TU), learner (L → sighted student: SS, non-sighted student: NSS) and knowledge (K), where the non-sighted student has particular position in the frame of didactical situation (DS), because of the way that s/he obtains the knowledge.

[1] Theory of Didactical Situation, which fundamental methodological principle is built upon: "a piece of mathematical knowledge is represented by a "situation" that involves problems that can be solved in an optimal manner using this knowledge". The characteristic situations for pieces of mathematical knowledge can be studied or even modelled within the framework of mathematics itself, which sometimes makes it possible to use computation to predict their evolution.

[2] Situation that enables to obtain new knowledge.

[3] The part of didactical situation that enables to the student to acquire new knowledge own and consequently s/he is able to put it to use in situations, which s/he will come across outside any teaching context, and in the absence of any intentional directions.

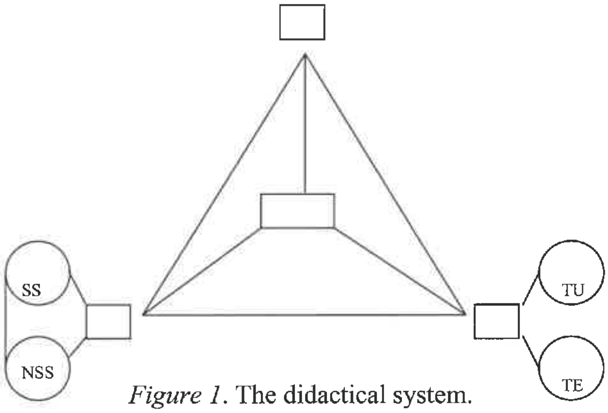

Figure 1. The didactical system.

In general, we think the process of acquisition the knowledge is different than the same process of sighted students. In follows, we describe the didactical triangle in form of sides (Sbaragli, 2004), which are stated by three main vertices, focusing on non-sighted student, taking into account personal experience of interviewed people – teacher of mathematics and her non-sighted student:

1. *Teacher (TE) – knowledge(K)*: This side is characterized by the verb "*to transpose*", where the main activity is the first part of the didactical transposition from the *scholarly knowledge* to the *knowledge to be taught.* The question is whether the teacher is conscious of necessity of different approach that s/he has to use when transposing the knowledge to the non-sighted student. It consists in extra preparation of a-didactical situation, so the non-sighted student can also participate in activity. When making analysis a-priori, the teacher has to take into account possible difficulties of understanding and questions that may appear at the lesson and so be able to react dynamically and clear, but not at the expense of other sighted students.

2. *Tutor (TU) – knowledge (K):* This side is specific by various ways of transposing the knowledge, which are not used when teaching sighted students. Sometimes the teacher is not conscious of all that might be a problem to understand for non-sighted student, s/he finds is out at the lesson and in analysis a-posteriori. So as a tutor, s/he has to select different view on given subject and try to transpose the scholarly knowledge to the knowledge to be taught in appropriate and customized way, sometimes by using models and tactile pictures.

3. *Student (NSS) – knowledge (K)*: This side is expressed by the verb "to learn", where the predominant activity is the involvement of the non-sighted student that is characterized by nouns "motivation – interest - volition". S/he accepts the devolution [4]

[4] Devolution is the action of the teacher on the student, by which the teacher makes the student to take responsibility for a learning situation or problem, and accept the consequences of this transfer her/himself.

and takes personal care of her/his proper knowledge. The knowledge is so constructed by the student and finally institutionalized [5] by the teacher/tutor.

4. *Teacher (TE) – student (NSS)*: This side can be represented by the verbs "to facilitate - to advice - to guide". When the non-sighted student faces with a-didactical situation, s/he produces her/his knowledge as a personal response to the didactical milieu by keeping the didactical contract. The teacher should allow a harmonization of the different phases of the learning process and intervene only in cases of possible misconceptions or false beliefs of the student.

5. Tutor (TU) –student (NSS): "Individual consultation" – it is the name of this side. Consultation between non-sighted student and tutor is held apart from lesson, depending on needs that have occurred at the lesson. Thanks to individual approach there is time and space for deeper understanding of the piece of knowledge and its consistent arrangement into the general knowledge structure.

6. Student(NSS) – student (SS): This side can be described by the verbs "to help – to correct" and by the noun "cooperation" of the non-sighted and sighted student at the lesson. In case of any doubt, incertitude or misunderstanding the non-sighted student can ask for help of her/his sighted neighbour. On the other side, the sighted student (who is good at mathematics) can correct some errors or mistakes if s/he sees them on the screen of her/his neighbour's computer.

Since there is no time for the teacher to engage fully in work with visual impaired student at the lesson (in sense not at the expense of sighted students), it is needed to accent that mentioned model of didactical system works on voluntariness and interest of the teacher/tutor; especially tutor works in her/his free time. So it seems like the integration of visual impaired student into common school is fictive, concerning the point of view of studying mathematics. The integration by itself has rather social aspect. Consequently, the question is: Who can be integrated and who not? Sometimes the integration might harm, on the contrary, it might help to some of non-sighted students to express them. The other remarkable thing is the question of limit. Since in Slovakia there is no standard for teaching mathematics of integrated visual impaired students on secondary level, the teacher has to determine requirements on these students by her/his own, on her/his subjective opinion.

University Level

Comparing to secondary school, there is quite a different situation in math for a visually impaired student at universities. The student is supposed to have skills necessary to study - make notes during lectures, read scientific text, perform complex calculations, communicate with teachers and other students in written form, etc. There is much more

[5] In institutionalization the teacher situates the student's production (the piece of knowledge constructed in a-didactical situation) in accordance with the scientific or cultural knowledge socially accepted.

independent work required. If s/he graduated at special school/class and used only Braille notation and spoken language for before mentioned purposes, s/he will have to overcome a lot of new challenges. There are very limited sources of scientific literature in accessible form for a visually impaired student. Therefore he should be able to read different mathematical notations.

Another way of delivering mathematical expressions in accessible, written form is electronic text document on personal/portable computer or special note taken for the visually impaired users. This sort of document usually contains linear mathematical notation with expressions built up of ASCII characters. Visually impaired student can access this type of notation in two ways. S/he can use refreshable Braille display and read line by line corresponding Braille cells (groups of 6 or 8 raised dots) by touching or listening to synthetic voice, which reads each written ASCII symbol for him/her. The second method is more difficult for reading complex mathematical expressions, although could be, however quicker for longer text with simple mathematical expressions. The ideal is combination of both methods, when student can choose appropriate method depending on current situation (what is s/he reading, writing or calculating). This way the student is able to take notes at the lesson, calculate or pass exams without any problem. Actually, it is not true in some cases. It is startling, as we find out, that some teachers who are not very familiar with computers refuse their usage as writing tool at exam, so the visually impaired student had to pass exam verbally where his sighted schoolmates answered in writing.

Some solutions, originally dedicated to electronic publishing of scientific text documents (TeX, LaTeX, AmSTeX, HrTeX, MathML), could be red and written by visually impaired student. Computer Algebra Systems (CAL Systems) are dedicated at the first place for algebraic calculations, e.g. differentiation or integration; solving of equations. They are also able to perform numerical calculations; visual graphs of functions, curves and 3- dimensional objects. Such software is Derive, MuPAD, MAPLE, MathCad or Mathematica. All of them contain as well a lot of functions of analysis, linear algebra, statistics, numerical analysis, number theory, graphics, etc. They are also useful for visually impaired students, especially by calculations. It has no sense to urge them to act calculations that are often just very tedious and mechanical. That is why CAL Systems are helpful. If the commands we put into command-line are linear, it means they are fully textual and therefore suitable for visually impaired students. The other advantage is that screen-reader does not have any problem to read linear text on the screen and so, access it to the student.

SPATIAL GEOMETRY AND NON-SIGHTED PUPILS

To answer the last of given questions (How do the visually impaired students perceive the space and how they are able to manage school's spatial geometry?) we have realized experiment in which we observed the perception of the space and its objects by non-sighted pupils. In addition, we compare it with sighted pupils in order to see the differences. Thus we are be able to propose possible solutions/tools for teaching space geometry that will be determined not only for visually impaired pupils and their teachers but also for teachers who are teaching integrated students at common schools.

Understanding of Geometric Figures

Van Hiele (1986) published a theory in which he classified five levels of understanding spatial concepts through which children move sequentially on their way to geometric thinking. Different numbering systems are found in the literature but the van Hiele's spoke of levels 0 through 4. At each level of geometric thought, the ideas created become the focus or object of thought at the next level as shown in Figure 2 (Van de Walle, 2001).

