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Ed. in Chief
Athanasios Gagatsis

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How to Promote Young Children's Mathematical Thinking?

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ABSTRACT: *On the basis of Vygotskij's cultural-historical activity theory of cultural learning and development, this article argues that the emergence of mathematical thinking is based on the introduction of tools for communication about mathematical aspects of reality, provided by adults or more knowledgeable peers in the context of young children's participation in everyday cultural practices. This mathematisation of young children's actions and utterances can be accomplished in early years classrooms by immersing pupils in a classroom climate that habitually supports questioning about mathematical aspects of reality, and by providing pupils with the appropriate communicative means that foster abstract thinking. The teacher plays an important role in theses process by taking part in children's playful activities and permanently spot for opportunities to engage children in new actions that may lead them to the required mathematical domains, tools and aims. The article discusses several classroom examples that illustrate these processes.*

Key words: *Young children, Mathematisation, Communication, Play, Abstraction, Schematization, Mathematical attitude.*

MATHEMATICS TODAY

Modern culture has become increasingly dependent on mathematical abilities of its citizens, not only in the sciences, but also in public and personal life. Understanding, for instance, simple diagrams in the newspaper and critically evaluating the conclusions drawn from them, require basic mathematical understanding of graphing, function, etc. People can't successfully take part in cultural practices without at least some basic mathematical proficiency. Modern states need an increasingly higher level of education in their citizens to keep up in international economic competitions and the innovating knowledge economy. Mathematics is part of this.

Mathematical abilities are basic exigencies for these cultural ambitions and it is not surprising that many states closely monitor the performance level of their students (TIMMS, PISA) and spend considerable (although varying) amounts of money in the innovation of mathematics education. In the wake of these developments and ambitions, policy makers and academics have advanced the idea that the improvement of mathematical culture requires strengthening of the foundations of mathematical thinking in early childhood education (see for example Backhouse, Haggarty, Pirie & Stratton, 1992; Hughes, Desforges, Mitchell & Carré (2000), Wright, Martland & Stafford, 2000; Wright, Martland, Stafford & Stanger (2002). Most of these approaches, however, focus on the teaching and acquisition of skills and strategies in individual primary school children (see also Siegler, & Svetina, 2006; Torbeyns et al., 2009). We may doubt, however, if this approach is applicable for very young children (2 – 5 year olds) whose thinking strategies have not yet developed to a high level and who haven't yet built up an idea of mathematising as an independent cultural activity. Moreover, this skills-based approach conceives of primary and secondary school mathematics as an individual endeavour of learning to find the correct answers to tasks and problems with quantity, number, variables and space.

In this article I will explore a semiotic activity-based approach to mathematics teaching and learning in early childhood. Starting out from Vygotskij's cultural-historical theory of human development and learning, we conceive of mathematics learning as a process of guided collaborative appropriation of means for communication about the quantitative, spatial and relational aspects of material or ideal worlds. Mathematics is seen here as a specialized communicative cultural faculty that helps people to participate in cultural practices (including –for some people- in the practice of mathematical experts). As I am mainly interested in this article in how this learning takes off in young children, I will confine myself to children from 3 to 8 years old.

A NOTE ON CONTEXTUALISATION AND DEVELOPMENT

The semiotic approach to mathematics education focuses on the development of mathematical meanings and implies due attention to the quality of the environments (contexts) in which meanings emerge and evolve. Since the 1970s mathematics education emphasized the context-based nature of learning processes. Like all meanings, mathematical meaning is dependent on context in order to get specific and comprehensible content. In the educational literature, however, not much attention is given to the clarification of the notion of context itself. Initially, and particularly so in the sciences and mathematics, 'context' was mostly seen as the real life *situation* in which learning was embedded. In my former analyses of the notion of context and contextualisation (van Oers, 1998a and b), I gave a reinterpretation of the notion of context on the basis of activity theory (Leont'ev, 1978), and I argued for "activity" as the real context for the meaning of actions, tools, objects. From this point of view, actions, tools, goals etc are imbued with meaning through their embeddedness in activities. However, 'activity' is just an abstract concept for psychological analysis of human behaviour. When we talk about real life learning processes, I would say today that concrete cultural *practices* are the contexts for meaningful learning processes.
However, access to practices always requires very specific prerequisites that have to be met before effective participation is possible and worth while. Mostly these prerequisites are determined by the practice's historically developed rules. Most of the time, a certain amount of knowledge and skills is required in order to be accepted as a participant in a cultural practice. Family education and schools play a very important

role in the development of the requirements for participation in specific practices. Families prepare children for participation in everyday cultural practices (like shops, restaurants, libraries, church, school etc). Schools may also contribute to the improvement of children's abilities to participate in cultural practices (like in learning to do simple arithmetic for shopping), but the school's major cultural task is preparing pupils for reflective participation in (specialized) practices that require specific knowledge, values, skill and interests.

Reasoning from this point of view, we can maintain that the development of mathematical thinking in ontogeny can only start off within the context of everyday cultural practices, which provide contexts for the meaning and relevance of mathematical operations and evaluations. In these contexts, mathematical words, expressions and meanings serve as tools for communication about the quantitative and spatial dimensions of reality. The improvement of mathematical thinking in this context then means the innovation and perfection of the mathematical tools (and their meanings) for communication about the mathematical aspects of that everyday practice. Mathematical education here implies guidance of pupils toward correct understanding and application of mathematical tools that may satisfy the needs that emerge in the everyday contexts (see Sfard, 2008 for further analysis of the relations between communication and mathematics).

Only gradually children can be prepared for participation (if ever) in specialized practices (mathematical, linguistic, historian, etc) by appropriating knowledge and skills that are deemed relevant and necessary for communication about (mathematical) meanings in specialized expert practices. Such preparation requires learning that takes into account the execution and communication rules of that expert practice (such as internal consistency, completeness etc), which are considered essential by the members of that expert community. The main fault of many curricula for primary school mathematics education is that pupils have been taken as participants in mathematical practices when these children don't have developed yet the mathematical interests and meta-cognitive abilities for successful participation in such expert practices. In such classrooms we find children confronted with assignments of formal calculations, or with forms of formal model-based reasoning that pupils should learn to master through instruction and practice, but which will not make much sense for their everyday lives. The transition from the development of mathematical thinking in everyday practices to developments in mathematical practices can only proceed on the basis of appropriation of tools and attitudes that have relevance for both everyday and later mathematical practices. Below I will give an example of such learning processes in young children (with regard to schematical representation) that may emerge under appropriate conditions in everyday practices and that can help making the transition to more strictly mathematical practices.

For the contextualisation of young children's mathematical learning we start out from the assumption that relevant mathematical aims can be achieved through guided learning in everyday contexts (like household, shops, museum, restaurant, factories, gardening, construction sites, etc). At this moment we don't have much knowledge yet about how specific mathematical interests and meta-cognitive abilities can be promoted in children in order to make the transition to participation in strictly focused mathematical practices. Maybe a differentiated answer is needed here, as some children may be able to make the transition in the upper grades of primary school, some may make the transition later or never at all. I will not go further into the details of this issue here.

MATHEMATISING YOUNG CHILDREN'S ACTIONS AND UTTERANCES

Although neurological research of the past decades has provided evidence that the human brain is wired to create mathematical tools for interacting with the environment (see for example Dehaene, 1997), there is no reason as yet to think that mathematical thinking will emerge automatically without social support. Penny Munn has nicely demonstrated that children between the ages of 3 to 5 do not naturally focus on number, but will do this when adults around them encourage them to do so and positively support this type of actions (Munn & Schaffer, 1993, Munn, 1998). The question then is, how to understand this process? How do we insert mathematical meanings into children's actions and language at a moment when the child hasn't yet developed mathematical objects, interests or intentions?

In one of his major works on child development Vygotskij has given a general description of the process of enculturation of children's actions (Vygotskij, 1984, pp. 222-227), beginning with the analysis of pointing gestures. In his view, the act of pointing is the outcome of an *interpretation* by another person of a child's (failing) action of grasping a distant thing. By interpreting the action of grasping as an act of pointing and acting upon this interpretation, the more knowledgeable other adds *new meaning* from another point of view to the child's original action. When the child finally understands this new meaning of the other person as pointing, then he has appropriated new cultural meaning.

Generalizing the stages in this process, Vygotskij points to three essential moments:

1. The stage of the spontaneous *action per se* (by itself), which may emerge unintentionally, or emerge from situational needs, field forces, habits, remembering, intuitions, etc.
2. The stage of the *action for the other,* which refers to the moment that another person interprets the child's actions from a cultural point of view; in this stage the adult may continue acting upon this interpretation and explicitly demonstrate the value of this meaning for the child, as happens when the object that the child has tried to grasp, is given to him.
3. The stage of *action for oneself*: the child understands the new meaning of his action as pointing and is aware of his ability to use it for the realization of his own (communicative) intentions.

As Tomasello (2008) has convincingly argued, the crucial moment in this process is the emergence of a shared intentionality, i.e. a joint attention for a shared object (in stage 2 in the description above). In this shared intentionality meanings (even from different points of view) can merge and enrich each other. Tomasello has elaborated a general theory of human communication from a similar point of view as Vygotskij's. However, it can be quite easily particularized for the communication about the quantitative and spatial features of reality, leading finally to the joint construction of *mathematical meaning* (see van Oers, 2000, 2002).

A few examples might be helpful here. In one situation a (Dutch) father was sitting on the ground in front of his girl child (about 18 moths old). They are rolling a ball back and forth and the child says "dee, dee, dee" (pronounced like in English "day-day-day"). The father gives this language back to the child (in Dutch language) as "een" (i.e. "one", uttered when he gently rolls the ball to the child) and "twee" (i.e. "two" when she tries to roll the ball back to him). After a while the child says "dee" each time a ball is rolled by the father or herself. Notice that in Dutch "dee - dee" sounds very similar to "een - twee". We have no reason to assume that the child initially indeed counted "een -twee" (but pronounced it wrongly). Presumably, the child was just practicing sounds, which was interpreted by the father as indicating the alternations with numbers. Interestingly, the child picks up the regularity to utter "dee" each time one of them rolls the ball. The initial seemingly meaningless act of the child got a cultural meaning for the first time through the verbalization of the counting words, inserted into the activity by the father. It is too far-fetched to assume that the child started counting here, but she experienced a very important characteristic of counting (the one – to – one correspondence between counting and objects to count, which is one of the fundamental counting principles, according to Gelman & Gallistel, 1978).

Another example with somewhat older children (5 to 7) goes like this: Children have piled up shoe boxes in the shoe shop they have set up in their classroom. One pile for mother shoes (12 boxes), one for father shoes (3 boxes), and another one for children shoes. The 7-year-olds have made a precise drawing of the boxes (for counting and book keeping reasons). One of the younger children (5-year-old) also wants to make a drawing, but probably for other reasons: there is no sign whatsoever, that the 5-year-old child is trying to count the boxes in the pile, or is matching the real piles with her drawing. There is only a rough phenomenal correspondence. One of the older children notices this child drawing, and offers help with counting. Obviously, this child *interprets* the 5-year-old child's actions on itself (drawing) as representational action (action for the other) and starts counting for the younger child, pointing at the individual boxes in the drawing with each counting word:

Seven year old child: One-two-three-four-five-six-seven-eight.....you have to draw some more.

Five year old child: (drawing 2 new boxes, and then looks at the older child).

Seven-year old child: OK, let's count again: one-two-three-four-five-six-seven-eight – NINE – TEN (emphasising the latter two counting words)....you have to draw two more"

Five year old child: (draws another box)

Seven year old: And another one

Five year old child: (draws another box)

Seven-year old child: OK, let's count again: one-two-three-four-five-six-seven-eight – nine (the child starts counting slower and with more emphasis on the counting terms) .... TEN .... ELEVEN .... TWELVE.....you see? !

Five year old child: Yeah, and now the father shoes, I can do that on my own.

In this example we nicely see how an action of drawing receives mathematical meaning as a result of another child's interpretation of this action as a mathematical act. A similar point of view was once also pointed out by Walkerdine (1988, p. 97), when she contended that the psychologist or teacher 'sees' mathematics *in* the activity in question. Through acting and interacting accordingly, the activity may become mathematical for the child as well. It is important to note that the original action doesn't need to be framed in mathematical intentions, but it should contain elements that can be taken by others as a starting point for mathematical activity that makes sense for the child. The example above also shows how mathematical notions of the older child served as means for communication about the quantitative aspects of the situation.

IMPROVING MATHEMATICAL COMMUNICATION IN EARLY YEARS CLASSROOMS FOR THE ENHANCEMENT OF PARTICIPATION IN CULTURAL PRACTICES

As we have seen above, young children's activities can be enriched with mathematical meanings, even if the activities did not start out from mathematical intentions. Mathematisation of young children's activities (including language activities) is one of the responsibilities of educators who try to help children to improve their abilities for participation in various cultural practices and to move from a peripheral participant towards more autonomous participation in those practices (Lave & Wenger, 1998).

In the Dutch classrooms following the Vygotskian educational concept of *Developmental Education* (see van Oers, 2003; in press), teachers get young children first of all involved in playful emulations of cultural practices. The curriculum is essentially a play-based curriculum that aims at helping children to attain the cultural tools of mathematics, literacy etc in meaningful ways, functional for their participation in the emulated cultural practices (van Oers, 2010a). In this approach, teachers participate in children's play and permanently spot for moments that open opportunities for the introduction of cultural tools (like mathematical concepts) into the children's activities, in such a way that these tools enrich children's play and makes sense for them. The children's activity in the emulated practices may from time to time arouse the need for special instructions on the use of some specific tool (like the use of counting, or dividing large quantities, etc). It is important, however, to emphasize that the Developmental Education curriculum is not an unstructured learning-teaching trajectory. Teachers committed to the Developmental Education view try to promote broad development in children, and with regard to mathematics this means that they try to lay a foundation for young children's mathematical development in a broad sense, including mathematical attitude development, appropriation of powerful tools for abstract thinking, and good operational mastery of the culturally relevant basic mathematical skills. I will give brief descriptions of each of these.

Fostering Mathematical Attitude

Following Vygotskij, it is important that teaching will result in learning outcomes that contribute to the formation of cultural identity in pupils, that is to say that learning must contribute to the person's abilities to participate meaningfully, self-dependently and critically in cultural practices. Persons must learn to account for their actions and utterances. With regard to mathematics, this means that a person at least should be able to explain or clarify the notions, procedures and evaluations he or she uses in communications about the mathematical aspects of reality (practices).

This requires a mathematical attitude which manifests itself in people's readiness to address the mathematical aspects of reality and deal meaningfully with them. From social psychology (see for example Ajzen & Fischbein, 2005) we know that the development of attitude is closely related to the values of a community, and particularly to the evaluations within that community of the type of behaviour of its members. A community culture that permanently attaches positive value to discourse on mathematical aspects of physical or cultural reality and positively values explanations of mathematical notions, provides a strong background for prompting a positive mathematical attitude in its members.

In the Development Education classrooms in the Netherlands, teachers incessantly foster an inquisitive culture that encourages children to explore and investigate, and to communicate with each other about their explorations. From the beginning the young children are ensured that it is good to ask questions and probe answers. One of the fundamental starting points of the Developmental Education-philosophy is the necessity of a climate in which children feel emotionally free and on their ease in order to maximize the chances that they will start to explore, speak themselves out, and open up for new possibilities of acting. With regard to early mathematics education, this means that children are encouraged to explore their situations and helped by the teacher (if necessary) to find interesting dimensions that eventually may lead to mathematical aims the teacher has in mind. One of the basis strategic questions that the teacher keeps asking is "Are you sure?" (or variation of this question). Such question often stimulate children to reconsider their actions, monitor each other, and discuss ways to find out how sure they could be about their answers and solutions (see van Oers, 1996 for further elaboration).

An example of this is demonstrated in the following: Two six year old girls have been building a farm with blocks in the corner of the classroom and they are placing miniature animals in the stables and in the adjacent paddocks. The teacher is joining them and asks questions like: what is this? Why did you do it this way? Then she shows amazement about the great number of blocks the girls used for the fences around the paddocks. "Do you know how many blocks there are?" One of the girls starts counting, and she has to recount several times as she gets confused about the numbers and blocks she has and has not yet counted. Finally she comes up with a number "35". "Are you sure about that?", the teachers asks. The girl starts counting again, but discovers a way to abbreviate the process, as the wall is constructed with every time three block on top of each other. And the girl goes: "three, ……six, eh…..nine, ..(it looks like she is counting the individual block perceptually), twelve……, fifteen,…….and then she goes back to an explicit counting strategy again ending up with 34. The other girl then says, "Hey,… that's a different number…Is it correct?" The teacher encourage the girls to figure out together how many blocks they have used. The girls start counting together checking each other and thus making sure they will find the right answer. In a later stage of this project, the girls will make a representation on paper of the numbers of the different blocks they have used (cubic, oblong, triangle) in their construction. This whole process of producing a diagram is based on counting and checking if they can be sure about the outcome. They discover that a vertical axis with the numbers on it is very useful: "we don't have to count all the drawn blocks again and again, but we can see it here!"

Of course, "Are you sure?" is just a metaphor symbolizing different ways teachers use to stimulate an inquisitive classroom, in which children feel free to explore and look for ways to convince each other and themselves that they have found the right answer. An inquisitive classroom culture is an important condition for fostering the development of a mathematical attitude based on self confidence and mastery of relevant mathematical tools.

Fostering Abstract Thinking

Communicating about the quantitative or spatial aspects of reality requires a focus towards the relevant objects. In the case of communication contextualised by everyday practices, it implies that children focus on quantity or spatial dimensions that are relevant for the message they want to transmit. In the case of communication contextualised by mathematical practices, it often implies a focus on mathematical objects or on the tools necessary for analysis and transformation of mathematical objects. As I announced above, this article concentrates on mathematical thinking in young children and will concentrate on children's everyday communication on quantitative and spatial issues. Part of the mathematical education of such young children is helping them to appropriate the tools for analytic communication on the mentioned real world aspects.

Thinking focused on one part of reality (at the expense of other parts) is often taken as a form of abstract thinking. As such we may conceive of mathematical thinking and communication as expressions of abstract thinking. However, young children are generally seen as concrete thinkers, who usually have problems with abstractions. An important question, therefore is: Is it possible to foster abstract thinking in young children in ways that make sense to them and that support their development of thinking consistently from one point of view?

Part of our studies with young children have been devoted to this question. Following the work of Davydov (1972) and Egan (1986) we found reasons to suppose that abstract thinking can be meaningfully promoted in young children when we find ways to give these children access to cognitive tools that help them focussing on specific parts of reality.

In our approach to abstract thinking, I redefined abstraction (following Cassirer, 1923. 1957) as a process of taking a specific point of view that gives a specific view on reality (or objects) while in the mean time neglecting others. Hence, viewing reality from the point of view of "roundness" organises perceptual reality in a way that makes round things pop up. In the classical (Aristotelian) concept of abstraction, roundness is thought to emerge as a common quality of different objects. In the new (Cassirer) conception of abstraction, roundness is conceived of as an abstraction that is *put into* reality (imposed onto reality) by the thinking individual through his decision to *take a specific point of view*. From this point of view it is clear that young children with the appropriate cognitive tools can indeed learn to view reality from a specific point of view, and – consequently- exhibit a form of abstract thinking.

In our previous research we found several pieces of evidence for the possibility to promote such abstract thinking in young children. Basically, language already provides a toolkit for children to categorise experiences and look at them from a specific point of view, and communicate about them accordingly. The child that talks about the play ground in his neighbourhood in terms of the excitement the play objects aroused in him, has made a first step towards abstract communication by taking this point of view and at the same time neglecting the colour of the objects, the presence of other children, the condition of the flowers, etc.

In our research we found out that children's drawings can serve as tools for increasingly specific communication. In different case studies with young children (see for example van Oers, 1997), we discovered that serious communication with children about their drawings can help them focusing on their communicative intentions and articulating what they were really trying to tell with their drawing. One of the means children of about 5 years old often use in their communicative concern to be as precise as possible, is the inclusion of written language or symbols in the drawing, in order to make sure that the viewer of the drawing indeed will "read" the intended meaning in it. Similarly, we and other researchers could also observe examples in which children start using numerals in their drawings in order to draw attention specifically to the number aspects in the drawing (van Oers, 1994; see also Carruthers & Worthington, 2006 for an unparalleled rich source of examples).

In a series of observational and experimental studies we used 5- and 6-year-old children's drawings to focus on the structural aspects of situations in their play. We asked, for instance, if the children could make a drawing of the railway track they had just made, a map of their play ground, a construction plan of the building they had just made etc. Through communication about these drawings in the children's play activity, the children discovered, for example, that it was unnecessary to draw all the sleepers of the railway track, if we just want to know the form of the trajectory or the number of stations along this track. Similarly, children discover that there is no need to draw the people or animals in a building, when the main aim is to guarantee that the building can be rebuilt later on. In that case it is important to make sure that the drawing indicates precisely how many blocks have to be put on top of each other when the building (a castle for example) should be made again on a later moment, or when we have to explain to other children how the castle should be constructed. Common feature in the examples above is that children take a point of view towards the situation and represent in a schematic way what they see from this point of view. In the previously given example of the shoe shop, the question "How many shoes do we have in our shop?" was such a point of view, which led to a series of specifications (what kind of shoes?) that could be represented in a diagram. Interestingly, through discussing the diagram, the children were also classifying, counting, estimating etc., and they discovered suddenly that there were many more "mama-shoes" than "papa-shoes". So the schematic representation was also a tool for them to explore the world from an abstract, mathematical point of view.