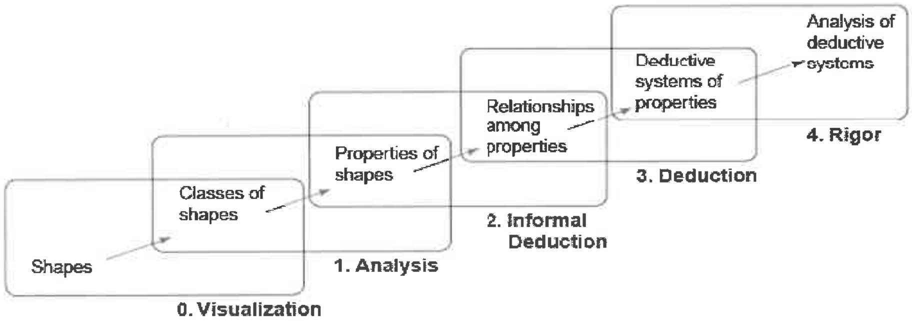

Figure 2. Van Hiele's levels.

According to Jirotková (2001) there are three levels of the quality of the mental picture of a perceived solid:

1. The solid is a 'personality' for the pupil,

2. The solid is unknown to the pupil, however, the pupil perceives some relationship between the considered solid and another solid which is a 'personality' for him/her,

3. The solid is entirely new for the pupil.

Analysis of the Activity

Activity theory originated in the former Soviet Union as a part of the culturalhistorical school of psychology founded by Vygotsky, Leontiev and Luria. Its unit of analysis is an activity that is being composed of a subject, and an object, mediated by a tool. In following model (see Figure 3) of an activity system, the subject refers to the individual or group whose point of view is taken in the analysis of the activity.

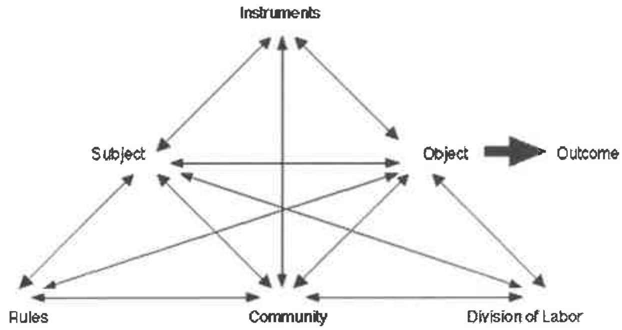

Figure 3. Model of activity system.

The object (or objective) is the target of the activity within the system. Instruments refer to internal or external mediating artefacts, which help to achieve the outcomes of the activity. The community is comprised of one or more people who share the objective with the subject. Rules regulate actions and interactions within the activity system. The division of labour discusses how tasks are divided horizontally between community members as well as referring to any vertical division of power and status. We have used this model as a tool for description and analysis of realized experiment.

The Research and Determination of the Hypotheses

We have placed various subjects of different shapes in the room. Except of typical office subjects (table, chairs, PC, cabinets) we put in the room the fit ball, air freshener, the clock of pyramid shape and flowers as well. The lamp on the table was on as well as the PC; water in the sink, which is in the closet, flow. That all because we wanted to observe what sense the person in the room will use while exploring the room. Before realizing the experiment, we consulted about the location of subjects in the room with visually impaired university student, who is experienced in exploration of new places. Final arrangement is shown in following figure.

Figure 4. Arrangement of the objects in prepared room.

Consequently, seven pupils took part in the experiment, the sighted pupils (SP) were selected at random and all pupils were of 7th - 9th grade. Pupils of these grades know 2-D and 3-D shapes and their characteristics; they have their personal experience and they have learned it also in the school[6]. However, the problem was the number of pupils who took part in experiment. We wanted to form pairs of all possible combination of sighted and non-sighted pupils (NSP), which means 4 pairs. It is needed to say we concentrated only on pupils who are non-sighted since birth and so do not have any visual imagination. That is why we were able to find only 3 non-sighted pupils (age 13-14) attending the special primary school for visually impaired children in Bratislava. Then we changed pairs for trinities and pairs as follows: NSP1-NSP2-SP1, NSP3-NSP2-SP2, SP3-SP4, where always the first one of the trinity/pair went in to the room and verbally described what s/he sees and the others of the trinity/pair built the model of the room on the basis of audio record. The first one of the trinity/pair built the model of the room as well, but on the basis of her/his memory. In the first and second trinity is the same person (NSP2) and we are conscious that it might influence the results, but NSP2 was not told that she is building the model of the same room in both cases.

During the experiment we observed:

1. The orientation in the space,

2. The way of description of the room and objects,

3. The relationship between the image in the pupil's mind and the vocabulary s/he uses in the communication,

4. What is the dominant attribute by description of the room,

5. Perception of the shapes, positions and dimensions,

6. What senses s/he uses,

7. What way s/he builds the model of the room,

8. Differentiation of the shapes and characteristics of the objects.

Consequently we have set following hypothesis:

H1: The sighted and non-sighted pupils perceive the space and its objects in different way. The point of view on geometry of the space of visually impaired people is point of perception and it is dynamic. The point of view on geometry of sighted people is static.

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[6] According to curriculum pupils of 1st grade of primary school should know to differentiate the geometric shapes as: triangle, circle, square, rectangle, cube, sphere and cylinder. Later on, in the 3rd grade they learn how to draw the circle, square, rectangle and triangle and learn name the edges and sides. In 6th grade pupils get in contact with cube, cuboid (they calculate volume and surface) parallelograms and trapezoids. In 7th grade they focus on prism, in 9th grade on cylinder, pyramid and cone.

H2: Based on the senses the non-sighted pupils are able to differentiate and name basic geometric figures and solids.

H3: When exploring new room and objects in it, the non-sighted are using several senses; sense of touch, smell and ear; while sighted rely only on sight.

H4: The non-sighted pupils will describe objects in the space (shape and position) better and more exact as sighted pupils.

H5: The non-sighted pupils have better imagination about position of objects in the space as sighted pupils and so they build more precise scale model of the room, even if they build it on the basis of given audio record.

Method and Description of the Experiment

As written above we have divided children into the trinities and pair. We call the one who goes into the room pupil A, pupil B is the one who doesn't go into the room. The tasks for the pupils were as follows:

Task 1

Pupil A: Enter the room. Within the twenty minutes explore it and tell me exactly what do you see. Tell me about everything, about all objects, their characteristics and their localization.

Task 2a

Pupil B: By using these packages and stuff try to build the model of the room on the basis of audio record of Pupil A. The caps of plastic bottles represent the chairs. Later on you can ask for more information, but only by asking questions to which Pupil A can only answer 'Yes' or 'No'.

Task 2b

Pupil A: By using these packages and stuff try to build the model of the room on the basis of your memory, on the basis of what you have seen. The caps of plastic bottles represent the chairs.

Applying the Activity theory we described two activities, one that has been carried out in the room (Task 1) and the second activity that has been realized out of the room (Task 2). In Task 1 we made audio records of Pupils A descriptions of the room. In Task 2a we made audio records of dialogs between Pupil B and Pupil A, in Task 2 pictures of all 8 models of the room.

The exploring and describing the prepared room is the activity that refers to the subject of Pupil A, who goes into the room. The object of her/his activity is the room and all objects in it. The expected outcome is the as precise verbal description of the room as possible; consequently we are going to analyze this description in sense of perceiving the space and its objects. There were no set rules concerning the progressing activity, just one restriction regarding the time was given. It has an implication that Pupil A can proceed as s/he wants, in the way s/he likes, so there are no horizontally segmented tasks of division of labour. Anyway, with respect to action of university student M. and our experience we have expected the following possible actions which Pupil A could make in the room:

1. To specify the shape of the ground plan and verify the dimensions of the room;

2. To seek points of the reference by means of the echo of the windows, of the doors, of the voice, etc.;

3. To individuate and memorize every possible obstacle;

4. To look for references in the noises and vibrations or in the odours;

5. To clapp one's hands to grasp the dimensions and the volume of a room;

6. To move with the white stick and perceive the space, objects and obstacles;

7. To perceive the obstacles by air pressure on the face;

8. To touch all objects and describe them.

Mentioned possible actions could be done by using the white stick, all senses, language, imagination, etc. and these are mediating tools or instruments by which the Pupil A can achieve the outcome of the activity. There is also no vertical division of status and power concerning the division of labour, since the community of this activity consists only of researcher who is present in the room in order to record the description and assist if necessary. It is needed to mention that the whole environment in which the experiment was realized, as well as the researcher was new for pupils, so that is the reason why we are conscious of pupil's doubtful and sometimes reserved behaviour. All that might influence the objectivity of the experiment.