Given our conception of abstract thinking as a process of focussing on the basis of a specific point of view, children's activity described above can be interpreted as a form of abstract thinking (see van Oers, 2001; van Oers & Poland, 2007) in which children learn to take a point of view that focuses particularly on the relational aspects of reality. We assumed that improving young children's abstract thinking with the help of such schematic representations in the context of their everyday play, might also support children's mathematical activities, as representing and reflecting relations between quantities and positions is an important element of mathematical thinking. In an experimental study, Poland tested this hypothesis, and it turned out that children who had learned to read and make schematic representations in everyday contexts outperformed children without those experiences in early mathematical activities (arithmetic) (Poland, van Oers & Terwel, 2007, 2009; see also van Oers, 2010b). These outcomes give ground to the earlier mentioned assumption that schemes may be an example of the type of tools that facilitate the transition from mathematical thinking in everyday practices to mathematical thinking in mathematical practice.

On the basis of our research we assume that the promotion of abstract thinking through schematising activities in the context of children's everyday activities might be a fruitful way of preparing children for later mathematical activities (van Dijk, van Oers, & Terwel, 2003; van Oers & van Dijk, 2004). In our schools for Developmental Education it is now common practice to make schematic representation with children in their everyday play activities.

Fostering Cultural Relevance

One of the returning critical questions regarding a play-based curriculum in primary school is always: do the pupils learn enough? The children may be playing enthusiastically, but school is also (or rather: primarily!) for teaching children the necessary knowledge and skills that enable them to take part successfully in later studies and work in a society that is increasingly conceived as knowledge-based.

In the play-based mathematics curriculum this question is addressed by professionalising the teacher, rather than implementing a strict curriculum syllabus that prescribes how and when specific mathematical operations should be taught to pupils. Basically, teachers should have good understanding of mathematising as a problem solving activity, and master the basic mathematical concepts and operations. In a play-based curriculum, the teachers should be able to stage cultural activities in the classroom that engage pupils in meaningful ways, but also provide opportunities for the teacher to introduce mathematical concepts that can be acknowledged by the pupils as meaningful tools for the solution of the problems that emerge in their emulations of these practices (see again my shoe-shop example above).

In the play-based curriculum of Developmental Education, cultural relevance of the curriculum contents is considered important. Hence, the math curriculum should cover the necessary topics and assist children with appropriating the relevant mathematical operations and concepts. Fostering the cultural relevance of children's learning is a major aim in Developmental Education. For young children (until the age of 7-8) it is generally assumed that they should at least get good mastery of activities of counting, calculation with whole numbers, and measuring/geometry. The role of the teacher is considered essential here. In order to accomplish this educational responsibility in the classroom, the teacher can employ a number of pedagogical-didactical tools for planning learning trajectories, choosing content, evaluation of outcomes. For mathematics education a major tool for teachers is the so called 3D-model (developed by Fijma, 2003, *in press*).

The 3D-model is a tool that helps teachers to plan mathematics education in a play based curriculum, by making decision concerning the mathematical **D**omain (number/operations, measuring, geometry), the aims (in Dutch: **D**oel) to be focused on, and the **D**idactics to be employed in the context of an ongoing classroom activity (cultural practice). In the teacher's planning of a classroom activity, related to a cultural context like the post office, the teacher may for example decide to focus on the domain of *counting*, take as an aim counting *backwards*, and decide to introduce the *number line* as a didactical tool. Of course, with other children the teacher could make other decisions and decide to make another 3D-analysis. The teacher could, for example, decide to take *geometry* as a domain for development and teaching, which is relevant in relation to the functioning of the post office (delivery of post). In this case the teacher may take as an aim "*routes*" (e.g. finding the optimal daily round of the mail man), and use as a didactical tools *grid and squared paper*. The teacher plans his/her classroom activities with the 3D-analysis, and with the outcomes of this analysis in mind she/he tries to find (or create) moments in which children's activities can be meaningfully focused on the chosen domains, goals and tools. By consistently planning, evaluating and registering the mathematical learning trajectory of individual pupils, the teacher warrants that the mathematics curriculum covers all relevant topics.

For achieving the aims and providing pupils with cognitive tools that lay a foundation for further mathematical learning, it is also necessary that pupils repeatedly encounter the relevant operations, tools and concepts, in order to master them proficiently. In order to track the pupils' learning processes the teachers registers pupils' performances in individual diaries and decides to create enough opportunities for practicing the relevant operations. It is important to note here that the play-based curriculum does not prohibit paper-and-pencil practices or special instructions, as long as it remains clear for the pupils why this is important for their autonomous participation in the ongoing classroom practices. In any of these situations (practice or instruction) the teacher will carefully see to it that pupils are aware that they are appropriating tools, symbols and operations for improving their ability to *communicate* with others and with themselves about the numerical or spatial aspects of reality. Practicing for mere mechanical mastery of operations that result in the right answers is to be avoided.

CONCLUSION

Optimising mathematical development in citizens and fostering mathematical autonomy in diverse cultural practices, is a basic aim of modern education. From a Vygotskian point of view, it can be argued that successful (mathematics) education requires the integration of cultural meanings (relevant cultural contents) and personal sense. In the Netherlands we have tried to achieve this with the approach of Developmental Education, which is based on Vygotskij's cultural historical theory. Characteristically, this approach tries to get young children involved in various cultural practices that make sense to them and in which they encounter problems of number and space that can be solved by collaboratively (including the teacher!) improving the means for communication about these problems, their possible solutions and the tools available for successful solution. As demonstrated above, a properly conceived play-based curriculum can create optimal conditions for such learning in young children. Several empirical investigations have been conducted to support this claim (see for example van Oers, 2010b for a summary).

In our approach we have deliberately chosen to start the process of mathematics learning from participation and communication in everyday practices. We avoided the (in our view: wrong) assumption that young children can learn to do early mathematics on the basis of interest in mathematics as such, or interest in formal thinking as such. We may even doubt if participation in such genuinely mathematical practice (driven by mathematical interest and attitude, and focused on communication about mathematical tools for the analysis of mathematical objects) is a feasible goal for all students. We believe, however, that good levels of mathematical achievement can also be reached by proper learning of mathematics in relation to practical practices and problems. We have been gathering empirical evidence for this claim in the past decade showing that meaningful mathematics learning in higher levels of education can successfully start off from participation in meaningful everyday cultural practices (van Schaik, van Oers & Terwel, 2010; Terwel, van Oers, van Dijk & van den Eeden, 2009).

At any educational level, promoting mathematical thinking is based on meaningful communication between novices and more knowledgeable others about the tools that may help finding solutions to the problems that emerge in the practices that really matter to them.

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The Influence of Narrative and Depictive Elements in Solving Mathematical Word Problems Realistically

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ABSTRACT: *In this article we report a study on the effect of task authenticity on pupils' realistic problem solving. We expected to increase pupils' use of their real world knowledge by accompanying problematic word problems (P-items, i.e., items that are difficult to solve because reality has to be taken into account to come to the correct solution) with narrative textual elements, depictive pictorial elements, or the combination of both. 155 sixth grade elementary school pupils received a paper-and-pencil task with ten word problems six of which were P-items, under four different conditions. In the first condition the word problems were presented in their traditional form, in the second condition, the problems were accompanied with narrative elements, i.e., the word problem was modified into a short story with a real life event and question, in the third condition with depictive elements, i.e., a representational illustration, and in the last condition with a combination of narrative and depictive elements, which resulted in a comic strip. The findings showed that there was only a main effect of narration on the number of realistic reactions to the P-items. Neither the addition of depictive elements nor the combination of narrative elements and depictive elements had an effect.*

Key words: *Illustrations, Narrative elements, Realistic problem solving, Word problems.*

INTRODUCTION

Mathematics is an important aspect of daily life. Everywhere we go, we consciously or unconsciously apply mathematics, for example when shopping in the store, when cooking a recipe, when playing card games, etc. In mathematics education, these real life situations in which mathematics is needed have been brought into the mathematics classroom in the form of mathematical word problems, i.e., "verbal descriptions of problem situations, typically presented in a school context, wherein a question is raised the answer to which can be found by performing mathematical operation(s) with the numbers in the problem" (Verschaffel, Greer, & De Corte, 2000, p. ix). The most common argument is that, through these word problems, pupils learn that the mathematics taught at school is important and applicable in everyday life, and get practice in applying their mathematics in real world without having to leave the mathematics class. A considerable amount of research now has shown that word problems do not play this application function well (Gravemeijer, 1997; Lave, 1992; Nesher, 1980; Reusser & Stebler, 1997a; Schoenfeld, 1991; Verschaffel et al., 2000; Verschaffel, Greer, Van Dooren, & Mukhopadhyay, 2009). Pupils do not use their everyday knowledge when solving these mathematical word problems. Instead they approach these problems as artificial puzzles that have no connection with the real world and develop superficial strategies to cope with these problems, such as doing a calculation that is suggested by a key word in the problem statement without seriously considering the reality described in that problem statement. They also develop accompanying inappropriate beliefs, such as the belief that all mathematical problems are solvable by means of a precise numerical answer or that there is little or no connection between the world of mathematics and the real world.

THEORETICAL BACKGROUND

Realistic Problem Solving

In the nineties, two pioneering studies by Greer (1993) and Verschaffel, De Corte, and Lasure (1994) demonstrated the tendency of pupils to exclude realistic considerations when solving word problems in the mathematics class. These researchers made a distinction between standard word problems (S-items) and problematic word problems (P-items). S-items were defined as problems that are easy to solve by performing one or more arithmetic operations with the given numbers, and P-items as problems that could only be solved correctly by seriously taking into account the realistic constraints of the situation described in the problem. For example, the following S-item "A man cuts a clothesline of 12m into pieces of 1.5m each. How many pieces does he get?" can be solved by dividing 12 by 1.5m (= 8 pieces), whereas the correct solution of the P-item "A man wants to have a rope long enough to stretch between two poles 12m apart, but he has only pieces of rope 1.5m long. How many of these pieces would he need to tie together to stretch between the poles?" requires that one considers that the pieces of rope need to be knotted together to be stretched between the poles, and takes this realistic consideration into account into one's answer, e.g. by answering that anyhow more than 8 ropes are needed to connect both poles. In Verschaffel et al.'s study elementary school children from 10 to 11 years old were confronted which ten such S-items and ten P-items. Their answers to the P-items were coded as realistic reactions (RRs) or non-realistic reactions (NRs). A reaction was scored as a RR when the given answer was based on realistic reasoning, or when a non-realistic answer was given together with a remark that the problem cannot be solved or with another kind of (realistic) query. Non-realistic reactions without any such (realistic) comment were coded as NR. The analysis of pupils' answers to the P-items revealed that only 17.0% of all reactions to the P-items were RRs. The authors concluded that pupils tended to exclude realistic considerations when solving problematic word problems. This conclusion was confirmed in several replication studies conducted in different countries, including Belgium, China, Germany, Hungary, Japan, Northern Ireland, Switzerland and Venezuela (see Verschaffel et al. 2000, 2009), with strikingly similar findings.

Some researchers have claimed that these unrealistic reactions to P-items, and their underlying strategies and accompanying beliefs, are caused by the culture of the traditional mathematic class in which the pupils have been involved (Gravemeijer, 1997; Greer, 1997; Reusser & Stebler; 1997a; Wyndhamn & Säljö, 1997; Verschaffel et al., 2000). For example, Gravemeijer summarizes the traditional socio-mathematical classroom norm concerning the role of realistic considerations in mathematical word problem solving as follows: "Don't bother with reality, just focus on the mathematics" (1997, p. 393).

Interestingly, a few studies have provided evidence showing that pupils are more engaged in realistic thinking when problems are presented outside the culture of the mathematics class. DeFranco and Curcio (1997) embedded a P-item about a division with remainder in a more authentic setting. More specifically, pupils from the 6th grade were asked to make a telephone call to a bus company to order minivans to take sixth graders to a party. Prior to this task, the pupils had already been confronted with a similar buses item in the restricted in-school setting. The authors found that in the restricted school setting pupils hardly gave appropriate/realistic answers involving a situationally correct interpretation of the remainder, while nearly all answers in the more authentic setting were realistic. Reusser and Stebler (1997b) gave pupils from the 6th and 7th grade Verschaffel et al.'s (1994) P-items, together with concrete materials, in the form of performance tasks. For example, to solve the above mentioned rope item, pupils received pieces of rope, scissors and a meter-stick. The authors found that performance was better in the performance-based setting with the materials than in the traditional paper-and-pencil setting. Finally, Säljö and Wyndhamn (1993) asked students aged 15 or 16 years, to determine the postage rate for sending a letter of 120 grams, by means of the official table of the postage rates of the Swedish post offices. This problem was presented to the students either in a mathematics lesson or in a social studies lesson. It was found that in the mathematics lesson most students calculated their answer (typically by applying proportional reasoning), leading to realistically inappropriate answers, whereas in the social studies lessons students tended to read their answer in the postage table and appropriately abstained from calculating. So, as the authors reported, "the overall context in which the participants find themselves tends to determine their interpretation of the task" (p.332).

In the above studies, the P-problems were taken *out of* the culture of the mathematics class. The results showed that stepping outside the mathematics classroom was an effective way to make the word problems more authentic and to elicit more realistic considerations among pupils. In the present study we also tried to increase the authenticity of the setting, however, while *staying within* the cultural setting of the mathematics lesson, by adding narrative textual elements, i.e., modifying the word problems in rich story problems, and/or by adding depictive pictorial elements, i.e. by adding an illustration to the problem.

Narrative Elements

The first factor that we manipulated to increase the authenticity of the setting related to the formulation of the word problems themselves. Several researchers argued that word problems presented in the mathematics class do not reflect real life situations. Palm (2006), for example, commented that many of the traditional mathematics tasks are "'dressed up' with an out-of school figurative context" (p. 42). He therefore proposed a theoretical framework that indicates what aspects should be taken into account when composing or modifying word problems, to strengthen the concordance between real life situations and the mathematical word problems. As shown in Figure 1, Palm's framework includes the following aspects: *event*, *question*, *information/data*, *presentation*, *solution strategies*, *circumstances*, *solution requirements*, *and purpose*. A word problem can be analyzed in each of these aspects in terms of how good it reflects an experientially real situation for a pupil and how that aspect should be revised in order to make it more authentic. For example, the aspect *event* refers to the question if the event described in the problem could also happen in pupils' real life. Likewise, the aspect *question* relates to the issue whether the question at the end of the word problem can be a real question that pupils could encounter in their everyday life.

A. Event	F. Circumstances
B. Question	F1. Availability of external tools
C. Information/data C1. Existence C2. Realism C3. Specificity	F2. Guidance F3. Consultation and collaboration F4. Discussion opportunities F5. Time
D. Presentation D1. Mode D2. Language	F6. Consequences G. Solution requirements H. Purpose
E. Solution strategies E1. Availability E2. Experienced plausibility	H1. Purpose in the figurative context H2. Purpose in the social context

Figure 1. The aspects of real-life situations considered to be important in their simulation (Palm, 2006, p. 44).

So, according to Palm (2008), in order for a task to be more authentic, "it must represent some task situation in real life, and important aspects of that situation must be simulated in some reasonable degree" (p. 40). He claimed that when word problems have a higher representativeness, the proportion of RRs should be higher (2006, 2008). Palm (2008) investigated that claim by making the P-items (adapted from Verschaffel et al., 1994) more authentic by means of his proposed framework. He confronted 161 pupils of 11 years old with a written text with seven word problems, and afterwards interviewed them to gather further information about their responses. Eighty-two pupils had to solve the problems in their original form, i.e. as in the research of Verschaffel et al. (1994), while the other 79 pupils received the problems in a more authentic version. For example, the problem "Anton has bought 4 planks of 2.5 m each. How many planks of 1 m can he saw out of these planks?" was modified into a more authentic problem; "You are building a cabin and as walls you want to use planks that are 1m long. You are at the moment short of thirteen 1-meter planks. A friend says that she has found 4 planks, each 2.5m long. You are wondering if that is enough to finish the walls. How many 1-meter planks can you saw out of the planks she found?" Palm found that increasing the task authenticity in such a way increased pupils' tendency to effectively use their real world knowledge. For example for the planks item described above, the percentage increased from 24.0% RRs in their original form to 32.0% RRs in the more authentic form. In the present study we also investigated if pupils answer more realistically to P-items when these items are modified into more authentic story problems, however, we added one factor; depictive elements.

Depictive Elements

A second factor that we manipulated in order to increase the authenticity of the setting, is the addition of illustrations. In their integrated view of learning from verbal and pictorial representations, Schnotz and Bannert (2003) made a distinction between descriptive and depictive representations. They define *descriptive representations* as representations consisting of symbols describing an object, for example text, and *depictive representations* as representations consisting of iconic signs, for example pictures. The "integrated model of text and picture comprehension" they propose (see Figure 2) has two branches: a descriptive branch and a depictive branch. As shown in Figure 2, when reading a text, a mental model of the subject matter described in the text is constructed. The reader first builds a mental representation of the text surface, then generates a propositional representation and eventually comes to a mental model (= the descriptive branch). To come to this mental model a transition of a descriptive to a depictive representation is necessary. When looking at an illustration, first a visual image is created and then a mental model is constructed or is included in the already existing mental model of text comprehension (= the depictive branch).

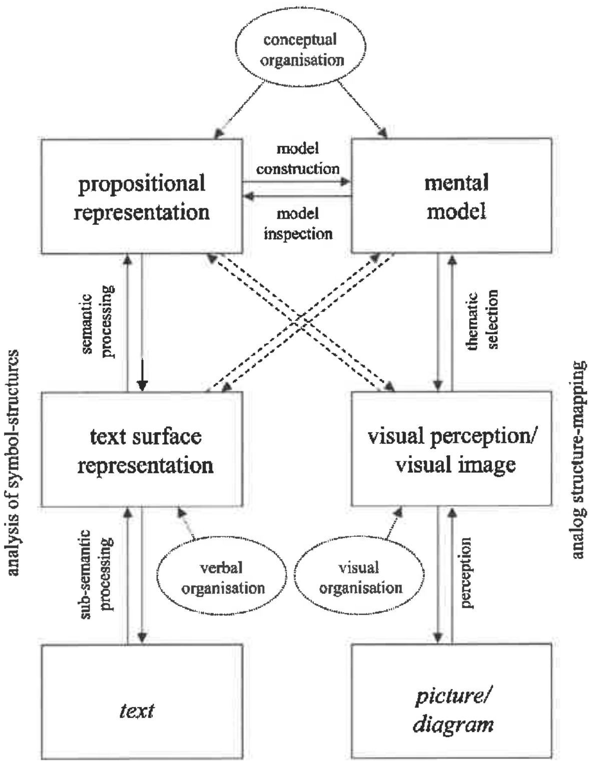

Figure 2. An integrated model of text and picture comprehension (Schnotz & Bannert, 2003, p.145).

Schnotz and Bannert (2003) investigated, based on their framework, how mental model construction is influenced when text and pictures are presented together. They concluded that pictures are beneficial when the subject matter is visualized in a task-appropriate way, which is in line with the structure mapping hypothesis; "structural features of the external graphic representation are mapped onto a mental model. Accordingly, different forms of informationally equivalent visualizations should result in mental models with different structures and, thus, with different computational efficiencies for different tasks." (Schontz & Bannert, 2003, p.148). For instance, In Schnotz and Bannert's study, participants who received a carpet diagram (i.e., the earth as rectangle moving across a time axis), performed much better than the group that received a circle diagram (i.e., the earth as a circle) on a time difference task but much worse on a circumnavigation task. Based on this empirical evidence, Schnotz (2005) concluded that text including pictures is only beneficial when the text and the picture are semantically related (*coherence principle*), be presented closely together in space and time (*contiguity principle*), and the picture should not be redundant with the text information (*redundancy principle*).

Pupils encounter various kinds of illustrations in their mathematics lessons. Some illustrations just decorate the pages of the textbook, while others are directly linked to the mathematical content. Empirical research concerning the influence of illustrations in mathematics and in particular in mathematical problem solving is scarce. One example is the study of Elia and Philippou (2004). Based on the categorization of Carney and Levin (2002), they proposed four functions that illustrations may serve in relation to a word problem: They can just decorate the page without any information concerning the word problem or its solution (*decorative pictures*), depict the problem situation partly or entirely (*representational pictures*), provide directions that support the solution procedure (*organizational pictures*), or provide essential information for solving the problem (*informational pictures*). Elia and Philippou observed and interviewed eight 6th grade pupils who were confronted with four word problems corresponding to the different functions of illustrations. It was found that decorative illustrations had no effect on mathematical problem solving, whereas representational, organizational and informational illustrations had a positive effect on problem solving. This study was however a small-scale, qualitative investigation of the effect of the four functions of illustrations. Moreover, it only involved standard word problems that can be solved correctly without making any realistic considerations (S-items). The present study involves a larger, quantitative investigation on the influence of representational illustrations on pupils' reactions to P-items.

THE PRESENT STUDY

In the present study we examined if presenting word problems in a more authentic way – without stepping out of the socio-cultural setting of the regular mathematics lesson – would positively affect the number of RRs to P-items. More specifically, we investigated the impact of two factors: the addition of narrative elements and/or depictive elements. This resulted in four conditions. In the first condition (the control condition), the problems were presented in their original form, without additional narrative or depictive elements, as in the research of Verschaffel et al. (1994), in the second condition (the narrative only condition) in the form of a short story with a real life event and question, in the third condition (the depiction only condition) together with a representational illustration that represents the problematic situation, and in the fourth condition (the combined condition) with both the narrative and the depictive elements, i.e., resulting in word problems presented in the form of a comic.

Our first hypothesis stated that when the P-items are presented in a narrative form, they will elicit more RRs than when presented in their original form (*Hypothesis 1*). This was expected based on the research of Palm (2008), where it was shown that, in the more authentic form, more pupils answered the P-items consistent with the constraints of the real world, than when the problems were in their original from. Our second hypothesis stated that when depictive elements are added, more P-items would be solved realistically than without depictive elements (*Hypothesis 2*). Based on the theoretical framework of Schnotz and Bannert (2003) and the qualitative study of Elia and Philippou (2004), we expected that the addition of representational illustrations would increase pupils' situational understanding of the problem situation and consequently increase the number of RRs to the P-items. Third, we hypothesized that the combination of both narrative elements and depictive elements in the form of a comic would elicit even more RRs to the P-items than merely the added effect of depiction and narration (*Hypothesis 3*), because the presence of one element may strengthen the expected effect of the other element (i.e., the presence of the narrative elements may cause pupils to pay more attention to the illustrations and process them deeper; likewise the presence of the depictive elements may assist the narrative element to elicit realistic considerations in the pupils).

METHOD

Participants

The participants were 155 elementary school pupils (6th grade), coming from four different schools located in the Flemish speaking part of Belgium. Pupils were assigned to the four conditions, taking into account their scores on a general mathematics achievement test (LVS, Dudal & Deloof, 2004). There were 34 pupils in the control condition (with an average LVS score of 36.3), 44 in the narrative only condition (38.0), 32 pupils in the depiction only condition (36.6), and 41 in the combined condition (37.0).

Material and Procedure

In each condition, pupils received ten word problems (six P-items and four S-items) in a paper-and-pencil task as a part of the mathematics class (see Table 1 for an overview of the P-items). These P-items and two of the S-items were adopted from the study of Verschaffel et al. (1994), the other two S-items were created by the researchers. All items were presented in a mixed order to rule out order effects, using two different sequences. The S-items in the task only served as buffer items and therefore were not included in the analysis.

To stay within the traditional culture of the mathematics class, the problems were presented in the form of a typical paper-and-pencil task that was given by the regular class teacher during the regular math teaching hours. Pupils were instructed to solve the task individually and to write down not only their answer but also their calculations and/or some additional explanatory comments. They had one hour to solve all items.

Table 1
An Overview of the P-items Used in the Present Research

P1	Carl has 5 friends and Georges has 6 friends. Carl and Georges decide to give a party together. They invite all their friends. All friends are present. How many friends are there at the party?
P2	Steve has bought 4 planks of 2.5m each. How many planks of 1m can he get out of these planks?
P4	450 soldiers must be bused to their training site. Each army bus can hold 36 soldiers. How many buses are needed?
P5	John's best time to run 100m is 17sec. How long will it take to run 1km?
P6	Bruce and Alice go to the same school. Bruce lives at a distance of 17km from the school and Alice at 8km. How far do Bruce and Alice live from each other?
P9	A man wants to have a rope long enough to stretch between two poles 12m apart, but he has only pieces of rope 1.5m long. How many of these pieces would he need to tie together to stretch between the poles?

* All P-items are adopted from Verschaffel et al. (1994). The original numbering was maintained.

Depending on the condition, narrative and/or depictive elements were added to the problems. In the first condition, the control condition, the problems were presented in their original form as in Verschaffel et al. (1994) (for an example, see Table 2). In the second condition, the narrative only condition, we added narrative elements. Based on the framework of Palm (2006, 2008) the word problems were changed into short stories with a clear description of a real life event and with a question that also could be asked in the real world. Transforming the word problem in a narrative story was expected to create a rich representation of the problem situation. In the third condition, depictive elements were added in the form of a representational illustration. These representational illustrations depicted the situation of the word problem, without adding additional information, to stimulate the creation of a rich representation of the problem situation. The illustrations were in black and white and were presented underneath the problem. In the fourth and last condition we combined the narrative and depictive elements, resulting in a comic consisting of several representational illustrations that pictorially described the real life event and the question but that also contained some speech bubbles. Each comic was composed of four or five frames, the last of which always contained the question.

Table 2
Example of the Plank Item (P2) in Each Condition

Condition 1 (control condition)

Steve has bought 4 planks of 2.5m each. How many planks of 1m can he get out of these planks?

Condition 2 (narrative only condition)

Steve walks out of a do-it-yourself store. In his shopping cart are 4 long planks. 'They only had planks of 2.5 meter. But with these planks I can also get to work' thinks Steve. Steve puts the material in his car and leaves. After a fifteen minutes drive he comes home. 'I just have to unload everything, and then I can start with the bookshelf. I'm already excited!' He says cheerfully. Everything is ready in his workplace. Steve looks at the plans of the shelf and thinks: 'I have 4 planks of 2.5 meter, but I need planks of 1 meter. How many planks of 1m can he get out of these planks?

Condition 3 (depiction only condition)

Steve has bought 4 planks of 2.5m each. How many planks of 1m can he get out of these planks?

Condition 4 (combined condition)

Analysis

The available data were pupils' answers on the different P-items, as well as their possible calculations and explanations. The data were coded in three steps, in exactly the same way as in Verschaffel et al.'s (1994) study.

First, the answers were coded in five answer categories: An answer was coded as Expected Answer (EA) when it was the result of a straightforward application of the arithmetic operation hidden in the word problem. When an answer was the result of a straightforward application of the operation, but with a technical mistake in the execution, it was coded as Technical Error (TE). An answer that resulted of the effective use of real-world was coded as Realistic Answer (RA). When pupils did not give an answer it was coded as No Answer (NA). When none of the above categories could be applied, we coded it as Other Answer (OA).

In a second step, the additional calculations and explanations were scored. A "+" was added to the answer code when a trace of the activation of real world knowledge was found in these additional comments, whereas a "–" was added when such a trace was not found.

Hereafter we give a few examples of the first two steps of our data coding. When a pupil answered the rope item with "8 pieces of rope", without any additional comment or with the notation "12/1.5=8", his reaction was coded as "EA–". When a pupil answered with "8 pieces of rope" but added the comment that "in fact, more pieces of rope would be needed to actually knot them together and knot them to the poles", his reaction was coded as "EA+". Likewise, when a pupil did not provide an answer to the rope item and also did not write any extra comment, his reaction was coded as "NA –", whereas a pupil who did not answer the item but added the realistic comment that he did not know how to answer the problem because he did not know how many cm of rope he had to take for every knot, received a "NA+" code.

Finally, in a third step, all RAs (RA+ and RA–) as well as all reactions containing a "+" sign (EA+, TE+, NA+, and OA+) were considered as Realistic Reactions (RRs), whereas all other reactions (EA–, TE–, NA–, and OA–) were considered as Non-realistic Reactions (NRs).

The coded answers were analyzed with a repeated measures logistic regression, with the factors narration and depiction as independent variables and the RRs as dependent variable.

RESULTS

First, we looked if pupils demonstrated the expected general tendency to respond unrealistically to P-items. The data showed that only 32.0% of the reactions were realistic. Compared with the findings of Verschaffel et al. (1994), where the percentage for the same six P-items was 14.6%, and with the findings of replication studies (see

Verschaffel et al., 2000, 2009), this percentage was relatively high. However, it still shows pupils' strong tendency to exclude real world knowledge when solving word problems in the mathematics classroom context.

To investigate the effect of the addition of the narrative and/or depictive elements, we compared the results of the four conditions. The data showed that in the control condition, where the problems were presented in their traditional form, only 22.0% of the reactions were realistic. In the depiction only condition with the representational illustrations, only 19.6% of the reactions were realistic. In the narrative only condition and the combined condition, the percentages were remarkably higher, respectively 45.1% and 37.0%. A logistic regression analysis revealed that there was a main effect of narration (X^2 (1,930) = 41.21 with $p<.001$) (See Figure 3), whereas there was no main effect of depiction (X^2 (1,930) = 2.68 with $p=.102$) nor an interaction effect between both factors (X^2 (1,930) = 0.34 with $p=.557$). The pairwise comparisons showed that the narrative only condition had significantly more RRs than the control condition (OR=2.87 and p<.001) and than the depiction only condition (OR=3.36 and p<.001). Also the combined condition (with narrative and depictive elements) had significantly more RRs than the control condition (OR=2.05 and p=.001) and the depiction only condition (OR=2.41 and p<.001). There was no significant difference between the control condition and the depiction only condition; neither was there a significant difference between the narrative only condition and the combined condition with both narrative and depictive elements.

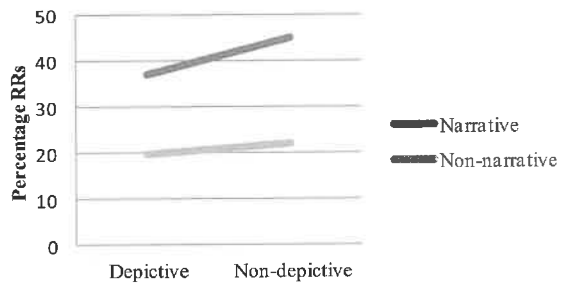

Figure 3. Main effect of narration.

We also looked for differences between items. In line what had been found in previous studies (Verschaffel et al., 2000), the bus item (P4) yielded the highest number of RRs (71.9%). This percentage was higher than in the study of Verschaffel et al. (1994) (49.3%) and in all other studies (between 11% and 67%, see Verschaffel et al., 2000). Another remarkable finding was that the rope item (P9), which yielded 0.0% RRs Verschaffel et al.'s (1994) study, and between 0% and 8% in the other studies (see Verschaffel et al., 2000), elicited 29.9% RRs. See Table 3 for an overview of the percentages per item.

Table 3
The Percentage of RRs for Each Item Separately per Condition

	Condition				Total	Total Verschaffel et al. 1994
	Control	narrative only	Depictive only	combined		
P1: Friends item	36.1	52.3	11.8	43.9	36.0	20.0
P2: Planks item	19.4	45.5	17.6	36.6	29.8	13.3
P4: Bus item	63.9	68.2	82.4	73.2	71.9	49.3
P5: Runner item	5.6	18.2	0.0	9.8	8.4	2.6
P6: School item	8.3	18.2	5.9	7.3	9.9	2.6
P9: Rope item	0.0	68.2	0.0	51.2	29.9	0.0
	22.0	45.1	19.6	37.0	32.0	14.6

The logistic regression analysis, showed for the plank item significant differences between conditions. A main effect of narration was found (X^2 (1,155)= 8.62, p=.003). There were significantly more RRs in the narrative only condition, than in the depiction only condition (p=.005), and than the control condition (p=.009). There was no significant difference between the narrative only condition and the combined condition (p=.404).

For the friends item there was a main effect of depiction (X^2 (1,155)= 5.34, p=.021) and of narration(X^2 (1.155)= 9.95, p=.002). There was no interaction effect. The depictive only condition had significantly fewer RRs than the control condition (p=.012), the narrative only condition (p<.001), and the combined condition (p=.001).

No significant differences between the conditions were found for the school item and the bus item. For the rope item and the runner item no statistical analyses could be performed due to the absence of RRs in some of the conditions. However, it must be noted that for the rope item, there tended to be a large number of RRs in the narrative only condition (68.2%) and the combined condition (51.2%), whereas in the two other conditions not a single RR was observed.

CONCLUSION AND DISCUSSION

In this study we investigated whether adding narrative and/or depictive elements would help pupils to create a rich and meaningful representation of the problem situation, and consequently to solve the problems more realistically. We confronted 155 primary school pupils (6^{th} grade), in their mathematics class with ten mathematical word problems: six P-items and four S-items. The word problems were complemented with narrative and/or depictive elements, resulting in four conditions. In the first condition (the control condition), no elements were added. In the second condition (the narrative only condition) the problems were presented in a rich narrative story, in the third condition (the depiction only condition), the problems were presented together with a representational illustration, and in the fourth condition (the combined condition) the narrative and depictive elements were combined, which resulted in a comic. Pupils' answers to the problems were analyzed and compared between the conditions.

Our first hypothesis stating that more P-items would be solved realistically when narrative elements are added (*Hypothesis 1*), was confirmed, as a main effect of narration was found. As in the research of Palm (2008), pupils solved the P-items more realistically when these problems were modified into more situationally rich story problems. In contrast, the second hypothesis stating that adding depictive elements to the P-items would result in more RRs (*Hypothesis 2*) was rejected, since no main effect of depiction was found. So, contrary to our expectation (based on the theoretical model of Schnotz and Bannert (2003) and the empirical study of Elia and Philippou (2004)), representational illustrations did not help pupils to create a rich representation of the problem situation, and consequently did not help them to involve realistic considerations in their answers on the word problems. Our third and last hypothesis that stated that the combination of both narrative and depictive elements would elicit even more RRs than the added effect of depiction and narration, was also rejected (*Hypothesis 3*), since no interaction was found between narration and depiction.

The present study replicated Palm's (2008) findings, as it showed that modifying the P-items into situationally rich story problems helps pupils to consider aspects of reality in their mental model of the problem and consequently react more realistically to these items. So, our findings confirmed that the word problems presented in the mathematics class are, as Palm (2006) argued, merely 'dressed up' problems that represented no 'real' simulation of everyday life. By transforming these problems into short narratives describing real world events involving protagonists who try to reach concrete goals and are confronted with experientially real questions, pupils are more prone to react in a realistic way. The question now arises why this effect was only present for some of the P-items (the friends item, the planks item and the rope item), and why the effect was, in general, rather small. After all, still 59.0% of the pupils in two conditions with narration still gave a NR to the P-items. A study with individual interviews instead of a paper-and-pencil test can shed some light at this question. In addition it can also be interesting to take individual differences into account and look for which pupils narration works and for whom it does not work. More specifically, the (positive) influence of narration may be caused by personality characteristics such as working memory, linguistic competence, gender, etc.

In contrast with the observed positive effect of narration to the problems, we did not find a positive effect of adding depictive elements. We had expected that the addition of representational illustrations would help pupils to build a situationally rich mental model of the P-items and consequently give more realistic responses to these problems, but this was not the case. A possible explanation can be that the representational illustrations that were used, were not sufficient to create a meaningful, experientially real representation of the problematic situation. In future research, it has to be investigated whether other illustrations, for example illustrations that more explicitly show the realistic modeling problem, can lead to more RRs. It also has to be investigated, for example by means of an eye movement research or individual interviews, whether pupils actually do use the illustrations when solving word problems. It is possible that they simply do not look at the illustrations, due to certain beliefs that illustrations in a mathematics textbook or test just decorate the page and have nothing to do with the problem solving activity.

Finally, as stated above, we did not find an interaction effect of the combination of narrative and depictive elements. As explained in section 1.4, we had expected that the combination of the narrative and depictive elements (in the form of a comic) would have elicited even more RRs to the P-items than merely the added effect of depiction and narration, because the presence of one element might strengthen the expected effect of the other. Moreover, both the theory of learning from verbal and pictorial representations of Schnotz and Bannert (2003) and Elia and Philippou's (2004) theoretical categorization of the different functions that illustrations may serve in relation to word problems, contain elements that may further support hypothesis 3. Indeed, it could be argued, on the basis of Schnotz and Bannert's theory, that whereas in the depiction only condition the illustrations that were juxtaposed to the text of a word problem could be easily ignored (see above), it is highly implausible that this also would happen in the combined condition, because of the strong contiguity of the pictorial and the textual information in the comics. Furthermore, one could argue that the function of the pictures – in terms of Elia and Philippou's (2004) categorization schema – slightly changed from being (more) representational in the illustrations only condition to being (more) informational in the combined condition, which should have contributed to a positive effect of the combined condition (because in the latter case pupils are, again, stimulated more to pay attention to and make active use of the illustrations). However, notwithstanding these theoretical arguments, we did not find the expected interaction effect. So, further research is needed to find out whether and how illustrations, not only merely juxtaposed to the text of a word problem but also integrated with the text and combined with narrative elements – are processed, and what their exact function is in pupils' understanding and solution of problematic word problems.

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Some Relations between Semiotics and Didactic of Mathematics

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ABSTRACT: *Between the years 1995 and 2010 we developed some researches within the Research Group NRD Bologna (Italy) which led to investigate various salient and relevant aspects of Mathematics Education; on these issues the NRD has published numerous books and articles, participating in international conferences. We present here some research questions and some results.*

Key words: *Semiotic, Treatment, Theories "external" of mathematics education, Pragmatic and realist theories, Misconception.*

THE PHENOMENON OF CHANGE OF THE MEANING OF MATHEMATICAL OBJECTS DUE TO THE PASSAGE BETWEEN THEIR DIFFERENT REPRESENTATIONS: HOW OTHER DISCIPLINES CAN BE USEFUL TO THE ANALYSIS

Background

In D'Amore and Fandiño Pinilla (2007a, b), we reported and discussed, exclusively from a structural semiotic point of view, episodes taken from classroom situations in which students are mathematics teachers in their initial training, engaged in facing representations problems. Some examples of the phenomenon have been given orally in Rhodes, on April 13th 2006, during a general conference (How the treatment or conversion changes the sense of mathematical objects) at the 5th MEDCONF2007 (Mediterranean Conference on Mathematics Education), 13-15 April 2007, Rhodes, Greece (D'Amore, 2007).

The task consisted in this: working in small groups the trainee teachers received a text written in natural language; such texts had to be transformed into algebraic language. Once they had come to the algebraic formulation, this was explained by the group and collectively discussed. Our duty as university teachers was to suggest the further transformation of the obtained algebraic expressions into other algebraic *expressions*, to face collective discussions on their meaning.

We present three examples below.

Example 1

[We omit the original linguistic formulation which, in this case, is not relevant];

The final algebraic formulation proposed by group 1 is: $x^2+y^2+2xy-1=0$, which in natural language is interpreted as follows: «A circumference» [the interpretation error is evident, but we decide to pass over]; we carry out the transformation which leads to: $x+y=\frac{1}{x+y}$ that after a few attempts is interpreted as «A sum that has the same value of its reciprocal»;

> question: But $x+y=\frac{1}{x+y}$ is it or not the "circumference" we started with?;
>
> student A: Absolutely no, a circumference must have x^2+y^2;
>
> student B: If we simplify, yes.

One can ask whether or not it is the transformation that gives a *sense*: from the episode it seems that if one would perform the inverse passages, then one would return to a "circumference". But it could also instead be that the meanings are attributed to the specific representations, without links between them, as if the transformation that makes sense for the teacher it does not make sense for the person who performs it.

Example 2

The text written in natural language requires the algebraic writing of the sum of three consecutive natural numbers and the proposal of group II is: $(n\text{-}1)+n+(n+1)$ [obviously the doubt remains in the case of $n=0$, but we decide to pass over]; we carry out the transformation that leads to the following writing: $3n$ that is interpreted as: «The triple of a natural number»;

> question: But $3n$ can be thought as the sum of three consecutive natural numbers?;
>
> student C: No, *like this* no, *like this* it is the sum of three equal numbers, that is n.

Example 3

We consider the sum of the first 100 natural positive numbers: 1+2+...+99+100; we perform Gauss classical transformation; 101×50; this representation is recognized as the solution of the problem but not as the representation of the starting object; the presence of the multiplication sign compels all the students to look for a sense in mathematical objects in which the "multiplication" term (or similar terms) appears;

question: But 101×50 is it or not the sum of the first 100 positive natural numbers?;

student D: That one, is not a sum, that is a multiplication; it corresponds to the sum, but it is not the sum.

In these episodes we witness a constant change of meaning during the transformations: each new representation has a specific meaning of its own not referable to the one of the starting representations, even if the passage from the first to the second ones has been performed in an evident and shared manner.

The Causes of the Changes of Meaning

What are the causes of the changes of meaning, what origin do they have?

We can start from this diagram that we appreciate a lot because of its attempt to put in the right place the ideas of *sense* and *understanding* (Radford, 2004a).

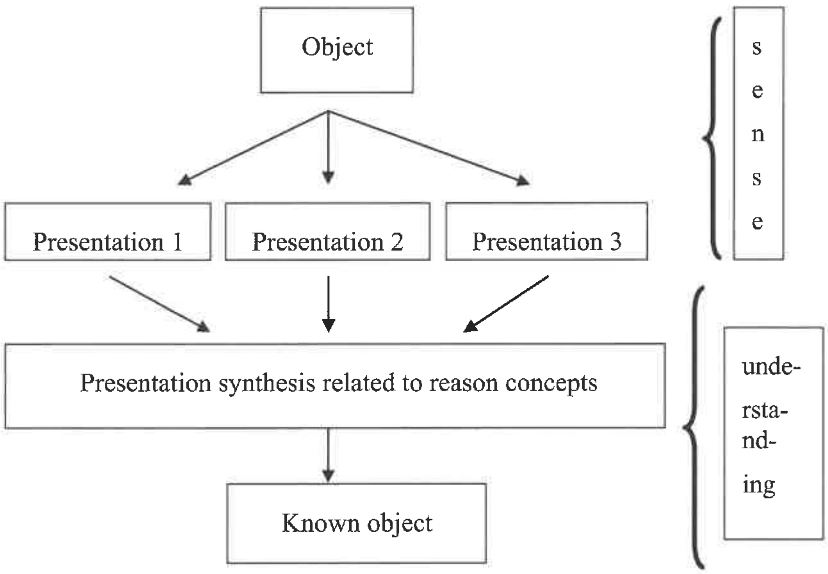


Figure 1. Diagram by Radford (2004a) about the changes of meaning, with the ideas of sense and understanding.