The second activity was realized out of the prepared room and its outcome is to interpret the room by building the model, which is also kind of description of the room and we can analyze it in the frame of perceiving and recognition of the space and its objects. The model of the room built by us is shown in following picture. This activity has to be distinguished with respect to the pupil who is building the model (Task 2a, Task 2b). In both cases the object of the activity is the prepared room and the rest changes.

In case of Pupil A, who is the subject of the activity, the rules are given only by saying that pupil should build the room by using given packages and stuff, moreover the bottle caps have to be used as chairs. In case of Pupil B we have two more rules about the building the model according to record and about the way of asking questions to Pupil A. All given packages and stuff of different shapes and sizes (playing cubes, packages of tea, matches, medicaments and cosmetics; tennis's and squash's balls, buttons, batteries, eraser, carton models) are for Pupil A and B instruments to build the model. The difference between Pupil A and Pupil B is that other instrument of Pupil A is her/his internal model of the room stored in her/his memory, while Pupil B has audio record of Pupil A at disposal. Pupil B can ask for more information that is becoming also his/her instruments. The community in both cases consists of researcher and her assistant and other pupils who took part in experiment. In case of Pupil A all community except of researcher is just side, unimportant effect; they were just observers, no interfering into the process of building the model. On the other hand, important role of community plays in case of Pupil B the researcher who moderates the conversation and Pupil A, who answers to the questions. Since the instructions of Task 2a say to Pupil B first to build the model of the room on the basis of audio record and later on to ask the supplementary questions, here we have horizontally segmented actions of division of labor (which is actually given by the rules). Also the succession: question, answer, and potential change of model represent partial horizontal division of the actions.

Qualitative Analysis of the Experiment

The sighted pupil really showed expected behaviour, right she entered the room she stated what is in there (sometimes very inexactly), while the non-sighted pupils detected the space gradually. So here we have development and dynamics of detection, which are actually facilitating the subsequent better description. If we would be able to bring the sighted pupils to such a dynamics, then the certain superficiality can be eliminated and hence also the superficial perception of the space.

In Task 1 pupils should describe the room and its objects, their characteristics and localization so pupil B in Task 2a can build the model of the room. We had seen that non-sighted pupils recognized and named many objects of different shapes (cube, cuboid, pyramid, cylinder, triangle, circle, trapezoid, square, and rectangle), so the second hypothesis seems to be true, although in some cases they used wrong terminology.

> *R2: Then there is cabinet, also shape of rectangle, classic cabinet with rectangular shelves.*
>
> *M27: So, in the middle of the room is the table in shape of rectangle.*

The hypothesis H3 has been confirmed only partially because the non-sighted pupils didn't use sense of smell, neither by finding the air freshener nor by flowers. The sense of touch is their leading analyzer and sense of ear is complementary analyzer. We can illustrate the usage of sense of ear by demonstrations from the protocols:

> *R22: I have heard water and then I went to see...*
>
> *M39: You can hear here the whirr of computer and like [...] as the water flows [...] or like that.*

Also in case of sighted pupil SP3 we cannot claim she only relied on sight. The true is she also didn't notice the air freshener, she saw the flowers, but she mentioned the sink in the cabinet even she couldn't see it since the door was closed; on the other side she didn't say anything about hearing.

J6: At the door are cabinets, where is for example the sink, in one there are books.

Since non-sighted pupils had to go over the whole room and touch everything, they described continuously and more exactly the objects in the room than sighted pupil, who stand in one point and described what she saw. Sighted pupil didn't mention lot of things, she didn't find it as necessary, even she was told to describe it precise. On the other hand, when building the scale model of the room, she did it very exact, which says about her strong visual memory. Based on these facts we can confirm hypothesis H4.

The fifth hypothesis wasn't neither acknowledged nor disproved since all Pupils A (sighted and non-sighted as well) built almost exact model of the room. In the case of pupils B we had noticed the ability to interpret the verbal description of the space and ability to create an image of solids and their location in the space. We cannot compare the results of sighted and non-sighted pupils who participated in Task 2 since there was the same non-sighted person participating two times in experiment. Anyway, regarding the mental representation of the space, the world of non-sighted is not different in comparison with that one of people who are sighted.

DISCUSSION AND CONLUSIONS

In the submitted paper we dealt with questions concerning the study and teaching mathematics to visually impaired people. The trend of the last years is to integrate the impaired students into common schools and so we have described the actual situation of teaching mathematics in Slovak schools at each level. Especially we have focused on secondary level and we present the change of classic didactic triangle in case of attendance of visually impaired student in the classroom. We have used for it the Theory of didactical situation and we have found out that integration in teaching process has rather social aspect.

As the next we analysed the perception of the space and its objects by non-sighted pupils and we compared it with sighted ones. Except of determined hypotheses we came also to the following conclusions that are applicable in pedagogic practice of the teacher.

1. Right in the experiment, concretely at Task 2, the visiting math's teachers from special school for visually impaired children pointed out that the same or similar tasks have considerable value as educational tools. They could be used for the diagnosis and assessment of pupils' levels of understanding of three-dimensional solids (van Hiele's levels) and metrics of the space and to develop their communicative skills about the solids. The Task 1 required the pupils to describe new space and its objects. This gave a very clear indication of level of vocabulary of the pupils and the communicative skills.

R48: ...this one side [...] front [...] If I hold it like this [...] it is longer than the other side. Actually, the horizontal side is longer than vertical. It depends how you hold it. I have it along, horizontally to me ...

R49: And under it is bigger packet which has shape of [...] it is also not the shape of cube [...] but it is shape of [...] what can I compare it to? It is shape of cuboid. Also the upper packet has had this shape. Yes [...] it is cuboid.

2. According to some similar experiments (Littler, Jirotková 2004) when authors observed sighted children in process of tactile manipulation with solids and their verbal communication, this analysis help us also to construct the process of building structure of geometrical knowledge or even the process of creating new knowledge by extending the existing structure or its restructuring.

3. At the first glance we saw the difference, while models of sighted pupils were large, the models of non-sighted were "small", tight, and all objects were close to each other. The reason for it might be on one side the necessity of the control of the model by hands, on the other side also the lack of experience with metrics. The other point is related to the estimation of distance and measure. It is shown in the protocols that non-sighted pupils compared the measures to their body.

R7: That cabinet is high about [...] something more than knees or like my thighs.

It could be meaningful to think about the usage and application of English system of measurements instead of metric system in their case.

4. Both sighted and non-sighted pupils built quite exact model of the explored room and thus, as regards the mental representation of the space, the world of non-sighted is not different in comparison with the one of people who are sighted. The difference is the way one gets information about the space. Through the sense of sight, one can obtain an overall knowledge of the environment, whereas one can achieve it through an analytic way, if s/he employs the haptic perception.

5. Except of some above mentioned proposals for future phase of the research we consider as interesting to observe the perception of the space and its object in connection with language as an individual tool. In what way the language and exactness of expression might influence the knowledge, but not only in the case of non-sighted pupils. The other improvement might be done in connection with realization of similar experiment with more pupils. However, we cannot influence the number of non-sighted pupils who will participate.

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An International Perspective between Problem Types in Textbooks and Students' Understanding of Relational Equality

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ABSTRACT: This study broadens the international knowledge base about second- and sixth-grade students' understanding of the equal sign and possible explanatory power of their textbooks affording insights about educationally relevant factors related to the presentation of the equal sign. Participants from China$_{(Beijing,\ Xia\ Peng)}$, S. Korea$_{(Jeju)}$, Turkey$_{(Istanbul)}$, and the U.S.$_{(central\ Texas)}$ (N = 1823) were administered a language-free instrument to determine their conceptualization of the equal sign. Textbooks used by each sample were coded for presentation of the equal sign. Results showed "operation equals answer" in S. Korean second- and sixth-grade textbooks (18%, 16%) was substantially lower than in Chinese (41%, 28%), Turkish (40%, 27%), and central Texas textbooks (54%, 18%), respectively. The achievement of students from Beijing and Jeju was substantially higher than the students from Istanbul and central Texas. The results from the equal sign instrument are discussed in relation to the presentation of equal sign in the textbooks.