The process of meanings endowment moves at the same time within various semiotic systems, simultaneously activated; we are not dealing with a pure classical dichotomy: treatment/conversion leaves the meaning prisoner of the internal semiotic structure, but with something much more complex. Ideally, from a structural point of view, the meaning should come from within the semiotic system we are immersed in. Therefore, in *Example 2,* the pure passage from $(n\text{-}1)+n+(n+1)$ to $3n$ should enter the category: treatment semiotic transformation. But what happens in the classroom practice, and not only with novices in algebra, is different. There is a whole path to cover, starting from single specific meanings culturally endowed to the signs of the algebraic language ($3n$ is the triple of something; 101×50 is a product, not a sum). Thus, there are sources of meanings relative to the algebraic language that anchor to meanings culturally constructed, previously in time; such meanings often have to do with the arithmetic language. From an, so to speak, "external" point of view, we can trace back to seeing the different algebraic writings as equally significant since they are obtainable through semiotic treatment, but from inside this picture is almost impossible, bound as it is to the culture constructed by the individual in time. In other words, we can say that students (not only novices) turn out bridled to sources of meaning that cannot be simply governed by the syntax of the algebraic language. Each passage gives rise to forms or symbols to which a specific meaning is recognised because of the cultural processes *through* which it has been introduced.

In Luis Radford's mathematical knowledge is seen as the product of a reflexive cognitive mediated praxis. «Knowledge as cognitive praxis (*praxis cogitans*) underlines the fact that what we know and the way we come to know it are underpinned by ontological positions and by cultural processes of meaning production that give form to a certain way of rationality within which certain types of questions and problems are posed. The *reflexive* nature of knowledge must be understood in Ilyenkov's sense, that is, as a distinctive component that makes cognition an intellectual reflexion of the external world in accordance with the forms of individuals' activity (Ilyenkov, 1977, page 252). The mediated nature of knowledge refers to the role played by tools and signs as means of knowledge objectification and as instruments that allow us to bring to a conclusion the cognitive *praxis*» (Radford, 2004b, page 17).

On the other hand, «the object of knowledge is not filtered only by our senses, as it appears in Kant, but overall by the cultural modes of signification (...). (...) the object of knowledge is filtered by the technology of the semiotic activity. (...) knowledge is culturally mediated» (Radford, 2004b, page 20). «(…) These terms are the semiotic means of objectification. Thanks to these means, the general object that always remains directly inaccessible starts to take form: it starts to become an "object of consciousness" for the pupils. Although general, these objects however remain *contextual*» (Radford, 2004b, page 23).

The approach to the object and its appropriation on the part of the individual who learns, are the result of personal intentions with which individuals express themselves through experiences that see the objects used in suitable contexts: «Intentions occur in contextual experiences that Husserl called *noesis*. The conceptual content of such experiences he termed *noema*. Thus, noema corresponds to the way objects are grasped and become known by the individuals while noesis relates to the modes of cultural categorical experiences accounting for the way objects become attended and disclosed (Husserl, 1931)» (Radford, 2002, page. 82).

In the cases we presented above, and in mathematics in general, it is clear that the objects are attended from the first moment in their formal expression, in our case in the algebraic language; the individual learns to formally handle these signs, but what happens to the initial mathematical object? What happens to the initial meanings? We suppose that these meanings are tightly bound to the arithmetic experience of the pupil and overall to the way in which such an experience becomes objective through its objective transposition into ordinary language. Deep understanding of algebraic or, in general, formal manipulation, holds a prominent position.

Through an interesting comparison, Radford expresses himself on this point as follows: «While Russell (1976, page 218) considered the formal manipulations of signs as empty descriptions of reality, Husserl stressed the fact that such a manipulation of signs requires a shift of intention, a noematic change: the focus becomes the signs themselves, but not as signs *per se*. And he insisted that the abstract manipulation of signs is supported by new meanings arising from rules resembling the rules of a game (Husserl, 1961, page 79), which led him talk about signs having a *game signification* (...)» (Radford, 2002, page 88).

After having shown the broad and complex significance of the phenomenon, we must refer to other disciplines in order to understand better and better the issue of the different meanings of algebraic expressions, that is, in order to give a significant contribution to this aspect of mathematics education.

Analysis of the Phenomenon Thanks to Theories "External" of Mathematics Education

We believe that some theories "external" of mathematics education can have, and in fact they already have, a strong influence on the analyses of various phenomena, like the ones described here, therefore giving a contribution to changing the theoretical frame of our discipline in its future research developments.

Philosophy. In section 1.2, we have seen how philosophy (Husserl's phenomenology) can have remarkable contribution and we will not repeat ourselves.

Learning is taking consciousness of a general object in accordance with the modes of rationality of the culture one belongs to.

More importantly we must face here the issue of the philosophical dilemma on concept and object, and even more the problem of the need of a previous choice between realist and pragmatist positions (D'Amore, Fandiño Pinilla, 2001; D'Amore, 2003; D'Amore, 2007).

In *realist theories* the meaning is a «conventional relationship between signs and ideal or concrete entities that exist independently of linguistic signs; they therefore suppose a conceptual realism» (Godino, Batanero, 1994). As Kutschera (1979) already claimed:

«According to this conception the meaning of a linguistic expression does not depend on its use in concrete situations, but it happens that the use holds on meaning, since a clear distinction between pragmatics and semantics is possible».

In the realist semantics that it derives, we attribute to linguistic expressions purely semantic functions; the meaning of a proper name (as: 'Bertrand Russell') is the object that such proper name indicates (in such a case: Bertrand Russell); the individual statements (as: 'A is a river') express facts that describe reality (in such a case; A is the name of a river); the binary predicates (as: 'A reads B') designate attributes, those indicated by the phrase that expresses them (in this case: person A reads thing B). Therefore every linguistic expression is an attribute of certain entities: the nominal relationship that derives is the only semantic function of expressions.

We recognise here the bases of Frege's, Carnap's and Wittgenstein's *(Tractatus)* positions.

A consequence of this position is the acknowledgement of a "scientific" observation (at the same time therefore, empiric and subjective or inter-subjective) as it could be, at a first level, a statement and predicate logic.

From the point of view we are mostly interested in, if we apply to Mathematics the ontological assumption of realist semantics, we necessarily draw a platonic picture of mathematical objects: notions, structures, etc. have a real existence that does not depend on human being, as they belong to an ideal domain; "to know" from a mathematical point of view means "to discover" in such domain entities and relationships between them. It is also obvious that such picture implies an absolutism of mathematical knowledge, since it is thought as a system of external certain truths that cannot be modified by human experience because they precede or, at least, are extraneous and independent from it.

Akin positions, although with different nuances, were sustained by Frege, Russell, Cantor, Bernays, Goedel,...; they also encountered violent criticisms [Wittgensteins' *Conventionalism* and Lakatos' *quasi-empirism* : see Ernest (1991) and Speranza (1997)].

In *pragmatic theories* linguistic expressions have different meanings according to the context in which they are used and therefore any scientific observation is impossible, since the only possible analysis is a "personal" and subjective one, anyway circumstantial and not generalizable. We cannot but analyse the different "uses": the set of "uses" in fact determines the meaning of objects.

We recognize here Wittgenstein's positions of the *Philosophical Investigations,* when he admits that the significance of a word depends on its function in a "linguistic game", since in such game it has a way of 'use' and a concrete purpose for which it has been precisely used: therefore the word does not have a meaning *per se*, but nevertheless, it can be meaningful.

Mathematical objects are therefore symbols of cultural units that emerge from a system of uses that characterise human pragmatics (or at least of individuals' homogeneous groups) and that continuously modify in time, also according to needs. In fact, mathematical objects and the meaning of such objects depend on the problems that we face in Mathematics and on their solution processes.

Table 1
Scheme about realist theories vs pragmatic theories

	"REALIST" THEORIES	"PRAGMATIC" THEORIES
meaning	conventional relationship between signs and concrete or ideal entities independent of linguistic signs	depends on the context and use
semantics Vs pragmatics	clear distinction	no distinction or faded distinction
objectivity or intersubjectivity	complete	missing or questionable
semantics	linguistic expressions have purely semantic functions	linguistic expressions and words have "personal" meanings, are meaningful in suitable contexts, but they don't have absolute meanings *per se*
analysis	possible and licit: logic for example	only a "personal" or subjective analysis is possible, not generalizable, not absolute
consequent epistemological picture	platonic conception of mathematical objects	problematic conception of mathematical objects
to know	to discover	to use in suitable contexts.
knowledge	is an absolute	is relative to circumstance and specific use
examples	Wittgenstein in *Tractatus*, Frege, Carnap [Russell, Cantor, Bernays, Gödel]	Wittgenstein in *Philosophical Invesigations* [Lakatos]

It is obvious and it would be easy to prove with philosophical examples, that the two fields are not fully complementary and clearly separated even if, for reasons of clarity, we preferred giving this "strong" impression.

With regard to the philosophical bases of Mathematics Education, we have decided to stay in the pragmatic domain that seems much closer to the reality of the empiric process of Mathematics teaching/learning. It seems that each specification that appears in the right column, cell by cell, is part of the same process and of its explicitation. It seems that focusing on didactical activity (and therefore research), on learning, and consequently on the epistemology of the domain that has the student as a protagonist, we are obliged to interpret each step of knowledge construction as responding to the *language game,* therefore admitting that the semantics blurs in pragmatics.

Sociology. In D'Amore (2005) and D'Amore and Godino (2007), we show how the results of the analyses relative to the behaviours of individuals engaged in an activity of conceptual learning of mathematical objects, their transformations of the descriptions of such objects from ordinary language to formal language, the manipulations of such formalizations can be framed within a sociological interpretation key: the learning environment is framed within a sociological interpretation key and the individuals' behaviours are interpreted through the notion of "practice" and its "meta-practice" evolution. Essentially the individuals shift from a shared practice, recognized as characteristic of the social group they belong to, to a meta-practice that modifies such a characteristic; the interpretative behaviour therefore ceases to be global and social and becomes local and personal; the notions that come into play in such interpretations are specific of the circumstance and not of the situation in its entirety.

We pass over this point, referring back to the quoted texts.

Anthropology. In D'Amore and Godino (2006, 2007) we go into strongly anthropological details in order to explain the nature of the choices of the individual who learns mathematics. In such articles we highlight how «Having obliged the researcher to point all his attention to the activities of human beings who have to do with mathematics (not only solving problems, but also communicating mathematics) is one of the merits of the anthropological point of view, inspiring other points of view, amongst which the one that today we call "anthropological" in the proper sense: the ATD, anthropological theory of didactics (of mathematics) (Chevallard, 1999; page 221). Why this adjective "anthropological"? It is not an exclusiveness of the approach created by Chevallard in 80s, as he himself declares (Chevallard, 1999), but an "effect of the language" (page 222); it distinguishes the theory, identifies it, but it is not peculiar to such theory in a univocal way» (D'Amore, Godino, 2006, page 15). The ATD is almost exclusively centred on the institutional dimension of mathematical knowledge, as a development of the research program started with fundamental didactics. The crucial point is that «ATD places the *mathematical* activity, and therefore *the study* in mathematics activity, *in the set of human activities and of social institutions*» (Chevallard, 1999).

This kind of analyses, although subjected to criticisms in D'Amore and Godino (2006, 2007), has opened the way to the use of anthropology as a critical instrument, as a new theoretical frame at research into mathematics education, in accordance with what has been already highlighted in the above quoted articles. It is the human being, strong of the acquired culture, strong of the specific expressive, communicative luggage, who handles formal writings and gives them a meaning that it cannot be anything else but coherent with his social history; every meaning of each formal expression is the result of an anthropological comparison between a lived history and a here-and- now that must be coherent with that history.

We pass over this point, referring back to the quoted texts.

Psychology. In D'Amore and Godino (2006) we show how the shift from the anthropological picture to the onto-semiotic one is made necessary (amongst other things) by the need of not trivializing the presence of psychology in the study of learning and, in general, classroom situations. In D'Amore (1999) we show, for example, how ideas on representation drawn from psychology, regarding the explanation of the passage from image (weak) to model (stable) of concepts (Paivio, 1971; Kosslyn, 1980; Johnson-Laird, 1983; Vecchio, 1992), can be placed as a unitary basis of the explanation of several didactic phenomena, as intuitive models, the shift from internal to external models, the figural concepts, up to misconceptions, studied mainly in the 80s. Also the ideas of frame and script (Bateson, 1972; Schank, Abelson, 1977) have been used for the same purpose.

CHANGES OF MEANING: AN ANALYSIS CONNECTING THEORIES WITHIN MATHEMATICS EDUCATION

A possible path we can follow to understand the phenomenon of "changes of meaning" is to network more than one semiotic approach (Santi, 2010). In this section, we present the issue of "changes of meaning" addressing two semiotic perspectives: Duval's structural and functional approach and Radford's cultural-semiotic approach. We show the complementarity of the two perspectives to give an encompassing interpretation of this didactical phenomenon. We use the connection of Duval's and Radford's perspectives to analyse a successful teaching experiment involving primary school pupils who do not change the sense of meaning when exposed to treatment transformations of figural representations of sequences.

A Conceptual Framework for Changes of Meaning

Duval's Structural and Functional Approach

Duval's (1995) approach stems from a realistic view point that considers mathematical objects a priori inaccessible ideal objects. Since mathematical objects are inaccessible entities, the theory pivots around the notion of semiotic systems and the coordination of semiotic systems through treatment and conversion transformations. A semiotic system is characterized by a set of elementary signs, a set of rules for the production and the transformation of signs and an underlying meaning structure deriving from the relationship between the signs within the system.

Mathematical objects, that cannot be referred to directly, are recognised as invariant entities that bind different semiotic representations as treatment and conversion transformations are performed. Duval identifies the specific cognitive functioning to mathematics with the coordination of a variety of semiotic systems. Both the development of mathematics as a field of knowledge and its learning are accomplished through such specific cognitive functioning.

Duval develops Frege's classical semiotic triangle (sinn-bedeutung-zeichen) and identifies meaning with the couple (sign-object), i.e. a relationship between a sign and the object it represents. The sign becomes a rich structure that condenses both the semiotic representation (zeichen) and the way the semiotic expression offers the object in relation to the underlying meaning of the semiotic structure sinn. Meaning therefore has a twofold dimension: sinn, the way a semiotic representation offers the object; bedeutung the reference to the inaccessible mathematical object. Meaning making processes and learning require to handle different sinns networked through semiotic transformations without losing the bedeutung to the invariant mathematical object.

The following schema represents the construction of meaning when several semiotic systems are coordinated to conceptualize a mathematical object.

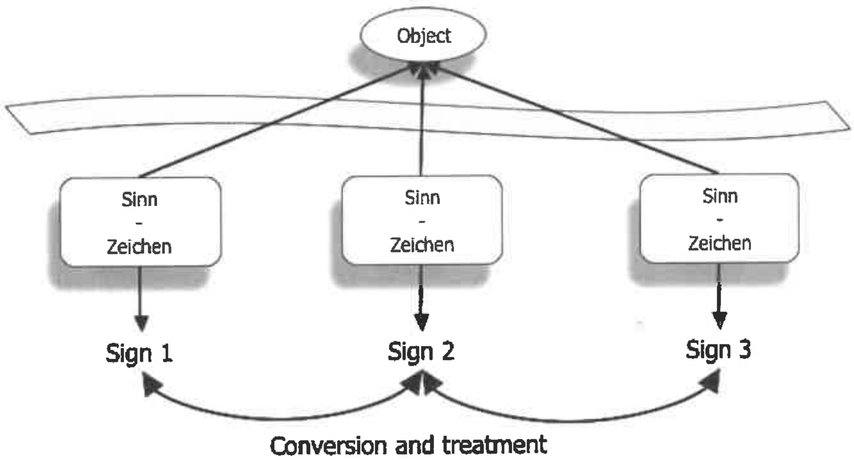


Figure 2. Meaning and changes of meaning in Duval's approach.

In this framework, the research issue is how students recognize the common bedeutung as the sinn changes through semiotic transformations. What we have above termed a "change of meaning", is a change of bedeutung as the sinn changes.

Radford's Cultural-Semiotic Approach

Within a socio-cultural and phenomenological standpoint, Radford's (2008) approach ascribes reflexive mediated activity, a central role both in cognition and in the emergence of mathematical objects. The reflexive activity entangles mathematical objects, semiotic resources, individuals' consciousness and intentional acts, within social practice and a cultural and historical dimension.

Mathematical objects are fixed patterns that emerge from the reflexive mediated activity. Mathematical objects lose any ideal and a priori existence but they are ontologically intertwined with the mediated activity from which they emerge. Nevertheless, mathematical objects acquire a form of ideality and existence in the culture that encompasses the reflexive activity.

Learning is considered an objectification process accomplished through a reflexive activity, a meaning making process that allows to become aware of the mathematical object that exists in the culture, but the student doesn't recognize. The complexity of the objectification process requires to broaden the notion of sign and go beyond its representational role, since signs culturally mediate activity and direct the individual's intention towards the mathematical object. Signs are termed as semiotic means of objectification and they include, artefacts, gestures, language, rhythm. Semiotic means of objectification stratify the mathematical object into levels of generality according the reflexive activity they mediate.

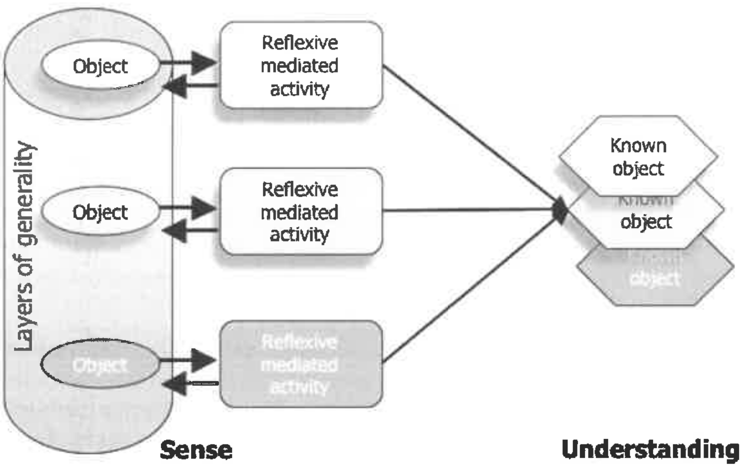


Figure 3. Meaning and changes of meaning in Radford's perspectives.

Meaning is no longer a mere relation sign-object, but is deeply interwoven with the reflexive activity, with intentional acts culturally mediated by semiotic means of objectification. Meaning is a double sided construct with a personal and a cultural dimension. The personal dimension refers to the individual's intentional acts directed towards a cultural unitary object. The cultural dimension refers to cultural and historical features that are condensed in the general and interpersonal mathematical object brought to the individual by teaching activities. The expected outcome of learning as an objectification process, is the alignment of the personal meaning with the cultural meaning.

Although both the theories we analysed are semiotic perspectives - if we look at the relationship with cognition - semiotics has a different hierarchical position in the respective system of principles. Therefore, the two theories have strong boundaries that separate them. This brings along also differences regarding the nature of mathematical objects and processes. In Duval's approach semiotics plays a representational role and it is the very substance of cognition that is identified with the coordination of semiotic systems. In Radford's perspective cognition is considered a process of objectification in which signs mediate a reflexive activity. Furthermore, the way signs are used is very different. In Duval's perspective, semiotic representations are used diachronically through treatment and conversion transformations. Whereas in Radford's perspective, a wide range of semiotic means of objectification are used synchronically organized around a particular mediator that changes as the level of generality changes. The different hierarchical position of semiotics allows Radford to broaden the notion of sign to include gestures, artefacts, rhythm, kinaesthetic activity etc. that Duval would never consider semiotic.

The different hierarchical position of semiotics stems from the different ontologies behind the two theories. The structural and functional approach has a realistic view of mathematical objects that ascribes to semiotics a representational role and to meaning a relation sign-object. The theory of objectification has a pragmatic standpoint towards mathematical objects that ascribes to semiotics the role of mediating a reflexive activity, the "substance" of ontology, meaning and cognition. Mathematical process are also differently positioned in the system of principle. Duval identifies the mathematical activity with the transformation of signs, subsumed in the robust structure of the semiotic systems that accomplish discursive and meta-discursive functions. Radford considers activity a form of reflection that involves the individual as a whole – his consciousness, feelings, perception, sensorimotor activity etc- immersed in a system of cultural signification that orients his intentional acts.

At a more profound level, any attempt to enlarge one of the theories subsuming elements of the other conflicts with its epistemological foundations. Nevertheless, the boundaries that separate the two theories do not imply an opposition between the two perspectives. Ullmann (1962) highlights two complementary features that characterise the development of mathematical objects: the operational phase and the referential phase. On the one hand mathematical objects and their meaning emerge from and are objectified by a reflexive activity, on the other hand it is necessary to linguistically refer to the entities that emerge from such practices. The dual nature of mathematical objects – as patterns of activity and as "existing" ideal entities in the culture – implies that also meaning and semiotics have a dual nature. In the connecting theories terminology, the strong differences result in a high level of complementarity that accounts for networking by coordinating the two perspectives, respecting their identity. Coordinating Duval's and Radford's theories allows to encompass the double-sided nature of objects, meaning and representations.