Key words: *Equal sign, International comparisons, Elementary mathematics, Early algebra.*

THEORETICAL BACKGROUND

In this study, the connection between textbook use and conception of the equal sign is examined on a basis firmly rooted in international mathematics achievement and curricula comparisons. While international comparisons are useful as benchmarks to measure progress and change as new innovations are implemented, their most important application is to explore what mathematics is taught and how it is taught across countries. This study contributes uniquely to the literature by examining student performance in four countries on similar tasks identified from previous studies and then carefully examining a textbook from each country for how students encounter the equal sign. The U.S. has been commonly criticized for developing standards and curricula that are a "mile wide but an inch deep" (Schmidt, Houang, & Cogan, 2002, p. 3), and the U.S. has not been able to secure top rankings in international studies. In fact, U.S. students scored below most Asian countries such as China and S. Korea but higher than some other Organisation for Economic Co-operation and Development (OECD) (2006) countries such as Turkey on the *Trends in International Mathematics and Science Study* (TIMSS) in 2007 (Mullis, Martin, & Foy, 2008). For more than 30 years, researchers have examined students' understanding of the equal sign in the U.S., however, it has not been widely explored as a factor that may possibly influence large-scale international comparisons. This study built on previous findings by incorporating items used in previous research on second and sixth grade children's conception of the equal sign.

The purpose of the current study was to a) broaden the international knowledge base about second and sixth graders' understanding of the equal sign within various uses and sentence structures that allows comparisons to earlier findings, b) provide international comparisons of second and sixth graders from specific regions in China $_{\text{(Beijing, Xia Peng)}}$, S. Korea $_{\text{(Jeju)}}$, Turkey $_{\text{(Istanbul)}}$, and the U.S. $_{\text{(central Texas)}}$ referred to in the rest of the paper as Beijing, Jeju, Istanbul, and central Texas, and c) explore student textbooks (curricula) underlying students' understandings of the equal sign in each region of each country to add to the existing framework for how the equal sign is presented to students.

Two reasons for selecting second and sixth grades was first, relational symbols have been initially taught at the second grade across the proposed samples and examination of the second-grade textbook would indicate how the equal sign was presented and to facilitate comparisons to earlier findings (cf. Li, Ding, Capraro, & Capraro, 2008). Further, research indicated a U-shaped development of equivalence knowledge (e.g., 3rd graders do worse than 2nd graders) (McNeil, 2007). Given we have an indication that 2^{nd} grades perform well we decided it would likely yield powerful insights. The second reason was several studies have examined aspects of the equal sign at the second- and sixth-grade levels, and it was these studies that provided the framework for the instrument and the study design. The items presented in previously published studies formed the basis and theoretical framework for designing this study's instrument. More specifically, the instrument contains items that examine the two broad categories of how the equal sign is traditionally presented in the U.S. (standard) and non-standard presentations (hypothesized by some to yield a relational understanding of the equal sign). Using second- and sixth-grade samples allowed us to build on the work of others and for our findings to build the next iteration of the theoretical framework.

This study was informed by regional performance within a country based on its national performance on international comparisons. Because international comparisons use complex sampling techniques to ensure adequate and proportional sampling (PISA, 2002, 2006) across each nation, we expect our sampling within specific regions to result in rankings from this study to mirror those from international studies but we did not attempt to induct effects from this study to major international comparisons nor vice-versa. It was important to understand how textbooks might have influenced the development of students' understanding of the equal sign concept; therefore, an analysis of the textbooks used by the student participants was conducted.

Equal Sign

The concept of equality and the equal sign including students' misconceptions about the equal sign has been studied for over thirty years (Behr et al., 1980; National Council of Teachers of Mathematics [NCTM], 2000; Sáenz-Ludlow & Walgamuth, 1998; Thompson & Babcock, 1978). The 2006 Programme for International Student Assessment (PISA) (Baldi, Ying, Skemer, Green, Herget, & Xie, 2007) and 2007 TIMSS (Mullis et al., 2008) revealed that students from China, Hong Kong, Japan, and Singapore remained among the top achievers in algebra, which required understanding of the equal sign.

Studies have shown that students have difficulty in understanding the meaning of the equal sign symbol (Bernstein, 1974; Ginsburg, 1977; Hiebert, 1984; Kieran, 1981; Li et al., 2008). These investigations were paramount because understanding of the equal sign has been linked to future algebraic success (Knuth, Stephens, McNeil, & Alibali, 2006) and continued success in higher mathematics (Usiskin, 1995). From earlier studies it has been articulated that some experiences convey to students that the equal sign functions as an operator. That is, a signal to do something, similar to how it works on a calculator. Considering the equal sign as an operator places it in the same class of symbols as the addition, subtraction, multiplication and division signs instead of with other relational symbols such as the greater than (>) and less than (<) signs. This operational interpretation has been considered responsible for functional misconceptions among them, the one we term "running equal sign" e.g., 2+3=5*2=10-2=8. While the actual answer may in fact be correct, the representational form is inaccurate and laden with incorrect statements. This phenomenon was also termed "equality strings" (Knuth et al., p. 310) and combined into a bigger categorization referred to as process difficulties where students did not consider opposite sides of the equal sign to represent the same quantity (Capraro, Ding, Matteson, Li, & Capraro, 2007). The running equal sign sets up a contrasting condition (2+3=5+2=7) to the relational purpose of the equal sign where its role is to establish a "relationship" between what is contained on either side.

Weaver (1971, 1973) investigated several factors that influenced success on various open-ended number sentences and found that student response was lower for items in which the operation was on the right of the equal sign. Ginsburg (1977) noted, "Most often, sentences do ask children to perform a calculation; if so, why should they interpret them otherwise?" (p. 85). In a study by Rittle-Johnson and Alibali (1999) only 31% of fourth- and fifth-graders correctly solved problems such as 3 + 4 + 5 = 3 + ___. Additionally, other researchers have found low percentages of students who developed accurate understanding of the equal sign: 20% (Baroody & Ginsburg, 1983); less than 10% (Falkner, Levi, & Carpenter, 1999); 28% (Li et al., 2008); and 18% (McNeil, 2005). Thus, students possessed a limited understanding of the equal sign as an operator, that is, a signal for "doing something" rather than developing a relational understanding. In a slightly different study, Knuth et al. (2006) found that 32% of sixth graders offered a correct definition of the equal sign.

Current researchers (Li et al., 2008; McNeil, 2008; McNeil et al., 2006; Seo & Ginsburg, 2003) specifically pointed out students' understandings of the equal sign depended on instructional conditions which have come to be used regularly in describing how the equal sign is used during instruction with mathematical sentences. The phrase "standard context" is used to describe the most typical instructional condition in presenting the equal sign. A+B=C is the standard context where the operation is on the left side of the equal sign and the answer immediately follows the equal sign. This format was also referred to as "operation equals answer". Another form of standard context is the use of the equal bar, $\begin{array}{r}13\\ \underline{+4}\end{array}$ is in the same form of operations equal answer, however, here the line underneath takes the place of the equal sign. The second phrase is "non-standard context" which places the equal sign in some other orientation than operation equals answer. The non-standard context encompasses a broad range of possibilities, however, historically it was only disaggregated into two sub-categories: operations on the right (opposite of the standard context) and operations on both sides. However, today researchers recognize many variants and have refined the categorization to accommodate many forms. Seo and Ginsburg examined the use of two U.S. textbooks in which second graders typically saw the equal sign and related students' understandings of the equal sign to how it was presented. McNeil et al. examined four U.S. middle school textbook series and suggested that students' interpretations of the equal sign were likely to be shaped by how it was presented. Ironically, even teacher preparation books in the U.S. lacked a clear description and strategies for preservice teachers to use in teaching students the relational meaning of equality (Li et al.).

Textbooks

"Many teachers rely heavily on the material contained in adopted textbooks and supporting instructional materials when determining content coverage in their classrooms" (Phillips, 2008, p. 3). In fact, in elementary school mathematics classrooms, 78% of the participating teachers had their students complete textbook/worksheet problems (Malzahn, 2002). On a broader scale, Tornroos (2005) used an item-based analysis of textbooks and found moderately high correlations between textbook presentations of the concepts and student performance at the item level on TIMSS implying that teachers taught the content of their textbooks. As Reys, Reys, and Chavez (2004) stated, "The choice of textbook often determines what teachers will teach, how they will teach it, and how their students will learn" (p. 61). Further, this implied that an analysis of textbooks could produce useful data when looking for explanations concerning student achievement in mathematics.

Equal Sign Research on Students from China, S. Korea, Turkey, and the U.S.