The emergence of a mathematical object and its objectification is described by the cultural semiotic perspective whereas the reference to the object is accounted for by the structural and functional approach. Meaning as a sense making process of the individual and as the activity culturally condensed in the institutional object are described by Radford's approach; meaning as the interplay between sinn and an bedeutung is framed by Duval's approach. The coordination of the two theories is in turn achieved by the dual nature of semiotics. On the one hand signs mediate reflexive activity on the other hand they represent objects and broaden our cognitive possibilities through semiotic transformations. Our general conjecture is that a successful outcome of mathematical learning processes rests on the dual nature of semiotic resources, i.e. as semiotic registers and semiotic means of objectification; as a semiotic mean of objectification a sign -synchronically interwoven with a rich arsenal of mediators – supports the reflexive activity; a sign belonging to a semiotic system can be diachronically transformed into another to connote and denote mathematical objects. They are two complementary and interwoven aspects of the same phenomenon.

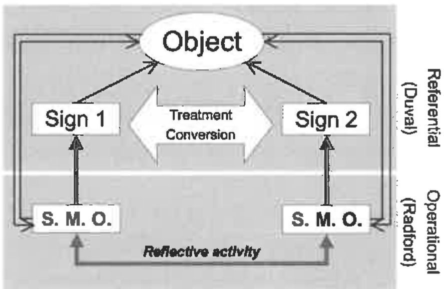


Figure 4. The complementary roles of Duval's and Radford's approaches in framing the meaning of mathematical objects.

If we disregard signs as semiotic means of objectification, learning is an empty and meaningless manipulation of signs, if we disregard signs as belonging also to semiotic systems, mathematical objects wouldn't have developed into the form of rationality we know today and their conceptual acquisition would be impossible. The changes of meaning can be traced back to semiotic transformations that are not sustained by a mediated reflexive activity that guarantees the relation to the common cultural meaning of the mathematical object. The technology of the semiotic system allows the transformations of signs, but meaning in its cultural and personal sense evaporates, thereby losing also the correct interplay between sinn and bedeutung.

Analysis of a Protocol

We present the protocol taken from an experimentation with primary school students working on sequences. According to a socio-cultural perspective students were immersed in a shared mathematical practice, working in small groups . We will analyse the sequence a_n=n^2+2n focussing on two different figural representations that are reported below.

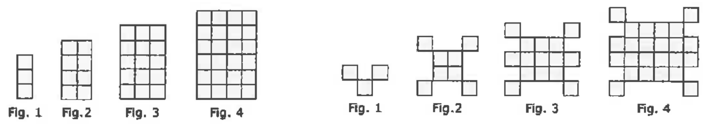
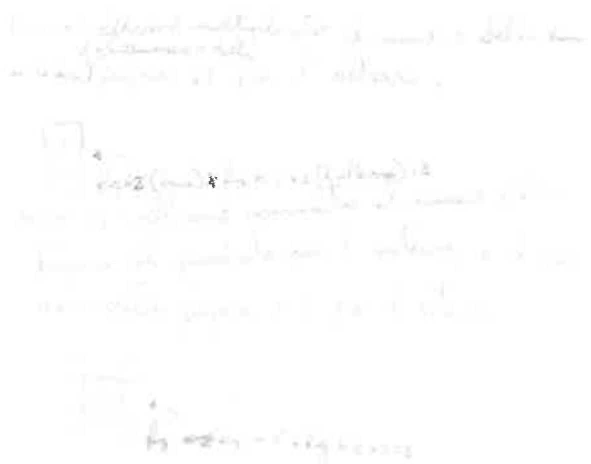


Figure 5. Two different figural representations of the sequence a_n=n^2+2n.

Most of the groups that were able to determine the general schema, also recognized the same sequence as the figural representation changed. Without any explicit request, some students even attempted a first algebraic symbolism to express the general rule Video tapes testify also students' rich sensory-motor activity, conveyed mainly by gestures, that I cannot relate here but it is clearly condensed in the explanation of the two schemas where the generality of the rule is expressed with spatial-geometrical properties as base, height, inside, outside. Below the protocol of group 5 with the general schema to determine the number of elements of any figure of the sequence.



We multiplied the number of the figure for the base and the number of the figure +2 as the height.

We added the square of the number of the figure for inside and the number of the figure × 2 for outside.

Figure 6. Protocol of group 5.

The pupils were successful in finding the schema for the general term of the sequence and they didn't change meaning when exposed to treatment transformation in the figural register. To understand how students accomplish this result we have to use the conceptual framework we constructed above. Using semiotic resources both as semiotic means of objectification and semiotic registers, pupils grasp the dual nature of mathematical objects and their meaning.

Cultural-semiotic interpretation. Students objectify the mathematical sequence within the sociocultural space of the classroom. The use of semiotic means of objectification pivots around the figural representation that allows also the synchronic use of gestures and the sensorimotor activity. The activity was extremely meaningful to the students because it was intimately connected to their embodied experience. As the students are more and more involved in the reflexive activity there is an increasing agreement between the personal meaning and the cultural meaning of the mathematical object, thereby accessing higher levels of generality. This accounts for both the recognition of the same sequence, as the figural representation changes, and the spontaneous attempt to introduce a syncopated algebraic notation for the general term of the sequence.

Structural and functional interpretation. Students carry out a complicated network of semiotic transformations that involve both treatment and conversion. The task proposed to students, requires to connect three semiotic systems: the figural register, natural language and the arithmetical register. First of all, a very difficult conversion is necessary to construct the function that associates the number of elements in the figure to the number of the figure. Also to recognize the general schema of the sequence, students perform a conversion that involves the above registers; they first find the number of elements for a small number then they generalize the schema to a big number, thereby arriving to the general term of the sequence. The conversions are carried out passing the following order: figural register-arithmetical register (to calculate the number of elements in the figure)-natural language (to represent the general term). The outcome of the coordination of such semiotic systems is that students recognize the common reference (bedeutung) as the figural representations (sinn) changes.

Our contention is that students are able to handle meaning correctly at the referential level because the semiotic transformation is supported, at the operational level, by a strong embodied reflexive activity that involves the students consciousness within a sociocultural space of signification.

THE TEACHERS CHOICES AS A CAUSE FOR MISCONCEPTIONS IN THE LEARNING OF THE ANGLE AS A MATHEMATICAL CONCEPT

Another research based on the *cultural-semiotic approach* is presented in Sbaragli and Santi (2011). Radford introduced the cultural-semiotic approach at the beginning of the 2000s and he ascribes to semiotics a central role within an anthropological viewpoint of mathematical objects and learning. This research shows how students' misconceptions on the concept of angle, highlighted by the broad literature on this subject, depend also on teachers' didactic choices relative to didactical transposition and didactical engineering. In particular, we focussed on the *incoherence of teachers' intentionality* with the cultural and conceptual aspects of the learning students should objectify. Such incoherences derive from a limited and unaware use of semiotics means of objectification.

The Cultural-Semiotic Approach

Referring to the phenomenology of Edmund Husserl (1913/1959), Radford (2006) associates objectification, regarded as the attribution of meaning, to an *intentional act* which places the subject in relationship to the object of knowledge and provides a particular understanding of such object. When considering scientific knowledge, particularly in mathematics, we have to face the issue of the interpersonal and general nature of mathematical objects. The subjective and situated meaning of intentional acts does not fully encompass the generality that characterises scientific knowledge.

According to the cultural-semiotic approach that we are following, we cannot reduce our individual experience to a solitary sensory and cognitive interaction with the world, but the way in which we intentionally enter into contact with reality is intrinsically determined by historical and cultural factors. The mediators, the artefacts, the gestures, the symbols, and the words which Radford calls *semiotic means of objectification* (Radford, 2003) are not only tools by which we manipulate the world, but mediators of our intentional acts, bearers of a historical consciousness built from the cognitive activity of the preceding generations. Such means determine and constitute the socially shared practices in which the processes of objectification develop.

Pupils and teachers find themselves immersed in a social and cultural context in which they find objects that are part of their culture. Institutionally, the teacher is in charge of guiding the pupil in the process of objectification, entrusting himself to the semiotic means of objectification and to the cultural ways of signification which culture and history have placed at his disposition.

It is useful for our analysis to take into account the fact that, according to Godino and Batanero (1994), and to D'Amore and Godino (2006), it is possible to attribute a personal and institutional dimension to the elements recalled above. The system of practices involves both a single individual and a group of institutionally recognised individuals, specifically the class. The same can be said for the mathematical object that exists both in a personal relationship with a subject and in an institutional relationship with the culture from which it emerged and with the social group that confers on it a knowledge value.

Learning, as a process of objectification requires an alignment between the personal dimension determined by the pupil's intentional acts and the institutional one that involves the historical and cultural aspects. The teaching-learning processes bring with them a dialectics between the personal aspects and the institutional ones bringing about the unification of the two dimensions towards a unified meaning. The construction of such a meaning, in which the unity of the individual with his culture is realised, is possible through the semiotic means of objectification that direct the intentional act of the individual towards the mathematical object. Such semiotic means, therefore, have their reason for being in that they are in the service of the intention of the individual and, at the same time, allow the embodying of knowledge and modes of rationality historically constructed by preceding generations. They, therefore, contribute to the creation of a shared meaning space that brings about the unity between the person and the culture, between personal meaning and institutional meaning, between individual intention and the object to which the intention is addressed.

It is necessary, therefore, to consider the complex network of individual and social practices, customs, beliefs, and convictions within which the teacher must daily orientate himself when he activates the mediators to encourage the learning of mathematical knowledge on the part of his pupils. This has to do with a network from which inconsistent behaviours can emerge on the part of the teacher.

It is from this point of view that it is possible to interpret the *avoidable misconceptions* (Sbaragli, 2005, pp. 56 and following) within the cultural semiotic perspective. In fact, such misconceptions depend directly on the choices of the teachers tied to the didactic transposition and the didactic engineering; two factors which, in the light of the cultural semiotic setting, turn out to be determining in the aligning of the personal meaning of the pupil and the cultural one, when the teacher manages the classroom practices.
In particular Sbaragli and Santi (2011) focus their attention on the mathematical object "angle", highlighting the incoherence between the cultural meaning of the angle and the intentional acts objectified by the semiotic means chosen by some teachers to their pupils. The existence of incoherence can lead students to avoidable (misconceptions); misconceptions that from a semiotic point of view hinder a correct coordination of different representations, when they are giving sense to the mathematical object.

Researches on the Angle

The research carried out by Sbaragli and Santi (2011) develops along two steps: the first is based on dialogues with 20 primary school teachers from different Italian teachers: the dialogues dealt with the concept of angle and their choices of the semiotic means of objectification they proposed to their pupils. The dialogues developed from the researchers' questions that aimed at triggering a discussion to highlight teachers' convictions on the angle and their educational choices. The second step is based on questions regarding the conceptual questions posed to primary school pupils (grade K-5) of the teachers mentioned above. We interviewed 8 pupils taken from each class of the 20 teachers for a total of 160 students. The students were chosen at random draw. We interviewed these pupils to understand in depth their convictions on the angle.

This research singled out incoherence between the teachers' intentionality and the mathematical concept their students should objectify. Incoherence can be traced back to an unaware and limited use of semiotic means of objectification. We analyse an example of such an incoherent behaviour.

An example of incoherence. Of the 14 teachers out of 20 who stated that the angle is the part of a plane comprised between the two half-lines with the common origin, 9 choose as a semiotic means the arc near the origin of the angle which limits a part of the plane, 3 choose the part of the plane coloured up to the arc, and 2 direct their attention to the unlimitedness of the part of the plane.

The 12 teachers who choose to indicate the arc or to colour the part of the plane up to the arc placed importance, with such graphic semiotic means of objectification, on the limitedness of the part of the plane and not on its unlimitedness; unlimitedness is instead contemplated in their definition because the part of the plane deriving from such definition turns out to be 'open'.

After the interview, the choices of these 12 teachers were divided into two categories: 5 relative to the *lack of awareness of the mathematical knowledge they bring into play* and 7 relative to *the lack of a critical sense with respect to their own choice.*

We report a part of the interview regarding the incoherent due to *lack of awareness of the mathematical knowledge.*

R.: Why did you choose this representation?

C.: Because the angle is represented like this.

R.: In what sense is it represented like this?

C.: When you want to talk about an angle, you draw it like this:

and the children know that we are talking about an angle.

In terms of the cultural-semiotic approach the there is no synchronic use of semiotic means of objectification. The restriction to the "little arc" fixes at a strong embodied level an incorrect objectification of the mathematical object keeping the student away from a rich mathematical activity that traces back the historical and cultural evolution of the mathematical object.

Note how this choice appears univocal in the eyes of that teacher.

The interview continues in the following way:

R.: Indicate, on this illustration, which angle you are speaking about.

(C. He indicates the part of the plane up to the arc).

R.: Up to where does the angle arrive?

C.: Up to here (he indicates the arc).

R.: Can you go beyond this arc?

C.: No, it goes up to here.

R.: Can't we go beyond the arc?

C.: In this case, no.

R.: And in which cases can we go beyond?

C.: If the angle is bigger.

(He draws another angle, apparently of the same amplitude, with longer half-lines and arc).

From this extract it emerges how misconceptions about the angle deriving from graphic representations, confirmed by classical research in the field and described in literature are present in some cases in the teachers themselves and therefore transferred to their pupils. The use of the "little arc" hinders the unlimited meaning of the angle that can be grasped at a higher level of generality that goes beyond the embodied meaning conveyed by this figural representation. The synchronic use of other semiotic means of objectification would allow to overcome this limit and access a disembodied meaning of this mathematical object.

The interview continued in the following way:

R.: Why did you choose this representation?

C.: Because this is the way to represent the angle.

R.: It is the way chosen by whom?

C.: By everyone, in all the books, it is like this.

R.: And do you like this representation?

C.: Yes, I have always done it this way, I don't see why I should change it.

R.: What, for you, is an angle?

C.: It is the part of the plane comprised between two half-lines that start from the same point.

R.: And how is this part of the plane?

C.: In what sense?

R.: What properties does this part of the plane have?

C.: I don't understand.

R.: Is this part of the plane of which you are speaking limited or unlimited?

C. He looks at his drawing, thinks a bit and then answers:

C.: It is limited by the half-lines.

R.: And here, how is it? (The researcher indicates the unlimited part of the plane).

C.: It arrives up to here (he indicates the arc).

R.: When I asked you what an angle is, why didn't you say that it arrives up to the arc?

C.: Because it isn't mentioned in the definition, but it becomes evident in the drawing.

We highlight that the graphic semiotic means of objectification is in contrast with the cultural meaning of the object conveyed by the verbal definition that the teacher expects her students to learn.

We highlight also that that the answer of the 160 selected students described in Sbaragli and Santi (2011) are not connected with the cultural and conceptual learning objectives of their teachers; in particular the graphic semiotic mean proposed by the teacher is stronger than her cultural and conceptual objective. In some cases, the graphic semiotic prevails to such an extent that it distorts the teacher's intention; for example when the extension of the angle is identified with the length of the little arc the little arc itself. Students confuse the graphic representation with the concept proposed by the teacher. Furthermore there are students' answers unexpected by their teachers deriving by everyday natural language (angle as synonymous of vertex) and a limited interpretation of the limited interpretation of the few and sometimes unique semiotic means of objectification proposed in the classroom.

The individual's (the teacher) intentionality plays a crucial role in the possibility of ascribing meaning to the mathematical object. Such an intentionality should be handled with awareness to be educationally effective. Referring to Husserl (1913/1959), the results of this research highlight that the teacher, in classroom practices, too often creates inconsistency between the intentional act that determines the way in which the object is presented to consciousness (noesis) and the conceptual content of the individual experience (noema). Consistency and unity of the different intentional acts of the teacher do not seem to be always present in the classroom practices, when dealing with the angle.

In fact, the inconsistency between the explicit intentionality of the teacher, through verbal means of objectification, and the graphicones, chosen to express this concept, can be the source of avoidable misconceptions in the mind of the pupil. The choice of the signs is not, in fact, neutral or independent. Radford (2005b, page 204) claims that «semiotic means of objectification offer several possibilities for carrying out a task, designating objects and expressing intentions. (...) It is necessary, therefore, to know how to identify the semiotic means of objectification to obtain objects of consciousness», such an identification should be managed with a strong critical sense on the part of the teacher.

Semiotic means of objectification should not be considered as *a priori* choices, that stem from outside the classroom without a critical analysis on the part of the teacher. To overcome *avoidable misconceptions* it is therefore essential to provide a variety of semiotic means that allow objectification processes within a social system of signification handled by teachers with awareness.

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Learning Rational Numbers: An Experimental Multi-Model Representation Approach Via Technology

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ABSTRACT: *The use of web-based instructional interventions for teaching and learning rational number concepts is common in today's middle-grades mathematics classrooms. Therefore, the purpose of this paper is to understand the cognitive effects of web-based instructional interventions with various representational components at different treatment durations. Students from multiple sixth-grade classes were randomly assigned to one of three treatment groups that differed by allocated session time (10, 20, and 30 minutes). The online manipulative tool (OMT), which was designed to scaffold learning in operations with rational numbers, allowed students to use in any order or sequence three different representational modes: (a) audio, (b) audio-visual, and (c) manipulatives. Participating students used OMT during their regular mathematics class (55 min.) so all the students received the same total amount of instruction. The regression analysis showed students who selected manipulatives had higher achievement than students who selected audio-visual and audio alone ($\beta = .437$, $p < .001$). Additionally, the 30-minute group used OMT the least and tended to use auditory and audio-visual representational modes (respectively, $d = -2.764$ and $d = -1.840$). There was a meaningful increase in the use of standard algorithm over manipulatives suggesting a transition from concrete to abstract thinking.*

Key words: *Online manipulative, Rational numbers, Representation, Treatment durations, Mathematics education, Technology, Middle school.*

INTRODUCTION

This study reports the effects of web-based instructional interventions on students' rational number learning and how various representational components function in student learning at different treatment durations. Previous research on the presentation of information in mathematics classrooms suggested the importance of multiple modes of representation (Goldin, 2003) and effective integration of computer activities (Kaput, 1998) to achieve meaningful learning (Ball, 1988).

MULTI-MODAL REPRESENTATION

A representation is defined as any configuration of characters, images, or concrete objects that can symbolize or *represent* something else (DeWindt-King & Goldin, 2003; Gagatsis & Shiakalli, 2004; Goldin & Kaput, 1996). Individual representations form larger representational systems that consist of characters or signs (i.e., letters, numbers, words, and real-life objects) as well as rules and practices for combining and operating on the signs (Goldin, 2003). Representational systems are both internal and external in nature. Internal representational systems are those that exist within the mind of the individual, whereas external representational systems can easily be shared with and seen by others. Internal representations consist of constructs that assist in depicting the processes of human learning and problem solving in mathematics (Goldin, 1998), and external representations consist of observable structures such as diagrams, formal language, and symbolic notations to express mathematics (Goldin & Shteingold, 2001). Using multiple representations to demonstrate the same concept helps to develop conceptual understanding (internal representations) and to strengthen one's ability to solve problems (Gagatsis & Shiakalli, 2004).

The National Council of Teachers of Mathematics (NCTM, 2000) in the *Principles and Standards for School Mathematics* and several researchers emphasized the importance of multiple representations for understanding mathematical concepts and relationships (e.g., Adiguzel & Akpinar, 2001; Elia, Gagatsis, & Demetriou, 2007; Goldin & Shteingold, 2001; Panaoura, Gagatsis, Deliyianni, & Elia, 2010; Panaoura, Gagatsis, Deliyianni, & Elia, 2009). There is strong evidence that students can grasp the meaning of mathematical concepts by experiencing different mathematical representations (e.g., Amato, 2008; Fennell & Rowan, 2001; Gagatsis & Shiakalli, 2004; Perry & Atkins, 2002; Suh, Moyer, & Heo, 2005) and making connections between and within these modes of representations (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). A student can demonstrate deep understanding of a concept as he or she translates among different representation of that particular concept. For instance, asking a student to restate a problem in his or her own words, to draw diagrams, to illustrate the concept, or to act out the problem are some ways for translating among representations. This translational skill among different modes of representation can support students' relational thinking and algebraic reasoning (Suh & Moyer, 2007).

An effective way of presenting students with multiple representations and providing opportunities for transitioning among representations is via virtual manipulatives. Although there is a substantial amount of research on the use of physical manipulatives (e.g., Green, Piel, & Flowers, 2008; Moyer, 2001), little is understood about how students interact with multiple representations and manipulatives in virtual worlds for computer-facilitated mathematics learning.

TECHNOLOGY AND MANIPULATIVES

Manipulatives are "physical objects specifically designed to foster learning" (Zuckerman, Arida, & Resnick, 2005, p. 859), and virtual manipulatives are "computationally enhanced versions of physical objects, created in an effort to expand the range of concepts that children can explore through direct manipulation" (Zuckerman et al., 2005, p. 860). The replication of physical manipulatives in the form of computer applications provides additional features and advantages over traditional manipulatives (Reimer & Moyer, 2005; Suh et al., 2005; Zuckerman et al., 2005).

Virtual manipulatives can be used in the same ways as concrete manipulatives. A computer mouse is the most commonly used interface for interacting with virtual manipulatives. With the mouse, students can flip, slide, and turn a virtual manipulative similar to the ways they can interact with a concrete manipulative. Moreover, virtual manipulatives can include additional features that make them more useful than concrete manipulatives for self-directed learning (Moyer, Bolyard, & Spikell, 2002). Instructional-support strategies incorporated into virtual manipulatives such as immediate feedback and help screens improve comprehension (Huang, Chern, & Lin, 2009; Yaman, Nerdel, & Bayrhuber, 2008) and self-efficacy (Wang & Wu, 2008). Some other affordances of virtual manipulatives include the safe environment they offer students to learn by guessing or trial-and-error (Suh et al., 2005). In fact, virtual manipulatives are identified as (a) helping students learn more about mathematical concepts by providing immediate and specific feedback, (b) reducing the amount of time it takes to learn to work with the manipulatives, and (c) enhancing students' enjoyment, attitude, and interest in learning mathematics (Reimer & Moyer, 2005).

Additionally, virtual manipulatives can provide a complete record of user interaction with the tool. For example, cursor movements and screen captures across time can be recorded so the teacher or the researcher can review students' processes as they attempt to answer each question. These archived data (screen captures) afford the teacher or the researcher the ability to examine, at length, the processes that may have led to errors or correct solutions even if the student has moved on to another question.

Multiple Representations in Virtual Manipulatives

Virtual manipulatives are advantageous in their capability to provide multiple mathematical representations. It has been shown that effective use of multiple mathematical representations helps students to increase their conceptual understanding. For example, integrating verbal information into visual information helps to reduce cognitive load for learners which makes information acquisition easier (Ertl, Kop; Kablan & Erden, 2008; Sweller, van Merrienboer, & Paas 1998). Connecting dynamic visual images with abstract symbols is another beneficial feature of virtual manipulatives. Unlike physical manipulatives, virtual manipulatives make use of graphics, numbers, and words on the computer screen to connect the pictorial with the symbolic mode (Suh et al., 2005). Linked representations ease the transitioning from one mode to another (Kaput, 2006).

Different modes in a virtual manipulative can include auditory, audio-visual (i.e., dynamic), and kinesthetic (i.e., interactive) components. Sweller et al. (1998) suggested utilizing audio in designing online manipulative tools to decrease the cognitive load that led to easier processing of information. In his recent review of research, Mayer (2005) concluded that students who received instruction supplemented with audio performed better than students who received instruction supplemented with on-screen text. Auditory representation complements the information presented in visual format. The audio-visual component in a virtual manipulative has a dynamic nature (Kaput, 2006). In other words, objects provided in audio-visual mode change with time. The dynamic feature of audio-visual mode facilitates transitioning between representations linking them together as a function of time. With the change in time, representations on the screen change simultaneously to make connections (Bolyard & Moyer, 2006). For example, when a fraction with a denominator seven is created algorithmically, the fraction strip associated with the fraction can be divided into seven pieces simultaneously. In addition to the dynamic feature, virtual manipulatives have interactive aspect that allows students manipulate objects to observe the change in different modes and make connections (Kaput, 2006). Unlike audio-visual mode, manipulatives will change according to students' inputs and interaction with the manipulatives.

LEARNING RATIONAL NUMBERS WITH VIRTUAL MANIPULATIVES

The NCTM (2000) stated that middle-grade students should have a deep understanding of fractions. However, rational numbers are one of the most difficult concepts for students to master (Perie, Moran, & Lutkus, 2005). There are various strategies for teaching rational numbers, and research has shown virtual manipulatives can be one of those (e.g., Reimer & Moyer, 2005; Suh et al., 2005; Witzel & Allsopp, 2007). However, teachers may avoid choosing virtual manipulatives as a strategy to teach rational numbers due to the lack of quality in the currently available virtual manipulatives (Donovan, 2008) or teachers' lack of training on the use of virtual manipulatives.

Time Needed for Learning with Virtual Manipulatives

Time needed for learning is an important consideration for teachers as they integrate virtual manipulatives both because students need enough time to master the software in addition to the concepts and because teachers have limited time to cover the curriculum. The effect of total amount of time devoted to instruction on student achievement has been investigated for more than 30 years. Research showed that the total amount of time allotted for instruction was a predictor of student success (Louw, Muller, & Tredoux, 2008; Wiley & Harnischfeger, 1974). That is, the more time allocated, the higher student achievement will be. This theory was later refined because simply allocating more and more time was inefficient. This gave rise to the importance of engaged time with the learning task versus the allocated time. Learners were not necessarily engaged in the instruction for the duration of allocated time. For optimal learning, students need not only be provided with necessary allocated time but also spend adequate amount of engaged time with the learning task. Cognitive scientists differentiate between allocated or engaged time and time needed for learning (Son & Sethi, 2006). The time needed for learning is related to individual differences as well as learning environment. Allocating or spending less time than needed for learning has a negative effect on student achievement (Gettinger, 1985).

The time needed for learning is also an important factor for improved achievement with virtual manipulatives (Rupe, 1986). Students need enough time to become proficient about the concepts they are being taught. In computer assisted instruction students need to master not only content knowledge but also the software used. Given that allocated time is fixed in middle-grade mathematics class, time spent learning software means that there will be less time allocated for learning the content. Thus, the design of software should be as transparent as it could be to avoid students struggling with the software but spending more time on the content. There is no standard time established in the literature for using computer assisted instruction for teaching mathematics (Bass, Ries, & Sharp, 1986; Salerno, 1995). However, Louw et al. (2008) found that more time on computer assisted instruction results in greater achievement. Per contra, more time does not always equate to greater achievement. Morrison (2008) and Son and Sethi (2006) suggest that there is an optimal time when learning reaches a peak and that design of instruction should consider this time frame.

EMPIRICAL RESEARCH ON VIRTUAL MANIPULATIVES

Even though there are several individuals and groups who are developing virtual manipulatives, there is limited research on virtual manipulatives' effectiveness. In general, of the available research some studies are on students' manipulative use in classrooms, and some others investigate teachers' perceptions of manipulatives. However, none of the studies have considered the time needed for learning with virtual manipulatives.

Classroom studies have mainly focused on the effectiveness of virtual manipulatives on mathematics achievement and student motivation. The findings from research on virtual manipulatives are somewhat mixed. Reimer and Moyer (2005), Suh and Moyer (2007), and Suh et al. (2005) showed statistically significant increases in students' achievement when the students used virtual manipulatives as compared to the students who used physical manipulatives or no manipulatives at all. However, in other studies no significant differences were found between students who used virtual or physical manipulatives (e.g., Dorward, 2002).

Research that focused on exploring teachers' perceptions of virtual manipulatives showed that to improve students' understandings of mathematics, teachers preferred using virtual manipulatives as cognitive technological tools (Moyer-Packenham, Salkind, & Bolyard, 2008). However, the frequency and allotted time for using virtual manipulatives differed among teachers (Mueller, Wood, Willoughby, Ross, & Specht, 2008). Teachers' pedagogical beliefs and confidence levels were mediating factors in their use of virtual manipulatives (Gadanidis, Gadanidis, & Schindler, 2003; Hermans, Tondeur, van Braak, & Valcke, 2008; Mueller et al.). Regardless of the grade level, teachers reported their students had higher motivation and more engagement when using virtual manipulatives (Crawford & Brown, 2003; Dorward, 2002; Hermans et al.).

In response to the available research discussed above, the author developed an online manipulative tool that incorporated different modes of representation of rational number concepts and assessed its effectiveness on students' transition from concrete to abstract thinking. In addition, the author investigated the optimal amount of engaged time with this tool for students to reach a peak in their learning.

RESEARCH QUESTIONS

Given the strong research support for the improvement of mathematical understanding with concrete manipulatives (e.g., Fennell & Rowan, 2001; Gagatsis & Elia, 2004; Goldin & Shteingold, 2001; Perry & Atkins, 2002; Suh et al., 2005), it is important to understand the effect of virtual manipulatives on student achievement. Using multiple representations via technology to present rational number concepts can provide evidence for understanding which representations students use and which representations are correlated with success (cf. Martin & Schwartz, 2005). In addition, the amount of time allotted with the online tool may have an effect on student selection of representational modes. Investigating this relationship between the time and the representations chosen can provide insight about students' transitioning between representational modes (i.e., between pictorial and symbolic representations). An important consideration in learning with virtual manipulatives is students' need to develop not only conceptual understanding but also understanding of how the manipulatives work. Thus, it is essential to investigate the optimal amount of time per session and number of sessions students need with virtual manipulatives to understand mathematical concepts.

Four major questions guided this study: (a) What is the relationship between students' representational selection (i.e., audio, audio-visual, and manipulatives) and student achievement?, (b) What is the relationship between treatment duration (i.e., 10-, 20-, or 30-minute) and the time spent on each representational mode selected (i.e., TRA, TRV, and TRM)?, (c) How does student achievement with OMT change between sessions in each group (i.e., 10-, 20-, and 30-minute groups)?, and (d) How does student achievement with OMT change during a session in each group (i.e., 10-, 20-, and 30-minute groups)?

METHOD

Participants

Sixth-grade students (32 female and 28 male) participated in this study. Eighteen students were Hispanic, 21 were African American, 11 were White, and 10 were other. This ethnic composition was similar to the district's which had 55% female, 33% Hispanic, 34% African American, 20% White, and 13% other. Table 1 presents the demographic information of the participants by the intervention group.

Table 1
Participants' Demographic Information by intervention Group

Group	Gender		Ethnicity				*N*
	Female	Male	African American	Hispanic	White	Other	
10-minute	13	7	7	6	4	3	20
20-minute	11	9	6	5	5	4	20
30-minute	8	12	8	7	2	3	20
Overall	32	28	21	18	11	10	60

Intervention

Online manipulative tool. The online manipulative tool was an interactive internet-based computer software program designed to present randomly generated addition, subtraction, multiplication, and comparison of rational number problems where each fraction was less than one, and the sums and products were all equal to or less than one.

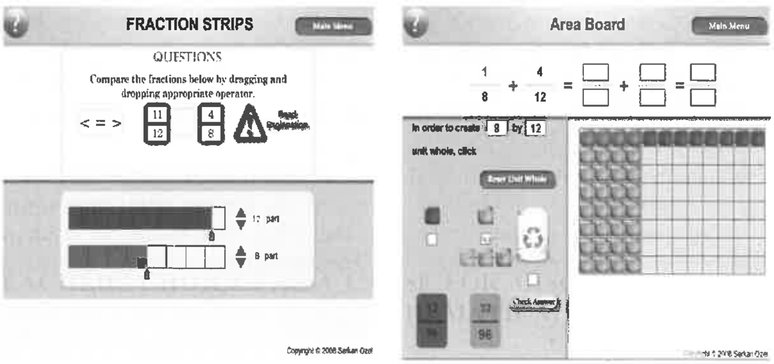

Figure 1. Screenshot of virtual fraction strips and screenshot of virtual area board.

The OMT contained two virtual manipulatives: (a) virtual area board (right) and (b) virtual fraction strips (left) (see Figure 1). The problems were provided on the same screen with virtual manipulatives and in algorithmic form. The OMT also included audio and audio-visual components that students used if they preferred to do so. The use of any representational modes in the solution of problems provided in OMT was optional. That is, students had the control over representations they preferred using in their solutions. The audio component consisted of instructions in the help menu and feedback for problems. In the help menu students could click any of the numbers, words, or symbols, and the tool read them aloud (See Figure 2). The feedback, which was provided in text as well as in audio format, included completeness of the algorithmic steps, each step's correctness, and if the answer was in the simplest form. The audio-visual component of OMT contained an instructional video on how to use the manipulatives. The video was provided to students to orient themselves to the online tool, and students could watch the videos as many times as they needed during their log in period.

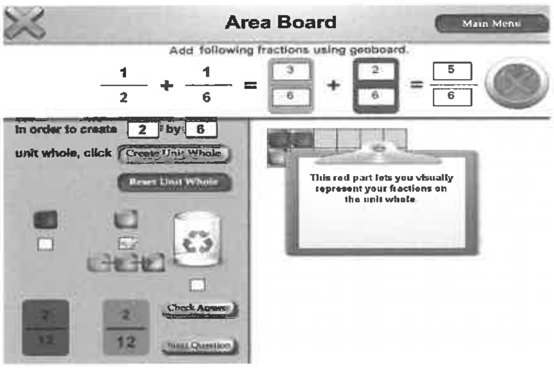

Figure 2. An example of clickable screen elements.

The tool employed several research protocols to improve the quality of the data collected. The OMT coded the content on the screen every 10 seconds. The coded information on the screen included every click, the question being solved, the current progress on manipulatives, the final solution in both algorithm and manipulative representations, and the feedback provided for students' answers. The purpose of screen coding was to provide precise information about students' progress to ensure a complete accounting of each attempt. Additional protocols to ensure data dependability and reliability were collection of data regarding the total time spent on each item and representation (i.e., audio, audio-visual, and manipulatives) and the internet protocol (IP) address. The time spent on each representation was recorded based on the activation and de-activation times of the representation.

Procedures

This study used experimental design where the intervention was an online manipulative tool (OMT) to scaffold learning of operations with rational numbers (i.e., addition, subtraction, multiplication, and comparison). Students from multiple sixth-grade classes participated in the study. The students in each class were randomly assigned to one of three treatment groups that differed by the allocated time per session: (a) 10-minute group, (b) 20-minute group, and (c) 30-minute group. Each group used the OMT three sessions per week over 3 weeks with the only difference among the groups being the time per session (see Figure 5). There were 20 randomly-assigned students in each group. This random assignment of students in each class to different treatment groups avoided the nesting issue of students within teacher (e.g. Raudenbush & Bryk, 2002) because in each group (i.e., 10-, 20-, and 30-minute groups) there were students from all of the teachers.

All groups used OMT without teacher or researcher assistance. The OMT allowed researchers to limit access for each student to a specific amount of time per day and three sessions per week. For example, a student assigned to the first group would be allowed to log in three times per week for 10 minutes per session. All students used their full allotment of time per week.

Participants used OMT for three weeks, which was the total time the teachers had allocated to learning rational number concepts covered by OMT. The mathematics class period was 55 minutes, and the intervention took place during this time only. All the students participated in direct instruction delivered by their teachers, but when the teachers assigned seatwork, participants logged onto the system. During the direct instruction the teacher followed the district curriculum and a textbook. The seatwork mainly consisted of teacher-prepared worksheets. All participants received their regular teacher instruction but were not held accountable for all assigned seatwork while they were using OMT. When students completed their computer time allotment, they returned to their class assignments. Participants in the same class were not all necessarily engaged with OMT for the same duration because students in the same class were randomly assigned to one of three conditions.

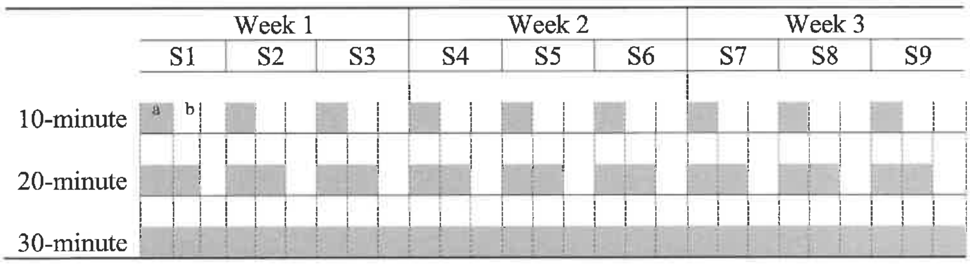

Figure 3. Online manipulative tool time allotment of each group by session and week.
[a] Each shaded cell represents a 10-minute period use of OMT. [b] Each white cell represents a 10-minute period of teacher-assigned activity.

Measures

Time. The OMT provided students with three representational modes, namely audio, audio-visual, and manipulatives. Students were free to use any of these representations. Three time variables were created for each representational mode to determine the time spent on each representation: time ratio for audio (TRA), time ratio for video (TRV), and time ratio for manipulatives (TRM). TRA, TRV, and TRM were obtained by getting the ratio of time spent using audio, video, and manipulatives to the total group time assigned to OMT.

Representational selection. Representational selection (RS) variable to determine students' representational preferences was created using the time spent on each representation. The representation that was used for the longest period determined the students' representational selections. For example, if a student used the audio component the longest, the RS for this student was coded as audio. The RS variable was coded as "1" for audio, "2" for audio-visual, and "3" for manipulatives.

Student achievement. Student achievement was assessed using the answers to problems presented by OMT. Students' answers to the problems were coded as either correct "1" or wrong "0." The total score for each student was calculated by getting the ratio of the number of correct answers to the total number of problems answered. It was reasonable to expect students who had more time would be able to attempt more items. Thus, the ratio provided a method for equitable comparisons across groups with different treatment durations. However, one caveat of calculating students' achievement scores by getting the ratio of the number of correct answers to the total number of problems answered is having extreme situations such as when a student who answers only one question which turns out to be correct may outperform another student who answers 20 questions correctly and 1 question incorrectly. However, the data were scrutinized for such cases, and none was observed.

Treatment duration. Treatment duration was a grouping variable. In other words, three different treatment durations (i.e., 10-, 20-, and 30-minute per session) determined three different groups.

RESULTS

Research Question 1

To determine the relationship between RS and student achievement, a multiple regression analysis was run. Students' representational selection variable, which had three levels, was coded into two dummy variables to compare (a) manipulative with audio-visual and audio and (b) audio-visual with audio on student achievement. The dummy variables were created to be independent from each other (r = 0). Overall the model was important, accounting for just over 37% of the variance in student achievement (see Table 2). Students who selected manipulatives had higher achievement than students who selected audio-visual or audio modes alone (β = .437, $p < .001$). When achievement of students who selected audio-visual was compared to the achievement of students who selected audio, the audio-visual representation was associated with higher achievement (β = .464, $p < .001$).

Table 2
Summary of Regression Analysis for Student Representational Selection Variables Predicting Student Achievement

Predictors	*B*	*SE*	β	*p*	r_s
Manipulative vs. Audio-Visual & Audio	.086	.020	.437	< .001	.672
Audio-Visual vs. Audio	3.104	.691	.464	< .001	.741

Note. $R^2 = .392$ ($p < .001$; Adjusted $R^2 = .371$).

Research Question 2

Spearman rho correlation was run to investigate the relationship between treatment duration and the time spent using audio, audio-visual, and manipulatives (i.e., TRA, TRV, and TRM, respectively). As the treatment duration increased (i.e., 10-minute to 30-minute), TRA, TRV, and TRM decreased (see Table 3).

Table 3
Non-Parametric Correlations (Spearman ρ) (N = 60)

	Treatment Duration	TRA	TRV	TRM
Treatment Duration	—			
TRA	-.801*	—		
TRV	-.612*	.873*	—	
TRM	-.342*	.761*	.829*	—

Note. TRA = time using audio, TRV = time using video, TRM = time using manipulatives.
*$p < .001$.

Therefore, more time on OMT was associated with less time with each of the representations. Because time spent on each of the three representations across groups was coded as a ratio of the time each representation was used to the total time, different treatment durations would not account for the obtained relationships.

Cohen's *d* effect size estimates were computed to determine the relative magnitude of difference in TRA, TRV, and TRM across different treatment durations. 10-minute group was used as the baseline, and all effect size estimates were calculated from that baseline. When comparing treatment duration by TRA, the use of audio (i.e., TRA) decreased in the 20-minute group with the obtained effect of d =-2.718 and in the 30-

minute group with the effect of d = -2.764. Thus, more time to engage with OMT was associated with less use of the auditory component. The TRV representation showed a similar pattern with the obtained effects of d = -1.291 for the 20-minute group and d = -1.840 for the 30-minute group. While not as dramatic, students used audio-visual representation less, as they gained experience with OMT. When examining TRM by treatment duration, the interest here was if students moved away from using manipulatives and went directly to the algorithm. The TRM followed a similar decreasing pattern as TRA and TRV, and as the treatment duration increased the TRM decreased with the obtained effects of d = -.574 for the 20-minute group and d = -.951 for the 30-minute group.

Research Question 3

To examine the change in students' success with OMT in each group the average percentages of correct answers in each session for each group were calculated and presented in Figures 4, 5, and 6. As seen in Figure 4, in the 10-minute group the average percentage of correct answers was almost stable around 28% from the first to the fourth session and started to increase from the fifth session on with a peak, 35%, achieved in the last session. In the 20-minute group, a slow increase from 32% to 35% was observed between sessions 1 and 7.

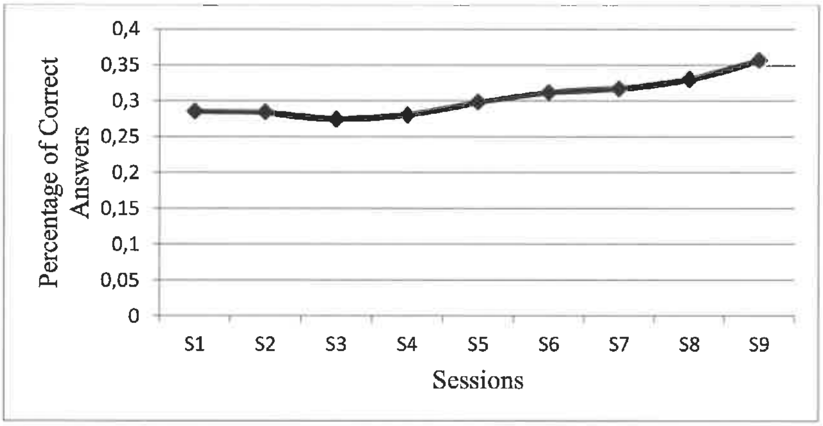

Figure 4. Percentages of correct answers in each session for 10-minute group.