The literature from each country identified that conceptual issues related to students' learning about the equal sign generally related to failing to understand that one side of an equation related to the other side (Choi, 2001 [Korea]; Ersoy & Erbas, 2002 [Turkey]; Fu & Wang, 2004 [China]; Kieran, 1981 [U.S.]; Research and Development Group of Mathematics Curriculum Standards, 2007 [China]). In fact, in three of the countries Korea, Turkey, and the U.S., students held an operational view of the equal sign at the elementary (Choi, 2001; Choi & Pang, 2008; Ersoy & Erbas, 2002), middle, and high school levels (Capraro et al., 2007; Do & Choi, 2003; Lee & Kim, 2003). Kieran (1981) pointed out that U.S. students believed that the equal sign was a "do something" signal, while Falkner et al. (1999) showed that students solved 8 + 4 = __ + 5 with 12 or 17. In Turkey and the U.S., conceptual misunderstandings with equalities were found to be related to students' thinking of equalities in standard context (Ersoy & Erbas, 2002; Li et al., 2006; Yaman, Toluk, & Olkun, 2003). It was reported in Turkey and the U.S. that students who possessed an operational view of the equal sign also believed that when an equation was written with the operation on the right side that it was reversed or "written wrong" (Baroody & Ginsburg, 1983; Dede, Yalin, & Argun, 2002; Diyifanwen, 2008; Nanchang Education Bureau, 2005). It was also reported in these two countries that students had difficulty when presented with equations with operations on both sides. In Turkey, students with an operational view of the equal sign considered the other side of the equal sign to be zero when nothing was provided on that side (i.e., 2 + 5x = ?; x = -2/5) (Dede, Yalin, & Argun, 2002). Moreover, students might have been predisposed to think of equality as calculating answers, and this kind of misconception could persist until they received direct instruction (Baroody & Ginsburg; Capraro et al., 2007). Across all countries when students had an operational view of the equal sign, they possessed one or more of the following equal sign misconceptions: the answer immediately follows the equal sign, an operation cannot be on the right side of the equal sign, the expression on one side of the equal sign is not related to the expression on the other side, or patterning strategies always provide suitable solutions. A patterning strategy is one where students can have success by finding a pattern and simply replicating the pattern, for example, 4+5+6= 7+<u>8</u>.

Research in each country showed that textbooks may account, at least in part, for the equal sign misconception. Textbooks have been reported as generally presenting problems that required an answer immediately following the equal sign, easily leading to over generalization (Choi, 2004; Ersoy & Erbas, 2002; Li et al., 2006; McNeil et al., 2006). Consequently, research in each country suggested various instructional strategies to address this curricular deficiency. The presentation of various problem types was suggested, for example, 25 = ___ + 8, 50 = ___ + ___, and 48 = 40 + 8 = 35 + 13 (Choi; Development Group of Mathematics Textbooks, 2001; Kim, 2002; Knuth et al., 2006; Mc Neil & Alibali, 2005; Xie, 2002).

In Turkey and Korea research supported the use of manipulatives to build a more robust understanding of the equal sign. Students who used manipulatives to solve equations overcame their limited understanding of the equal sign as an operational symbol (Yaman, Toluk, & Olkun, 2003). In addition to the importance of symbolic representations of the equal sign, research in S. Korea showed that students tended to better understand the concept of the equal sign through concrete manipulative activities (Choi, 2004; Choi & Pang, 2008; Do & Choi, 2003).

The synthesis of the independent literature from each country showed that students exhibited similar manifestations of the equal sign misconceptions in each country. While the literature from China lacked any direct connection to the misconception, the literature was specific about processes and procedures for ensuring students' accurate internal representations for the concept of equality. The common threads across countries were that students (a) viewed the equal sign as an operator, (b) interpreted equations with operations on the right side only as being structurally incorrect, (c) performed the operation on the left side and inserted the answer into the blank on the right, and (d) believed the equal sign did not relate one side to the other. Thus we identified differences in textbook treatments of the equal sign in these regions that would allow us to comment on pressing conceptual issues that have emerged from prior research and form the basis for the following questions: (a) How do central Texas second- and sixth-grade students interpret the equal sign as compared to their peers in Beijing, Jeju, and Istanbul? and (b) How do country specific textbooks treatments potentially influence student understanding of the equal sign?

METHODOLOGY

Instrumentation

Two different equal sign tests (ESTs) were developed to measure second and sixth graders' understanding of the equal sign respectively. Both tests included two classification matching items with no equal sign intended to evaluate students' basic arithmetic knowledge in order to eliminate students who did possess basic proficiency with arithmetic operations, thereby preventing the confounding effect of arithmetic deficiencies in the assessment of understanding of the equal sign. The classification items were not used in the analysis. To prevent redundancy, the test items and not the matching items were presented along with the results in Tables 3 and 4 for second and sixth graders respectively. For analytic purposes item 8 on the second grade test and item 9 on the sixth grade were disaggregated into component parts, which yielded 11 and 13 items, respectively.

It was important to remove language as a possible mediator of the obtained results; therefore, the tests were designed to be language independent, thus no words were used in any of the mathematical tasks or directions. For the two classification matching items, students were instructed to match the quantity on the left with the same quantity on the right. The spoken directions for the main portion of the test were to complete each item by putting the appropriate number or numbers in the blanks provided. The items represented presentations of the equal sign that have been used in previous research on the equal sign such as operations on both sides (i.e., 4 + 2 + 3 = 4 +__), operation on the left or right side only (i.e., 4 +__= 5 or 7 =__ + __), or reflexive, an equality without any operations (6 =__). These three forms are those that were used by other researchers who have examined student understanding of the equal sign and can readily be seen on standardized tests.

The items on the test could have been solved either using quantitative computations (using the numbers and operation to obtain equal quantities on both sides of the equal sign; 4+___ = ___ - 8) or qualitatively through reasoning (i.e., without needing to perform an operation in order to solve the problem). So when students solve a problem qualitatively we cannot make assumptions about their ideas of quantitative sameness. An example of qualitative solutions in the reflexive case, students could obtain the correct answer of 6 without performing a quantitative process by simply putting a six on the other side of the equal sign. Similarly, problems such as 3+5=5+___ could be solved by making both sides look alike (also referred to as a carry strategy, cf. Garber, Alibali, & Goldin-Meadow, 1998) without any quantitative processes or understanding of the equal sign. Qualitative items were added to the test for comparative purposes to previously published tests. Cronbach's alpha internal consistency reliability estimates for the second- and sixth-grade students in this study were .87 and .90, respectively. Internal consistency reliability estimates are constrained by score variance, therefore, homogeneity whether scoring all correct or any other possibility markedly diminishes the obtained estimates (Capraro & Capraro, 2003; Capraro, Capraro, & Henson, 2001).

Participants

The ESTs were administered to N = 699 second-grade students (Beijing = 169, Jeju = 152, Istanbul = 270, and central Texas = 108) and N = 1124 sixth-grade students (Beijing = 200, Jeju = 193, Istanbul = 334, and central Texas = 397). This sampling was not intended to characterize the performance of the country as a whole but the provinces or regions within a country. Participants were purposefully chosen to represent different SES groups, ethnicities, and genders, and to represent the composition of students at the second and sixth grades within the regions of interest (Gay, Mills, & Airasian, 2009). Therefore, the sampling technique was designed to allow generalization to students within a specific region of each country. Major demographics for each region were identified, and the samples were matched to the larger population. In central Texas, ethnicity, SES, and gender were critical. In Beijing and Jeju, SES was often controlled by urbanicity and there was only one ethnicity, therefore, only gender was important across these two countries. In Istanbul, there was one major ethnicity so SES and gender matched students to the population.

Students were sampled from schools within the region; however, some schools within regions chose not to participate. Choosing not to participate did not seem related to any factor important to this study. Because it was important to explicitly compare the sample to the population under consideration (American Educational Research Association, 2006; National Research Council, 2005a; Zientek, Capraro, & Capraro, 2008), we provided the comparable information by region composite. Sample and population demographical information for each region and ethnicity for the central Texas sample is displayed in Table 1.

Participants took the EST without prompting or assistance, and due to the algorithmic nature of the test there were no translation concerns for answering the items. All participants in the current study used the Arabic numeral system. Students were given unlimited time to complete the tasks. Students (n = 33; Beijing = 7, Jeju = 2, Istanbul = 12, central Texas = 12) were excluded based on their low scores on the two classification matching items.