However, the average percentage of correct answers increased at a higher rate between sessions 7 and 9 with a peak, 43%, achieved in the last session (see Figure 5).

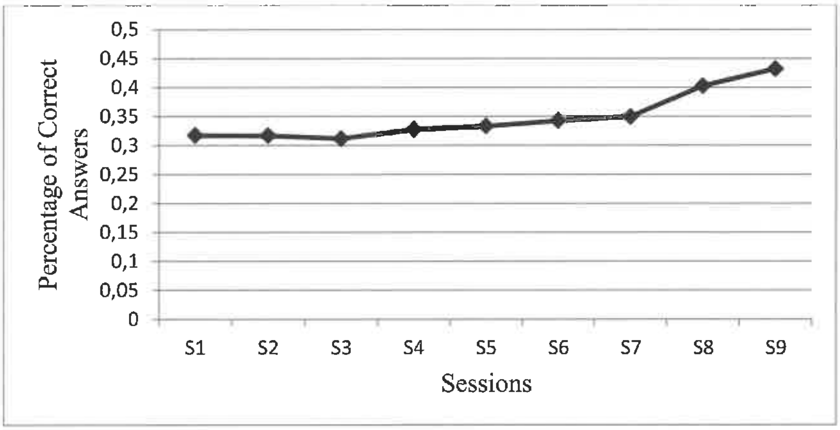

Figure 5. Percentages of correct answers in each session for 20-minute group.

The achievement pattern between sessions in the 30-minute group was substantially different than the other two groups. Students in the 30-minute group started with 74% correct response rate, which was almost twice as high as the other groups. However, this percentage decreased to as low as 69% in the fourth session (see Figure 6). From the fourth session on, the average percentage of correct answers started to increase at a decreasing rate till the last session, where a slight improvement was achieved compared to the initial correct answer rate.

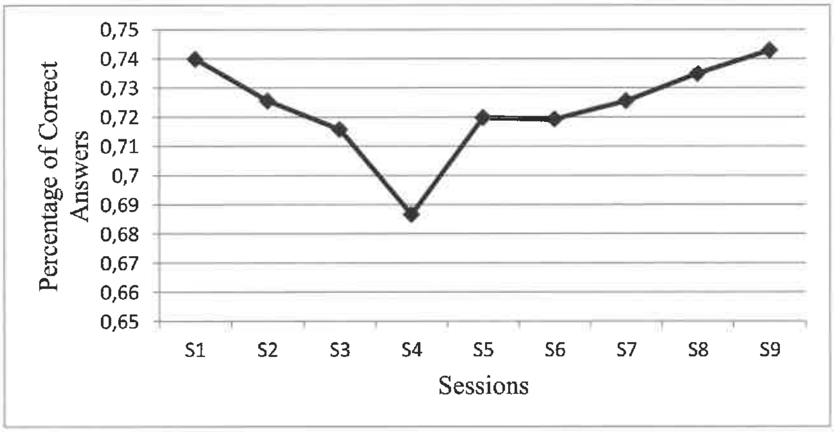

Figure 6. Percentages of correct answers in each session for 30-minute group.

Because the achievement pattern of the 30-minute group differed considerably from 10- and 20-minute groups, solution methods in the 30-minute group were examined closely (see Table 4). Students in the 30-minute group used the algorithm method increasingly more from the first to the last session with an effect size of $r = .69$ ($p < .001$). During the first four sessions, the incorrect response percentage when the algorithm was used was more than two times as much as the percentage of incorrect responses when the manipulatives were used. Starting from the fifth session the magnitude of the difference between the incorrect answer rate when the algorithm was used and when the manipulatives were used decreased till the eighth session when students became more likely to provide a correct answer when they used the algorithm.

Table 4
Frequencies of Answers by Method and Session in 30-Minute Group

			Answer		
Session			0	1	Total
1	Method	0	67.4%	21.5%	34.1%
		1	32.6%	78.5%	65.9%
2	Method	0	66.7%	19.0%	34.1%
		1	33.3%	81.0%	65.9%
3	Method	0	68.4%	19.5%	35.0%
		1	31.6%	80.5%	65.0%
4	Method	0	68.3%	25.2%	39.3%
		1	31.7%	74.8%	60.7%
5	Method	0	51.9%	34.1%	39.3%
		1	48.1%	65.9%	60.7%
6	Method	0	45.5%	42.7%	43.5%
		1	54.5%	57.3%	56.5%
7	Method	0	42.2%	48.2%	46.7%
		1	57.8%	51.8%	53.3%
8	Method	0	39.6%	57.2%	52.7%
		1	60.4%	42.8%	47.3%
9	Method	0	40.0%	65.3%	58.9%
		1	60.0%	34.7%	41.1%

Note. Method represents the mode used: Manipulatives = 1; algorithmic mode = 0. Correct answer = 1; incorrect answer = 0.

Research Question 4

Figures 7 present the mean cumulative percentage of correct answers over nine sessions taken at 2-minute intervals in 10-, 20-, and 30-minute groups. The data from nine sessions were averaged for each group using the following 2-step procedure: First, the cumulative correct answer percentages at 2-minute intervals were calculated for each session. Then, the averages of the percentages for each 2-minute interval over nine sessions were calculated. For example, percentages of cumulative correct answers at minute 2 from each session were averaged, and this procedure was repeated for each 2-minute interval in each group.

As seen in Figure 7, the correct response percentage in the 10-minute group increased during the sessions up to 30.4% with a decreasing rate in the second half of the sessions. In the 20-minute group the percentage of correct answers increased up to 37.5% till approximately 12th minute and then decreased till 16th minute to 34% where it virtually plateaued (see Figure 7). The percentage of correct responses during the first 20 minutes of the 30-minute group displayed a similar pattern as the 20-minute group (see Figure 7). The correct answers increased in the first half of the sessions, although to a larger percentage (i.e., 64%) than the 20-minute group, and then decreased and reached plateau till the 20th minute. After the 20th minute the correct response percent started increasing for approximately 6 minutes at a lower rate than it did during the first 10 minutes and then a plateau pattern appeared again.

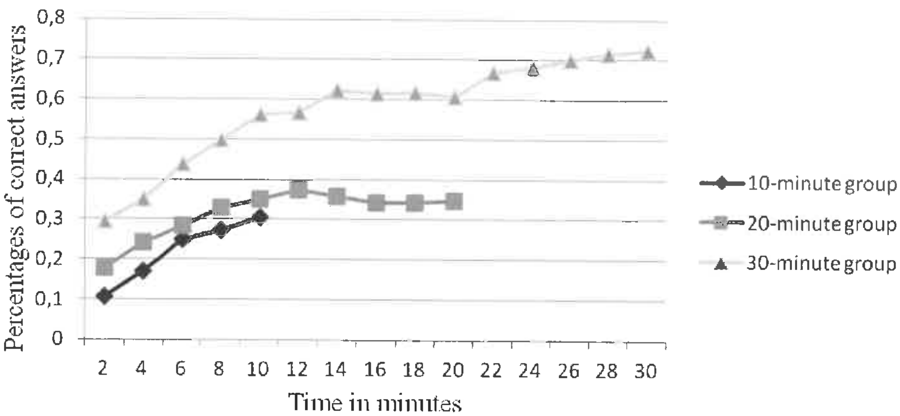

Figure 7. Mean cumulative percentage of correct answers over nine sessions taken at 2-minute intervals in 10-, 20-, and 30-minute groups.

DISCUSSION AND CONCLUSION

When the achievements of students with different representational selections were compared, it was found that students who selected manipulatives achieved better than students who selected audio or audio-visual. Auditory and visual learning are more passive learning and associated with perceived information; whereas, kinesthetic learning is more active and associated with both information perception and information processing (Felder & Silverman, 1988). Thus, students who used the manipulatives most might have not only perceived but also processed the concepts related to operations with fractions as they experimented with manipulatives.

The analyses showed that as the treatment duration increased, students became less reliant on audio and audio-visual components of OMT. Moreover, the longer treatment duration was associated with less use of the manipulatives. In other words, students in 30- and 20-minute groups used the manipulatives less frequently than the students in 10-minute group. It is especially interesting that even though students' more frequent use of the manipulatives was associated with better success, there was a meaningful increase in effect for students showing a transition away from the manipulative and toward the standard algorithm. This finding suggests that with longer use of the OMT students could be able to solve the problems only using the algorithmic procedures but not relying on audio, audio-visual, and manipulative components of OMT. This result supports the current literature on the effectiveness of virtual manipulatives helping students with transitioning from concrete to abstract thinking (Kaput, 2006).

Technology integration facilitates multiple modes of representations in a way that improves transition from concrete manipulatives to abstract thinking and provides a foundation for continued learning. Because OMT presented students with the algorithm and the concrete representation within the manipulatives simultaneously, students found a support and scaffold for their transitions from one mode to another. However, early transitions from one mode to a new one may result in a decrease in students' achievement. The finding that students' success initially dropped in the 30-minute group led to a closer investigation of this group's online tool use. One of the hypotheses for this finding was that students started with a mode (i.e., algorithm or manipulative) that was familiar to them in the first session and changed to the other mode, at which they were not as proficient, resulting in a decrease in their percentage of correct answers. As the students became experienced with the new mode, their success started to increase till they became as proficient with the new mode as they were with the initial mode. Further analyses supported this hypothesis as discussed in the third research question.

Time is one of the major problems that teachers encounter in their classrooms. Teachers need to carefully allocate their instructional time for each lesson. To completely cover curriculum, they have to address objectives within a limited window of time. Therefore, making optimal use of time for each activity is an important factor. In the case of an online manipulative tool, this optimal time contains the time needed for learning the tool and the content. When the student achievement in the 10-minute group was explored, there was a steady increase and no plateau pattern. This suggests that the students in 10-minute group were still in the process of learning the content. An interesting finding when the student achievement was examined across time within each group was that the similarity between the 20-minute group and the first 20 minutes of the 30-minute group. Both graphs almost plateaued approximately between 14th and 20th minutes. One possible explanation for this finding is students might have lost their motivation after working on OMT for almost 15 minutes. Thus, students could have either suspended answering questions or showed no improvement in their achievement with OMT. Even though students' learning might plateau some time after 30 minutes, there are time periods within 30 minutes found in the current study when students' learning was suspended. This finding suggests different factors to be considered when deciding the optimal time for learning. When considering optimal time, student motivation should be taken into account in addition to students' reaching a peak in their learning as Morrison (2008) and Son and Sethi (2006) suggested. When students lose their motivations, even if there is room for improvement in their learning, the lack of interest in learning can interfere with their improvement.

This study has useful implications for teaching and learning practice as well as for instructional designers. When teachers integrate such technology tools in their classrooms, they need to be cognizant of engaged time with the tool. Teachers need to provide students with enough time to become familiar with the technology and learn the content. However, teachers need to be attentive to students' motivations so that students would not get bored and start playing with the tool without learning the content. Students' individual characteristics and learning environments have impact on learning and motivation. Such tools might monitor student progress and provide the content with different difficulty levels. In addition to monitoring learning progress, students' motivation and interest can be tracked by getting input from students about their affect as they work with the tools. The tools can include different features that relate to different affective states of learners such as instructional games when they get bored.

This study is not without limitations. We had a limitation with the time frames provided for students. Because of logistical reasons and not to interfere with the teachers' lesson plans, the longest time period given students to work with the manipulatives was 30 minutes. In the current study, the plateau effect was identified in the 20-minute group and even more so in the 30-minute group. However, further studies might investigate if the plateau will continue after 30 minutes or if students will continue improving their achievement. Also, the sample size in this study was not large. Future studies with larger sample sizes further investigating the current finding that students transition from concrete to abstract thinking as they use OMT are encouraged.

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Self-Reported Reasons Developmental Mathematics Students Provided for Not Completing a Mathematics Course during their Senior Year of High School

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ABSTRACT: *Placement in developmental mathematics courses has been a barrier for some students in their educational pursuit. Completing high school mathematics courses that are at an academic level higher than Algebra II has been shown to be a predictor of college success. Previous studies suggest that both the rigor and the time when these courses are completed are related to college readiness. The present study sought to collect reasons developmental mathematics students were not completing a mathematics course during their senior year of high school. Findings indicated the primary reason for not enrolling in a senior-year mathematics course is because of a lack of interest. Findings also suggest some students are not being advised to complete a course and some students, particularly university students, were concerned about lowering their grade-point average. These results suggest improvements are needed for helping students develop an interest in mathematics, and all students need to be advised about the importance of completing more advanced mathematics courses.*

Key words: *Developmental mathematics, Interest in mathematics, Advisor*

INTRODUCTION

The need for students to understand and use mathematics in today's society has been a focus of the mathematics education community (National Council of Teachers of Mathematics, 2000). College and career readiness standards have been developed to help clarify the skills high school graduates need to be successful in society upon their high school graduation. Conley (2005) defined college readiness "as being able to meet the expectations they encounter in entry-level college courses" (p. xi) compared to college-eligible, which can be defined as being "able to meet admission requirements" (p. xi). Completing high school mathematics courses that are at an academic level higher than Algebra II has been shown to be a predictor of college success (Adelman, 1999).

In addition to the rigor of the mathematics courses, when these courses are completed has been shown to be related to college readiness (ACT, 2007; The College Board, 2006; Hill, 2006). The discussion of when mathematics courses should be completed can be traced back to over 100 years ago (Bidwell & Clason, 2002). In the 1899 *Report of Committee on College-Entrance Requirements*, it was stated that if a college-bound student is not being offered advanced algebra, "he should defer some or all of the mathematics at the eleventh grade until the last year of his school course, or be given opportunity for mathematical reviews in that year" (as cited in Bidwell & Clason, 1970/2002, p. 194). Educators want to encourage students to enroll in more mathematics courses; yet reasons for why students choose not to complete a senior-year mathematics course have rarely been collected. The purpose of the present study was to investigate reasons that some developmental mathematics students did not complete a mathematics course during their senior year of high school.

COURSE ENROLLMENT

High-Level Mathematics

Data from the High School Effectiveness Supplement (HSES) of National Educational Longitudinal Study: 88 (NELS:88) indicated a strong association between completing high-level mathematics courses and students' mathematical achievement at the end of high school (Lee, Burkam, Chow-Hoy, Smerdon, & Geverdt, 1998). Enrollment in higher-level mathematics courses also has been shown to help close the achievement gap (Secada, 1992; Walker, 2007). However, the High School Transcript Study (HSTS) indicated that although more students were enrolling in higher level mathematics courses, disparities existed between ethnic groups with advanced mathematics courses primarily comprised of Whites or Asians (Shettle et al., 2007, Walker, 2007). Lubienski (2002) found from an analysis of the National Assessment of Educational Progress (NAEP) 2000 data that more White students than Black students were enrolled in advanced high-school mathematics classes (i.e., trigonometry, precalculus, and calculus). However, Lubienski (2002) concluded that gaps identified on NAEP could be more closely linked to SES rather than race.

In a study that examined contributors to bachelor's degree completion, Adelman (1999) concluded success in college was impacted by the number of credits above Algebra II. Teitelbaum (2003) also suggested levels of mathematics and science courses students took were important in advancing students' proficiency in these subjects, and requiring more credits in mathematics and science by itself may not be enough to be effective. Thus, in addition to increasing the number of mathematics credits required, courses that contain both advanced topics and rigor will be needed to improve mathematical proficiency. Moreover, high-quality teachers must teach these courses.

In a study of graduation rates from 1991 to 2002, Greene and Winters (2005) found an increase (i.e., 9%) in the percent of students who were considered qualified to attend college (i.e., 34% in 2002). Over the same time period, graduation rates had remained unchanged (i.e., 72% in 1991 and 71% in 2002). Greene and Winters (2005) concluded that

> Flat high school graduation rates and increasing college readiness rates is likely the result of the increased standards and accountability programs over the last decade, which have required students to take more challenging courses required for admission to college without pushing those students to drop out of high school. (p. i)

Greene and Winters (2005) also found White students (40%) were more likely than Hispanics (20%) and Blacks (23%) to receive a regular diploma and be eligible to be admitted to college. In Texas, where the state university and three community colleges in the present study were located, the percentage of high-school graduates who were considered college-ready increased from 25% in 1991 to 32% in 2002 (Greene & Winters, 2005). The new legislation in Texas requires that students planning to graduate under the recommended or advanced high school programs must successfully complete four credits of mathematics from an approved list of courses (Texas Higher Education Coordinating Board, 2010). Using SAT scores and background information of 2006 high-school graduates, the College Board (2006) found that the average SAT reasoning score of students who took 4 years of mathematics in high school was 63 points higher than that of students who took only 3 years of mathematics. ACT's (2007) report showed that substantially larger percentages of students who took mathematics in their fourth year were considered college-ready compared to the students who took three years of core mathematics courses. ACT (2007) found only 16% of the students with 3 years of mathematics courses in high school were considered prepared for college-level mathematics; whereas, 38% of students with 3.5 years of mathematics and 62% of the students with 4 years of mathematics courses were ready for college-level mathematics courses.

High School Senior Year

Research is lacking on the impact of completing a mathematics course during a students' senior year of high school. As noted by Teitelbaum (2003), while more states are specifying courses students are required to complete for high school graduation, few if any, specify *when* mathematics courses should be completed. Teitelbaum (2003) concluded from NELS:88 data "that increasing the number of credits students have to earn in mathematics and science to graduate from high school by itself may not be sufficient to improve student proficiency in these subjects" (p. 31). Peter D. Hart Research Associates (2005) found when high-school graduates were asked about remedies to better prepare them for college and work, 74% suggested that requiring a fourth year math would make a difference.

However, in a study focused on entering freshmen at California State University, Chico, Ford (2007) suggested the number of mathematics courses completed above the Algebra II level was a more important factor than enrollment in a mathematics course during senior year in placing students into remedial courses. In fact, Ford found the number of students requiring remediation was only slightly more for the students who did not complete a mathematics course during their senior year. In contrast, Hill (2006) concluded from university data that "students who take no math their senior year are likely to place low and do poorly" and "generally, students who place into remedial math took no math their senior year, took an algebraically undemanding course their senior year, or did poorly in whatever course they took" (p. 18).

CONTRIBUTING FACTORS TO COURSE ENROLLMENT

Course enrollment in mathematics has been shown to be impacted by a number of factors including curriculum structure, school composition, students' background characteristics (e.g., SES, ethnicity, and gender), academic advisors, parents, teachers, interest, and motivation. Longitudinal data from the HSES of NELS:88, which contained samples from areas that are in or around US largest cities, indicated that curriculum structure – that is the level and number of courses offered – had direct and indirect impacts on subject-area student achievement and the trajectory of mathematics courses completed by students (Lee et al., 1998). When low-end course options were limited, students typically moved farther through the mathematics curriculum. Lee et al. (1998) also found "social as well as academic composition is important; the social-class level of the school influences achievement, even when several academic characteristics of students and schools were taken into account" (p. 33). Background characteristics were related to the number of mathematics courses completed, and students who completed lower-level mathematics courses were (a) Black or Hispanic, (b) female, (c) from lower-SES families, or (d) low-performers in mathematics. Lee et al. (1998) noted their "results suggest that where students go to high school has implications for their success" (p. 38). Students at schools that consisted of larger number of minorities from low-SES families or low achieving students were less likely to progress through higher-level mathematics courses. In a study conducted in Germany, Köller and Baumert (2001) found girls were less likely than boys to complete advanced courses suggesting gender differences still existed in academic choices.

Interest and Value

Updegraff, Eccles, Barber, and O'Brien (1996) stated that "course enrollment decisions are among the most influential self-regulatory behaviors students exercise in school because these decisions directly affect the opportunities students have to learn new material" (p. 239). Results from the Updegraff et al.'s (1996) study supported the need to impart the value of mathematics to students, particularly for improving mathematics enrollment practices.

Updegraff et al. (1996) found "perceived utility of math" was "the critical attitudinal predictor of the number of mathematics courses taken" (p. 256), even more so than interest in mathematics. However, interest in mathematics and perceived utility were related at a noteworthy level. Updegraff et al. (1996) stated, based on their findings, that "given the importance of the perceived utility of math as the critical attitudinal predictor of the math courses taken, interventions need to focus directly on increasing the perceived utility of high school math courses" (pp. 256–257). The perceived supportiveness of teachers has been shown to have an impact on junior high students' value of mathematics (Midgley, Feldlaufer, & Eccles, 1989). Therefore, the teacher is important in transforming students' value of and interest in mathematics.

Schiefele (1991) provided an overview of empirical findings that supported the importance of students' *interest*. Students who were interested in a subject area were more likely to utilize deep-level learning strategies versus surface-level learning such as rehearsal or memorization. In Updegraff et al.'s (1996) study, perceived utility was correlated with interest in mathematics ($r^2 = .35$). In the Köller and Baumert (2001) study consisting of students enrolled in Germany, interest in mathematics in Grade 10 was found to a have a substantial direct effect on subsequent enrollment in mathematics courses while achievement in Grade 10 did not have an additional direct effect on course selection.

Traditionally, precalculus or calculus has been the fourth-year mathematics courses available for students. However, not all students will find traditional fourth-year mathematics courses such as precalculus of interest or understand the utility of these courses. Therefore, they may elect not to complete these courses in high school. Providing alternate courses might aid in improving students' self-regulatory decisions to enroll in more mathematics courses. Charles A. Dana Center (2008) suggested that a high-school capstone course maintains and extends "prior mathematical knowledge, enhances the application of process skills, encourages the development of academic discipline and a positive attitude toward learning mathematics, and connects mathematics with varied student interests" (p. 1). Accordingly, Charles A. Dana Center (2008) developed fourth-year capstone courses.