Textbook Analysis

Mathematics textbooks, issued to students, from each country in both second and sixth grades were coded page by page to determine the different ways for how equal sign tasks were presented. The textbooks coded for this study were those adopted for use by the student participants. In Jeju and Istanbul, the public school textbooks selected for coding were the only ones approved by the Ministries of Education aligned with each national curriculum and used by the students in the sample. The S. Korean textbooks coded were Mathematics 2-ga (Ministry of Education and Human Resources Development [MOE&HRD] 2000a), Mathematics 2-na (MOE&HRD, 2000b), Mathematics 6-ga (MOE&HRD, 2002a), and Mathematics 6-na (MOE&HRD, 2002b). The Turkish textbooks were Ilkogretim Matematik Ders Kitabi 6 [primary mathematics textbook 6] (Eden, 2006) and Ilkogretim Matematik Ders Kitabi 2 [primary mathematics textbook 2] (Ozgun, Pektas, & Serficeli, 2007). The textbooks that were used by the second- and sixth-grade Beijing students and analyzed for this study were Peoples Education Press (PEP) Elementary Mathematics Book (Lu & Wang, 2005) and Jiang Su Education Press (JSEP) Elementary Mathematics Book (Su & Wang, 2005). All of the central Texas second-grade students in this study used the Scott Foresman-Addison Wesley (Charles, Crown, & Fennell, 2007) text reported as being the most popular second-grade textbook (Malzahn, 2002). All of the central Texas sixth-grade students in this study used the Holt, Rinehart and Winston (Bennet et al., 2007) textbook and was reported as being the most widely adopted textbook in major markets. The coding was divided into two main categories – standard and non-standard contexts. These categories were explicated in previous research by Knuth et al., 2006 where problems in standard contexts led students to view the equal sign as an operator (place the answer in the blank or box) in contrast to those in non-standard context which conveyed a relational meaning of the equal sign that encouraged students to try and balance both sides of the equal sign. Traditional problems, *operation on left side* and the missing number on the right of the equal sign only (2 + 3 = ___) and equal bar, i.e., $\begin{array}{r}11\\+3\\\hline 14\end{array}$, were considered as standard contexts for the equal sign. All other presentations of

Table 1
Percentages for Samples by SES, Gender and Ethnicity and Comparison to the Population

		Females		Males		SES		Asian		Black		Hispanic		White	
		Sample	Pop	Sample	Pop	Sample Low	Pop Low	Sample	Pop	Sample	Pop	Sample	Pop	Sample	Pop
Grade 2	Beijing	48	47	52	53	42	40	*	*	*	*	*	*	*	*
	Jeju	50	51	50	49	27	29	*	*	*	*	*	*	*	*
	Istanbul	47	50	53	50	68	66	*	*	*	*	*	*	*	*
	central Texas	53	51	47	49	56	71	7	6	32	34	38	37	23	24
Grade 6	Beijing	47	46	53	54	52	45	*	*	*	*	*	*	*	*
	Jeju	45	48	55	52	23	26	*	*	*	*	*	*	*	*
	Istanbul	51	51	49	49	56	58	*	*	*	*	*	*	*	*
	central Texas	46	50	55	50	70	68	2	2	37	38	40	39	22	22

Note. A = Asian, B = Black, H = Hispanic, and W = White. The population represents 2nd and 6th grade students who attended public schools (2007-2008) in the same regions as the sample. * not represented in the sample or the population there was a consistent decrease in the percentage of use of the equals bar in S. Korean, Turkish, and central Texas textbooks and a noteworthy increase in no explicit operations on either side, operations on the right side only, and operations on both sides presentations. We also observed that the S. Korean textbooks did not use the name part of the operation context at either grade level. Filling in missing numbers was the only context that had a recognizable increase in S. Korean textbooks from grade 2 to grade 6, whereas, there was a substantive decrease in the frequencies of the same context from grade 2 to grade 6 in Beijing, Turkish, and central Texas textbooks. It is important to note that S. Korean textbooks had the largest increase in the operations on both sides from second to sixth grade (17 percentage point increase) while the central Texas textbook had a .19 percentage point increase. The instances of operations on both sides in Turkish textbooks increased 11 percentage points.

Equal Sign Test Analysis

Each task on the language independent EST was scored dichotomously, either correct or incorrect. One item (__+3 = 5 + 7 = __) on both the second- and sixth-grade tests, numbers 8 and 9, respectively, had two parts because each blank required a unique answer. Thus in total there were 11 possible correct answers on the second-grade test and 13 on the sixth-grade test. The first two items on each test were not part of the analysis.

Because on international comparisons the U.S. outperformed Turkey, one might expect central Texas to outperform Istanbul on the EST, but the opposite was true. The means for the second (n = 270) and sixth (n = 334) grade Istanbul samples were 6.22 (SD = 2.88) and 9.36 (SD = 3.75), respectively. The means for central Texas second (n = 108) and sixth (n = 397) graders were 4.94 (SD = 2.90) and 8.31 (SD = 4.05), respectively.

The results were more interpretable in pictorial format and easier for readers so graphic displays of confidence intervals (CIs) were used (e.g., Capraro, 2005; Sullivan, 2001). Figure 1 shows there were statistically significant differences between samples for the second-grade EST scores for Beijing and all others, Jeju and both Istanbul and central Texas, and between Istanbul and central Texas (see Cumming & Finch, 2001 for interpretation of CIs). The variance or measurement error for the Beijing sample was much smaller than for the other three regions. The variance was similar for both the central Texas and Jeju samples. The effect size estimates were calculated using Istanbul as the control group (baseline) because international tests consistently placed Turkey below the other three countries in this study. The Cohen's *d*s for the second-grade samples were 1.89 (Beijing), 0.27 (Jeju), and - 0.44 (central Texas). The 95% CIs for Cohen's *d*s for the second-grade samples were [1.60 – 2.18], [-0.03 – 0.57], and [-0.78 – -0.11] for Beijing, Jeju, and central Texas, respectively.

Table 2

Percentage of Equal Sign by Presentation in Beijing, S. Korean, Turkish and Central Texas Textbooks: Grades 2 & 6

	Grade 2				Grade 6			
	Beijing	S. Korea	Turkey	central Texas	Beijing	S. Korea	Turkey	central Texas
Standard Contextual Presentation	**40.55**	**18.29**	**39.60**	**54.29**	**28.20**	**16.43**	**26.86**	**17.84**
Operation on Left Side Only (e.g. 3 + 15 = $\underset{11}{\_\_\_}$)	40.55	5.19	29.37	27.60	28.20	7.96	21.84	14.92
Equivalency Bar (e. g. $\frac{+3}{14}$)	*NA*[a]	13.10	10.23	26.69	*NA*[a]	8.47	5.02	2.92
Nonstandard Contextual Presentation	**59.45**	**81.71**	**60.40**	**45.71**	**71.80**	**83.56**	**73.14**	**82.17**
Without Equal Sign (e.g. 7 + 3; match to an equivalent quantity)	31.85	65.08	2.20	2.93	15.65	16.22	14.72	34.80
Name Part of the Operation (e.g. 4__4 = 8; place a + sign on the line)	5.20	0.00	0.88	1.25	0.00	0.00	0.40	0.00
Using Arrow to Connect (e.g. 7 --→ 3 + 4)	7.50	7.92	15.61	2.00	1.65	6.73	2.75	4.84
Filling in Missing Numbers (e.g. 5 + ____ = 9)	5.90	3.69	24.60	17.91	7.25	18.77	0.73	0.96
No Explicit Operations on Either Side (Reflexive) (e.g. 12 inches = 1 foot; 150=__)	0.70	0.88	8.20	2.64	12.45	16.94	27.59	22.09
Operation on Right Side Only (e.g. ___ = 7 + 9)	0.15	0.00	0.18	2.89	3.90	4.49	9.14	7.63
Operations on Both Sides (e.g. 6 + __ = 7 + __)	1.80	1.06	0.97	1.86	22.75	18.37	12.14	2.05
Use/Insert Relational Symbols (e.g. 6 ___9; insert <, >, or =)	3.50	0.88	2.20	7.63	8.15	1.33	4.21	4.70
Verbal Representation (e.g. 7 + 3 *is the same as*___; 2 x 5 –possible solution)	*NA*[a]	2.20	5.56	6.60	*NA*[a]	0.71	1.46	5.10

Note. Chinese book coding results are from Li et al. (2008), [a] *These categories were not represented in Li et al.*

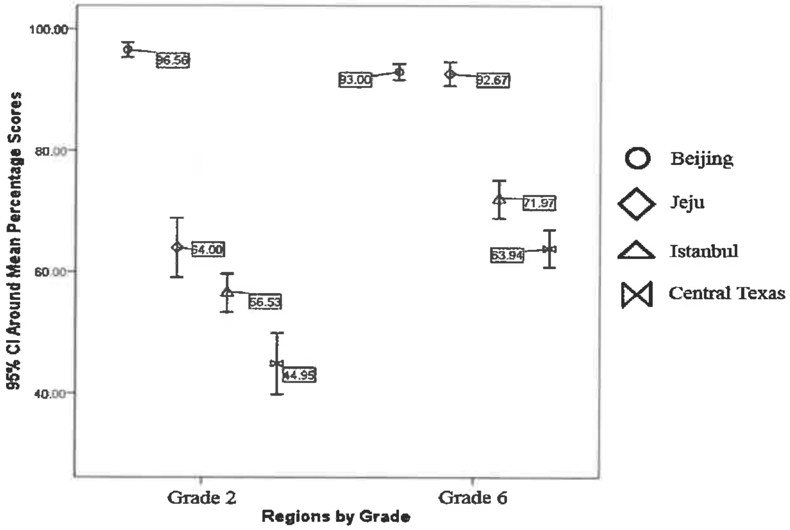

Figure 1. 95% CI for second- and sixth-grade students' percentage scores on EST by region.

Figure 1 also shows there were statistically significant differences between the sixth-grade samples from Beijing and both Istanbul and central Texas, and Jeju and both Istanbul and central Texas, but not between Beijing and Jeju on the EST. There was a statistically significant difference between the Istanbul and central Texas samples. The variances were similar for the samples from Beijing, Istanbul, and central Texas. For sixth grade, the Cohen's *d*s were 0.89 (Beijing), 0.85 (Jeju), and -0.27 (central Texas). The 95% CIs for Cohen's *d*s for the sixth-grade samples were [1.10 – 0.57], [1.12 – 0.56], and [-0.05 – -0.49] for Beijing, Jeju, and central Texas, respectively.

Item-based results for the EST are reported below in the following categories: standard, operations only on the right, operations on both sides, and carry/pattern strategy. The results of grade 6 and grade 2 are summarized in Tables 3 and 4 for all presentations of equality.

Second grade, item 6 (4+ __ = 5) had the highest success rate across all samples (see Table 3). Item 9 on the sixth-grade test and item 8 on the second-grade test (__+3 = 5+7 = __) consisted of two equal signs connecting two equations, containing two presentations in one question: (a) operations on both sides and (b) standard context. Only 26% of the second-grade central Texas sample answered the item correctly, with 17% of those who answered it incorrectly correctly putting a 9 in the first blank, but 70% of those who got it wrong correctly placed a 12 in the second blank. When considering only the standard context in item 9 on the sixth-grade test, it had high correct response rates from every sample (see 9b in Table 4), which was aligned with results from our textbook analysis. The correct response rate for the standard context part of item 8 on the second-grade test was also high across all countries. Similarly, the majority of sixth-grade students in this sample across the four countries answered the question in standard context (i.e., 47 + __= 63) correctly (i.e., the percentage of correct answers was above 89% in each country) as displayed in Table 4.

Table 3
Grade 2 Percent Correct Item Analyses by Region

Items	Context	Sample			
		Beijing	Jeju	Istanbul	central Texas
1) 3+5 = 5+ __	NS	97	75	78	66
2) 4+2+3 = 4+ __	NS	95	61	40	27
3) 3+ ___ = ___ +1	NS	97	56	56	36
4) 6 = __	NS	95	67	77	65
5) 8+3 = __ + 5	NS	92	49	36	18
6) 4+ __ = 5	NS	99	93	80	86
7) 8+3 = 5 +__	NS	95	54	49	26
8) __+3 = 5+7 = __	NS	88	41	17	12
[a] _*_ +3 = 5 + 7 =__	NS	92	43	30	17
[b] _ +3 = 5 + 7 =_*_	S	94	78	70	70
9) 7 = ___ + ___	NS	97	55	49	43
10) 3+5 = 4+ __	NS	97	62	49	33
Overall Total Correct		74%	22%	8%	8%

Note. [a] considers only the first portion of item 8, [b] considers only the second portion of item 8

NS = nonstandard presentation; S= standard presentation.

Table 4
Grade 6 Percent Correct Item Analyses by Region

Items	Context	Sample			
		Beijing	Jeju	Istanbul	central Texas
1) 13 + 51 = 51 + __	NS	95	93	75	69
2) 6 + 3 + 7 = 5 + __	NS	93	89	67	63
3) 8 + ___ = ___ + 7	NS	96	97	77	70
4) 160 = ___	NS	93	100	82	70
5) 15 - 7 = __ + 5	NS	96	92	61	49
6) 6 x __ = 40 - __	NS	88	92	59	51
7) 47 + __ = 63	NS	95	96	91	89
8) 15 - 7 = 5 + ___	NS	95	91	69	60
9) __ + 3 = 5 + 7 = __	NS	84	86	55	43
[a] __ + 3 = 5 + 7 = __	NS	89	87	60	48
__ + 3 = 5 + 7 = [b] __	S	91	93	86	83
10) __ + 5 = 2 x 8	NS	93	95	79	70
11) 13 + 51 = 24 + __	NS	93	94	71	58
12) 8 = __ - 8	NS	94	86	60	52
Overall Total Correct		49%	60%	28%	16%

Note. [a] considers only the first portion of item 9, [b] considers only the second portion of item 9

NS = nonstandard presentation; S= standard presentation.

Operation on the right side only. Students from Beijing did not have apparent difficulties with this presentation at the second or sixth grade level. Istanbul, and central Texas students' performances were low on this item at both grade levels. Although the percentage improved from second to sixth grade, this item remained as one of the most difficult at the sixth-grade level for these two countries.

Operations on both sides. Operations on both sides were found to be an effective presentation for promoting relational understanding of the equal sign (Carpenter, Franke, & Levi, 2003; Kieran, 1981). At the second-grade level, the correct response rate for the operations on both sides was low within each country respective to other item types for that country. Although the second-grade Beijing sample achieved better than the other samples, their achievement for operations on both sides when not including carry pattern items was lower than that for other problem types. The Istanbul and central Texas samples had apparent difficulties with the operations on both sides presentation. At the sixth-grade level, the Beijing and Jeju samples had the highest correct answer rates. The average performance for the Istanbul and central Texas samples on operations on both sides was low although better than their average performance on the operation on the right side only.

The difference between items 5 (8+3 = __ + 5) and 7 (8+3 = 5 +__) for second grade was the blank was moved from immediately following the equal sign to following the operator. Performance on item 7 was better than on item 5, either because of positioning of the equal sign or because of item ordering providing an important experience. While doing a more fine-grained analysis of specific incorrect solutions within the second-grade sample, it was discovered that (Beijing, 33%; Jeju, 44%; Istanbul, 52%; and central Texas, 47%) showed similar rates of the equal sign misconception by placing an 11 in the blank for item 5.

Carry/pattern presentation. To determine if students were more likely to answer items correctly that did not require them to compute quantitative sameness, the items that comprised this group were analyzed again in addition to their respective group analysis. The second-grade test items 1, 2, 3, and 4 and items 1, 3, and 4 on the sixth-grade level test could be answered correctly using a carry/pattern strategy. The carry/pattern items 1, 3, and 4 at both grades had some of the highest correct answer rates within each country. On item 2 of the second-grade test (4+2+3 = 4+ __), none of the Beijing sample, 12% of Jeju, 22% of Istanbul, and 36% of the central Texas samples placed either 2 or 3 in the blank possibly applying a carry/pattern strategy simply placing a 2 or 3 in the blank because there was one on the other side. However, across all samples, only one student answered with 2+3 in the blank (carry). On the third item (3+ ___ = ___ +1), students could have simply used a carry/pattern strategy simply placing a 1 on the left and 3 on the right because of the pattern or because there was one on the other side. Because no students chose to solve the problem using some other solution strategy, it is not clear if a carry/pattern was being used or if students would have been able to use some other numbers to make the sentence true such as, 2 and 4.

The fourth item on both the second- and sixth-grade tests could have been solved using either a qualitative or quantitative approach. At the sixth-grade level, the Beijing sample (26%) had the highest percentage of quantitative answers for 160 = __ (e.g., 80x2, 40x4, 160/1), followed by Jeju (18%), central Texas (14%), and Istanbul (10%). Of all the second-grade participants only one student from Beijing provided a quantitative answer for 6 = __ (i.e., 7-1).

To determine if there was a difference in the items hypothesized to be able to be answered qualitatively - that is without the need for computation, and quantitative items - ones requiring computation, the correct response rates were compared. Only 10% of sixth-graders in the Istanbul sample and 14% in the central Texas sample who answered all qualitative items correctly scored at or above the 25th percentile for the quantitative items. However, all of the Beijing and all but two of the Jeju students who answered all the qualitative items correctly were at or above the 25th percentile on the quantitative items. This indicates that student performance was different when students could use some form of patterning to answer the item (qualitatively) as compared to when they had to use some form of computation to solve the item. The second-grade results in Beijing and Jeju were similar to their sixth-grade results. In comparison the central Texas percentage dropped to 5%, whereas, in Istanbul it increased to 12%.

CONCLUSIONS AND DISCUSSION

In thinking meta-analytically about how the results from this study fit with those of other studies, one must consider the outcome effect (NRC, 2005b). Similarly, when researchers examined studies about the effect of cigarette smoking and longivity the studies differed in quantity, duration, and mediating and moderating variables – however, the point was clear a 2% effect across studies. So it is essential to examine the effects of studies purporting to report the effects related to the study at hand. The Texas sample did not perform as well as the international comparison groups, it is also evident that the magnitude of understanding the equal sign is not as high (second grade: 8%, sixth grade: 16% answering all items correctly) as other reported estimates (cf. Baroody & Ginsburg, 1983 [20%]; Falkner et al., 1999 [<10%]; McNeil, 2005 [18%]; Rittle-Johnson & Alibali, 1999 [31%]). These results are more modest than contemporary research studies showing improved understanding (cf. Knuth et al., 2006 [32%]; Li et al., 2008 [28%]).

Important differences emerge related to what and how textbooks might present the equal sign. The Beijing textbook uses operations on both sides nearly twice a much at six grade than the Istanbul or central Texas textbooks. Commensurately, in the Beijing text both without an equal sign and insert a relational symbol are decreased by six grade, both of which have been linked to equal sign misconceptions. Perhaps the relationship between decreasing the use of implied operations or implied equivalence and bolstering the use of operations on both sides of the equal sign have bolstered performance in both of the two higher achieving samples. Then it would be wise for textbooks publishers in the U.S. to decrease instances where students are expected to compute without the use of the equal sign, and insert the correct relational symbol (implied operations) in favor of operations on both sides of the equal sign. Regardless of the potential outcome for dealing with the equal sign misconception, removing most if not all of the instances of asking students to compute and answer without the use of an equal sign would be a benefit to students when they transition to formal algebra where they learn that equations and expressions are two very different things and removing expressions from both the second and sixth grades where they learn to compute from them might lead to unexpected benefits later.

Equal Sign Misconception

Among various equal sign presentations, operations on both sides was the most effective for understanding equality (Carpenter et al., 2003; Kieran, 1981). This presentation transmits the relational meaning of the equal sign more effectively than the standard or non-standard presentations and allows students to equate two arithmetic identities (e.g., 6 + 3 + 5 = 7 + __). The equalities that have an operation on one side and a single number on the other may emphasize operational understanding of the equal sign (i.e., do the operation). Even though the operation on both sides presentation supports the relational understanding of the equal sign, the absence of emphasis on this presentation in textbooks may result in students' failing to develop the appropriate understandings (McNeil & Alibali, 2005).

Recently, McNeil (2008) found that the arithmetic presentation of the equal sign hindered student performance; "Given the same verbal lessons, children solved fewer math equivalence problems correctly on average after receiving lessons in the arithmetic condition than after receiving lessons in the [non-arithmetic] condition" (p. 1530). This terminology refers to the ways students balance both sides of the equal sign without having to perform arithmetic computations to arrive at admissible solutions. When considering our international comparisons and the textbook analyses, the samples from Istanbul and central Texas exhibit the lowest performance on the EST even though their textbooks present this reflexive (non-arithmetic) use of equality more frequently than textbooks used by the two samples. It is also important to note that the Beijing sample has the highest performance on the EST, but the textbook analysis shows that in second grade there is a greater percentage of standard arithmetic problem types as compared to the other countries.

It is possible that a greater frequency of textbook problems with operations on both sides of the equal sign may account for the stark differences between samples. Operations on both sides is one of the least frequent presentations in second grade textbooks in all participating countries. However, in Beijing, S. Korea, and Turkey, operations on both sides increases at the sixth grade level. Only in central Texas textbooks does this frequency remain consistent and lower than comparison textbooks.

The misconception that the answer follows the equal sign is evident in all second-grade samples although it is limited in the samples from Beijing and Jeju. Among the students who missed the item 8+3 = __ + 5, 33% in Beijing, 44% in Jeju, 52% in Istanbul, and 47% in central Texas placed 11 in the blank, displaying an answer follows the equal sign misconception. However, in Beijing only 8% of the students answered this item incorrectly and 33% of that group placed an 11 in the blank, a marker for the misconception. This same misconception is evidenced in the Istanbul and central Texas samples to a greater degree and with a larger effect.

Additionally, the coding scheme suggested from the extant literature and items on previous tests proved insufficient for a complete and satisfactory coding of the current textbooks. There are several reasons for why this might be. First, it is possible that previous items were focused narrowly on the position of the equal sign and not on the general idea of equality. Therefore, the need to consider items where students are expected to match equivalent representations without an equal sign were not considered. Secondly, most previous textbook analyses used a random selection of pages in textbooks for coding. While a random sampling remains a prudent way to examine large volumes, it is possible that items are missed where the idea of equality is being conveyed in the material. Finally, we believe that a major contribution to the literature is the expansion of the categories for coding and future instruments designed to examine the equal sign should include items representing each of the nine categories.

Emergent Factors for Future Research Consideration

There are three factors that arise as part of this study that we present here for thought in formulating future studies. The first factor, seemingly one of syntax, is the use of the blank. The blank is expected to hold the place for a single number and not to be a placeholder for an expression. However, one second grade student when answering the non-arithmetic reflexive problem, 6 = __, wrote 6 = 7 – 1, while a larger percentage of sixth grade students use an expanded notion syntax (160 = 100+60) in answering 160 =__. These relationships were scored correct as long as the equality was true. It was more likely for students to have answered most of the items correctly (88%) than not (11%) when they inserted an expression in the blank.

For the second factor, the non-arithmetic items yield limited insights about students' solution patterns. One would expect students who incorrectly answer most arithmetic items with operations on both sides to still be able to answer all the non-arithmetic items. However, the non-arithmetic items are not consistently the easiest items. In central Texas and Istanbul item 10 (__ + 5 = 2 x 8), on the sixth-grade test, an arithmetic item, was answered correctly by a greater percentage of participants than the non-arithmetic items 1 (13 + 51 = 51+___) or 3 (8 + ___ = ___ + 7). Less than one-quarter of these sixth graders who missed the non-arithmetic item 1 (13 + 51 = 51 + __), placed 64 in the blank. This suggests that the arithmetic solution was employed (i.e., 13+51 = 64; so 64 was placed in the blank) resulting in an incorrect answer of 64 as compared to the anticipated correct solution of 13 that would have been obtained through a non-arithmetic balancing solution. The over application of an arithmetic solution seems to be related to the equal sign misconception.

Thirdly, the use of the equals bar in the textbook analysis appeared as a conundrum. In this study, we consider the equals bar as part of the standard context because its basic interpretation is the same as the horizontal presentation of problems. That is, the equals bar is read the same way as the operation equals answer. However, another presentation of the equals bar is revealed in long division when students encounter multiple equals bars (e.g., $\begin{array}{r} 17 \\ 7\overline{)123} \\ -7 \\ \hline 53 \\ -49 \\ \hline 4 \end{array}$). In the long division process, the bar implies a running equal sign interpretation. However, long division is introduced after conceptualization of the equal sign should take place. One general understanding of the equals bar is a "command to compute" and is not intended to convey equivalence. However, in early grades, students encounter the vertical presentations of basic addition and subtraction problems that are read in the same way as horizontal problems reading the command to compute "bar" as equals. Might this nexus in mathematical development between simple vertical addition or subtraction problems and the "command to compute" in long division fuel students' equal sign misconception? Regardless, the implied misconception by the use of the equals bar in long division may be important and require further study to understand how students interpret the equals bar. This may explicitly link the equals bar to the emerging theoretical model.

LIMITATIONS

Our sampling techniques as compared to that of international samples differ in important ways. The "National Project Managers for international studies must identify appropriate stratification variables to reduce sampling variance when appropriate" (PISA, 2006, p. 6) and the operations manual describe the sampling technique as a standard PISA two-stage stratified cluster design (PISA, 2002). Therefore, the nature of the international design would provide a reasonably well-drawn sample that would represent the country. Thus, well-drawn samples from smaller areas of the country (regions) should mirror the national performance at the ranking level of analysis. We are not saying the smaller well drawn samples can be used as proxy measures for national sampling, what we believe is that at the ranking level –well drawn smaller samples should adequately mirror the national trend. Therefore, we only compare at the ranking level between our study and national samples. We report our estimates of effect based on the regions within country. It is important to note that small poorly drawn samples can be highly idiosyncratic and produce greatly fluctuating estimates of an effect.

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