PURPOSE

Recent figures have indicated that the percent of students enrolled in developmental mathematics courses remains higher than percentages of students in developmental reading or writing classes (Parsad, Lewis, & Greene, 2003). Studies have indicated a possible link between completing more mathematics courses and producing students who are better prepared for college and work (see Charles A. Dana Center, 2006a, 2006b). The decision to require a fourth-year of mathematics course will increase the enrollment in a senior-level mathematics course; however, we would hope that students would self-regulate into higher-level mathematics courses. The origin of the present study came from the first author's desire to collect evidence from students across various institutions of higher education beyond discussions with students and colleagues on reasons for not completing a mathematics course in their senior year of high school. The present study builds off a pilot study where students selected their reasons for not completing a senior-year mathematics course in high school from a list of items provided by the researchers. The top four reasons identified in the pilot study along with the percentages of students who chose those responses were as follows: (a) not interested (45%), (b) not advised to do so (33%), (c) other reasons (32%), and (d) did not want to lower GPA (20%). In the pilot study, students were not asked to specify other reasons.

The purpose of the present study was to collect evidence from developmental mathematics students for choosing not to complete a mathematics course during their senior year of high school and identify possible differences by race and gender. Research questions for the present study were (a) Why were students not completing a mathematics course during their senior year?; (b) Did noteworthy differences exist between Whites, Blacks, and Hispanics on their reasons for not completing a mathematics course during their senior year?; (c) Did noteworthy differences exist between females and males on reasons for not completing a mathematics course during their senior year?; and (d) Did noteworthy differences exist between university students and students enrolled at community colleges on not completing a mathematics course during their senior year because of a concern about lowering their GPAs?

METHODOLOGY

Sample

The convenience sample consisted of developmental mathematics students (n = 1802) whose teachers agreed to disseminate the survey request to their developmental mathematics classes. The students then self-selected to participate. We decided a priori not to survey community college students enrolled in the lowest level developmental mathematics courses. The students in the lowest level course were learning basic arithmetic skills indicating that those students encountered serious problems with mathematics; therefore, completing a mathematics course during their senior year of high school would not have prevented their placement in developmental mathematics courses. Students from one state university (n = 624) and six CCs (n = 1178) in Texas were represented. Participation varied from all teachers at one CC to only two teachers at two of the CCs.

Comparisons between the Sample and National Figures

Because student participation was contingent on teachers' willingness to participate and then their students' willingness to participate, caution should be warranted about generalizing the results to the general population. However, the results provide insight into reasons students provided for not completing mathematics during their senior year. Wilkinson and the Task Force on Statistical Inference (1999) recommended comparing population characteristics with sample characteristics. In a national report based on data collected by Integrated Postsecondary Education Data System (IPEDS), 58% of the national postsecondary population of students was females, combining gender for public 4-year and 2-year colleges (Knapp, Kelly-Reid, & Ginder, 2009). Only 83% of the present sample reported their gender; however, of the students who reported gender, there were slightly more females than the national percentage (i.e., 69% in the present study compared to the national figure of 58%). In regards to number of hours students worked, the community college sample was comparative to national figures. Results from the Community College Survey of Student Engagement (CCSSE, 2008) showed that 56% of community college students were employed more than 20 hours per week, and 53% of the present sample of community college students were employed more than 20 hours per week.

Except for the students enrolled at the one CC that required race and gender to be collected from the research department, race and gender were self-reported by the students and were optional items. Adelman (2005) discussed the importance of disaggregating student data by age. As noted by Adelman (2005),

> the differences between backgrounds, family and job commitments, and consequent academic behavior and progress of traditional-age (18-24) students . . . and those who start out at later points in life are so different that mixing the age populations does considerable disservice to understanding. (p. xiv)

In the present study, analyses investigating race were limited to students who were under the age of 26 and who identified themselves to be of only one race (i.e., not two or more races) (n = 1161). Our definition of traditional-age student differed from Adelman's definition in that we included 25 year-old students as traditional-age students. Of these 1161 students who were 18 - 25, 56% were White (n = 648), 25% were Black (n = 291), and 19% were Hispanic (n = 222). Of the White students under the age of 26, 44% were enrolled at a state university and 56% were enrolled at a CC. Of the Black students under the age of 26, 54% were enrolled at a state university and 45% were enrolled at a CC. Of the Hispanic students under the age of 26, 43% were enrolled at a state university and 57% were enrolled at a CC. Analyses investigating differences by gender were also limited to students who were under the age of 26 and who reported their gender (n = 1127).

The sample primarily consisted of developmental mathematics students who passed the course. Letter grades were collected from teachers. However some teachers did not respond or some students input incorrect college identification numbers so their grades could not be obtained. In the present sample of *all* students, 23.6% earned an A; 22.1% earned a B; 21.8% earned a C; 18.9% earned a D or an F; 1.5% withdrew from the course or received a grade of no credit or an incomplete; and 13.1% did not have a grade reported. In the sample, six students received credit for the course without a letter grade.

Instrument

Students were asked to complete a questionnaire regarding their mathematics enrollment practices during their high-school senior year. One option on the item that asked the students the course they completed during the senior year was "did not complete a mathematics course during my senior year of high school." If they did not complete a mathematics course during their senior year, they were asked to identify reasons for not completing a mathematics course. The reasons provided in the results section were developed by two members of the research team based on their collective number of years of experience teaching developmental mathematics courses at community colleges and research findings about the contributing factors to mathematics course enrollment practices. If a student marked *other* reason, then they were asked to specify further in open-ended responses. If a reason was provided frequently in open-ended responses, then a new category was created for that particular reason during data analysis.

Analysis

The present analyses were primarily descriptive in nature. Frequencies were reported to examine students' reasons for not completing a mathematics course during their senior year. Crosstabs were conducted in SPSS to examine the first three research questions. Chi-square statistics with Cramer's *V* coefficient were reported for examining differences between Whites, Blacks, and Hispanics on their reasons for not completing a mathematics course during their senior year. Phi statistics were reported to examine differences between females and males on reasons for not completing a mathematics course during their senior year. Differences between traditional-age community college and university students on not completing a mathematics course during their senior year because of a concern about lowering their GPAs were examined using chi-square statistics accompanied with Cramer's *V* coefficient.

RESULTS

Of the non-traditional age students, 58% did not complete a senior-year mathematics course. Of the traditional-age students who were under the age of 26, 48% (n = 619) did not complete a senior-year mathematics course; 4% (n = 44) completed a course below the level of Algebra II; 14% (n = 182) completed Algebra II; 4% (n = 51) completed Geometry; 5% (n = 70) completed Mathematics Models; 4% (n = 45) completed Algebra III; 14% (n = 181) completed Precalculus; 2% (n = 31) completed Statistics; 2% (n = 30) completed Calculus; 1% (n = 13) completed other courses at or above the level of Algebra II; 1% (n = 11) completed College Algebra, and 1% (n = 9) did not identify their completion practices during their senior year.

Reasons for Not Completing a Mathematics Course

Results for why developmental mathematics students self-reported they did not complete a mathematics course during their senior year of high school are provided in Table 1. Of the students who did not complete a mathematics course during their senior year, 2% did not provide a reason and 68% chose one reason. The majority – 94% of the 51 students over the age of 25 and 100% of the 19 students under the age of 26 – who self-reported that they did not complete a mathematics course during their senior year did not do so because either they (a) obtained their GED or (b) did not graduate decided not to select any of the other choice options as reasons for not completing a mathematics course. The large percent of students in this category who did not choose any other reasons might have believed these reasons did not apply to them because they did not attend high school during their senior year. Therefore, in addition to the initial disaggregation of the data by age (i.e., students under the age of 26 and students who are 26 or older) we also disaggregated the data by whether or not students had indicated they did not complete high school.

When considering all students under the age of 26, *not interested* was the primary reason for not completing a mathematics course during their senior year of high school (see Table 1). The next most popular reasons for not enrolling in a mathematics course among students under the age of 26 were because they (a) were not advised to take a course and (b) did not want to lower their GPAs. For students older than 25, including high-school graduates and drop-outs, not being advised to complete a course was the most popular response followed by not interested, dropping out of high school/obtaining their GED, and not thinking it was important. Among students older than 25 who did not indicate they did not complete a course their senior year because of either not graduating or acquiring their GED, the most-cited reasons for not completing a mathematics course in their senior year were not being advised to do so, not being interested, and not thinking it was important.

For students who had not dropped out of high school, crosstab results showed that statistically significant differences existed between students under the age of 26 and students older than 25, as seen in Table 1, on five reasons for not completing a mathematics course. These reasons included not completing a mathematics course because they were (a) not interested, (b) not advised to take a course, (c) did not want to lower their GPA, (d) did not think it was important, and (e) were advised to complete a course but decided not to. Percentages of traditional age students who did not complete a mathematics course during their senior year because of a lack of interest, concerns about lowering GPAs, other required courses, or decisions not to take a mathematics course although they were advised to were statistically significantly larger than percentages of students above age 25. On the other hand, larger percentages of students above age 25 than traditional age students did not complete a mathematics course during their senior year because they were not advised to do so or they did not think it was important.

Table 1
Reasons Developmental Mathematics Students Indicated for Not Completing a Mathematics Course During Their Senior Year Disaggregated by Age

	All Students			Excluding Students Who Indicated They Received Their GED or Dropped Out of High School		
Reasons Provided by Students:	Younger than 26 (*n* = 619)	Older than 25 (*n* = 288)	*Phi*	Younger than 26 (*n* = 600)	Older than 25 (*n* = 234)	*Phi*
Was not interested	42.81%	26.39%	.158***	44.17%	31.20%	.119**
Was not advised to take a course	24.39%	34.38%	–.104**	25.17%	41.88%	–.164***
Did not want to lower my GPA	22.46%	3.13%	.244***	23.17%	3.85%	.227***
Did not because I needed to work	11.95%	12.15%	–.003	12.33%	14.53%	–.029
Did not because of other required courses	9.50%	4.62%	.077*	9.17%	5.56%	.059
Did not think it was important	8.56%	13.89%	–.082*	8.83%	16.67%	–.112**
Other reasons	6.62%	5.56%	.020	6.83%	6.84%	.000
Had already met the graduation requirements[a]	6.30%	5.90%	.008	6.50%	7.26%	–.014
Advised to complete a course but decided not to	5.65%	1.04%	.107**	5.83%	1.28%	.098**
Dropped out of HS and/or obtained my GED[a]	3.07%	18.75%	–.268***	-	-	-
Tried to complete a course but dropped the course[a]	1.78%	0.69%	.042	1.83%	0.85%	.035
Graduated early[a]	0.81%	0.35%	.026	0.83%	0.43%	.022

Note. *** $p < .001$; ** $p < .01$; *$p < .05$. Students could mark multiple responses. Eight students younger than 26 and 8 students older than 25 did not respond.
[a]A theme created from open-ended responses.

Extra-curricular activities (n = 6)
Don't do well in mathematics or dislike mathematics (n = 4)
Teacher (n = 4)
Not offered or not enough to form a class (n = 2)
Not enough time (n = 3)
Lazy (n = 1)
Math is HARD! (n = 1)
Career prep program (n = 1)
Corrections Academy (n = 1)
Could not pay for college advancement (n = 1)
Didn't want the stress while pregnant (n = 1)
Focused on future major (n = 1)
Focused on TAKS (n = 1)
Health (n = 1)
High GPA (n = 1)
Home school (n = 1)
I'm an artist and a writer, neither of which require advance knowledge in math. (n = 1)
I had been taking AP classes since my freshman year and wanted a break especially since I was involved in clubs and worked over 40 hours per week. (n = 1)
I was advised that I did not need a math course, I had good grades, and was put on the work program during school. (n = 1)
No senior had a math unless they failed. (n = 1)
Wanted the free time (n = 1)

Note. Spelling errors were corrected.

*Figure 1.*Other reasons identified by students (n = 35) who were under the age of 26 as to why they did not complete a mathematics course during their senior year of high school.

Needing to work was among the most cited reasons by both traditional age (i.e., 11.95%) and students above age 25 (i.e., 12.15%) for not completing a mathematics course during their senior year. Figure 1 contains *other* reasons students (n = 35) under the age of 26 indicated for why they did not complete a mathematics course during their senior year of high school. Four students reported they did not do well in mathematics or they disliked mathematics, and 6 students reported they were taking extra-curricular activities as reasons for not completing a mathematics course during their senior year.

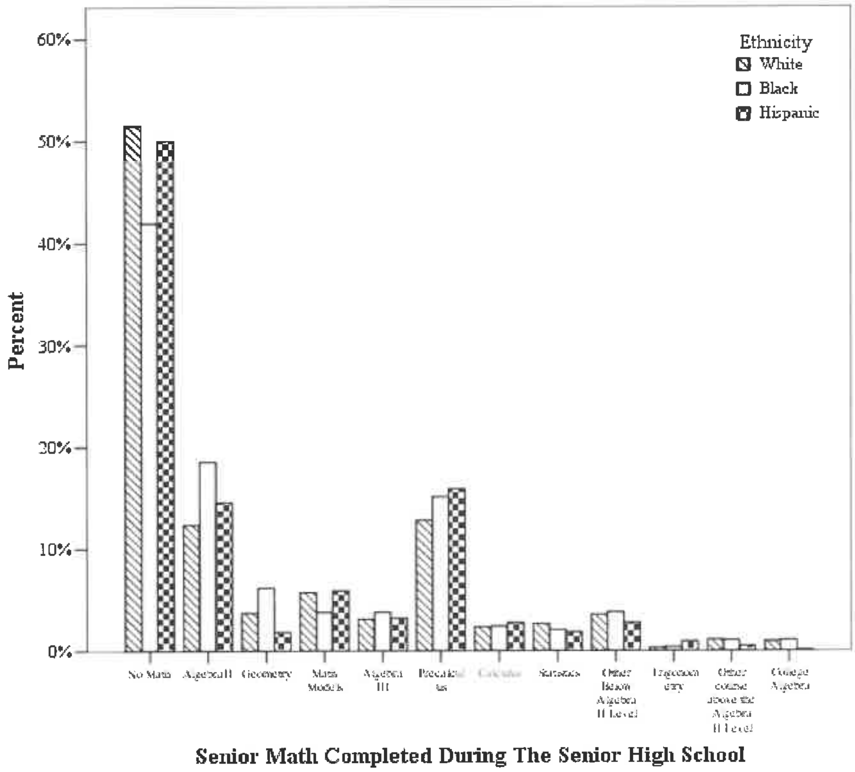

Figure 2. Enrollment practices of Hispanic, Black, and White developmental mathematics students in the present sample who were under the age of 26.

Race and Gender

Because recent graduates can better inform about the present population of high school students, we limited the data to traditional age students for our examination of the race and gender related differences in the reasons provided by the students. For students under the age of 26, statistically significant differences existed between races on completing a mathematics course during their senior year [$\chi^2(2, N = 1161) = 7.69, p = .021$] although the effect size was small (Cramer's $V = 0.08$). More Whites (51%) and Hispanics (50%) did not complete a mathematics course during their senior year of high school than Blacks (42%). Figure 2 illustrates how enrollment practices varied among Blacks, Hispanics, and Whites by course. We classified Mathematics Models at the Algebra II level because of our experience that Mathematics Models is often considered a remediation course. Other courses categorized as below the Algebra II level were described by students as basic or fundamental mathematics courses. In the present sample, there were no statistically significant differences between males and females who were below the age of 26 on completing a mathematics course during their senior year (*Phi* = .021, *p* = .483).

Table 2
Reasons Developmental Mathematics Students Who Were Under the Age of 26 Indicated for Not Completing a Mathematics Course During Their Senior Year Disaggregated by Race and Gender

	Race				Gender		
Reasons Provided by Students:	Whites (*n* = 334)	Blacks (*n* = 122)	Hispanics (*n* = 112)	Cramer 's V	Male (*n* = 163)	Female (*n* = 365)	*Phi*
Was not interested	44.31%	36.89%	44.64%	.062	50.92%	40.00%	–.102*
Was not advised to take a course	25.15%	26.23%	25.00%	.011	19.63%	25.75%	.066
Did not want to lower my GPA	22.16%	24.59%	25.00%	.031	22.70%	25.21%	.027
Did not because I needed to work	14.37%	8.20%	8.04%	.095	8.59%	12.60%	.058
Did not because of other required courses	9.28%	6.56%	8.93%	.039	5.52%	10.41%	.079
Did not think it was important	9.28%	5.74%	8.93%	.051	6.75%	10.41%	.058
Other reasons	7.49%	4.92%	5.36%	.047	6.75%	5.21%	–.031
Had already met the graduation requirements	5.39%	5.74%	7.14%	.029	4.29%	7.40%	.058
Advised to complete a course but decided not to	4.79%	5.74%	5.36%	.018	4.91%	4.66%	–.005
Dropped out of HS and/or obtained my GED	4.19%	1.64%	2.68%	.059	1.23%	3.01%	.053
Tried to complete a course but dropped the course	2.10%	1.64%	0.89%	.035	1.23%	1.92%	.025
Graduated early	0.90%	0.00%	0.00%	.061	0.61%	1.10%	.023
Did not provide a reason	0.60%	1.64%	0.89%	0.60%	1.23%	0.55%	–.036

Note. Some students reported either gender or race but not both. Gender also included students who marked two or more races. Therefore, some students were not represented in both gender and race. $*p < .05$.

Table 2 provides crosstab results for the reasons traditional-age students provided for not completing a mathematics course during their senior year, which were disaggregated by race (n = 568) and gender (n = 528). Remember that almost half of the sample completed a mathematics course their senior year; therefore the sample size is approximately half of the original values 1161 and 1127 reported earlier. Some students did not report either their race or gender; therefore, some students were represented in the race or gender results but not both. Although chi-square tests with Cramer's V statistics indicated no noteworthy differences between ethnicities and crosstab results with *Phi* statistics indicated no noteworthy differences between genders on reasons provided by students, comparisons of percentages suggested some minor differences existed on two items.

Approximately 37% of Blacks marked they did not complete a mathematics course during their senior year of high school because they were not interested in mathematics compared to approximately 44% of Whites and Hispanics. Females were slightly less likely than the males to mark not interested and slightly more likely to mark they were not advised to take a course.

Type of Institution

For our investigations between institution types, we limited the data to students who were under the age of 26 and who had not completed a mathematics course during their senior year of high school. Universities would have admission policies, and we wanted to determine if GPA was a factor in students' reasons for not choosing to complete a mathematics course during their senior year. Chi-squared results indicated statistically significant differences existed between the state university and community college students on choosing not to complete a mathematics course because they did not want to lower their GPAs ($\chi^2(N = 622, 1) = 15.79, p < .001$, Cramer's $V = -0.16$). Of the students under the age of 26, who had not completed a mathematics course during their senior year of high school, 30% (n = 85) of the university students and 16% (n = 55) of the CC students marked they did not complete a mathematics course during their senior year because they did not want to lower their GPAs.

DISCUSSION

Listening to students is important to facilitating change in academia. Requiring more mathematics courses will increase the enrollment; however, we hope that students will self-regulate into completing more mathematics courses. If educators are to help students make better academic choices regarding high school mathematics courses and increase their desire to complete more rigorous mathematics courses, reasons students forego completing senior-year mathematics courses need to be investigated. Results from the present study support the need to increase students' interest in mathematics, reach all students prior to their senior year of high school regarding the importance of completing a senior-year mathematics course, and convey the message that GPA may help students become college eligible but not college ready. In addition, a large percent of the developmental mathematics students in the present study completed a course during their senior year of high school, which suggests completing a mathematics course during the senior year of high school will not be a guarantee that students are college ready.

Interest in Mathematics

The present study suggests that developmental mathematics teachers' classrooms are comprised of students whose primary reason for not self-regulating into more mathematics courses during high school is a lack of interest in mathematics. We realize not all students will be interested in mathematics, but developing an interest in mathematics is important to removing the mathematics barrier that prevents many students from receiving their degrees or certificates. Innovative college curriculum and teaching are needed to engage student's mathematical interest. The need to make curriculum changes at institutions of higher education has been realized. Providing alternate courses might aid in improving students' self-regulatory decisions to enroll in more mathematics courses. The Charles A. Dana Center (2008) developed a fourth-year capstone course that "encourages the development of academic discipline and a positive attitude toward learning mathematics, and connects mathematics with varied student interests" (p. 1).

Two new pathways through developmental mathematics are being developed and supported by The Carnegie Foundation for the Advancement of Teaching (CF) in collaboration with the Charles A. Dana Center. The first pathway is the Statistics Pathway (Statway), and the second pathway is Quantway. According to the CF (2012), "both the Statway™ and Quantway™ pathways include an intensive student engagement component within the classroom environment focused on increasing student motivation and tenacity" (p. 1). Moreover, "both pathway designs will build on effective current student support practices, increase communication between students and their mathematics faculty, and make access to college support services more visible and user-friendly" (p. 1).

Updegraff et al. (1996) found perceived utility of mathematics as the best indicator for choosing to complete more mathematics courses, even more so than interest in mathematics. Some students, interest in mathematics and perceived utility were related at a noteworthy level. "Perceived utility" or "value of mathematics" was not a choice that was included in the current study. Some students did not know completing a mathematics course during their senior year of high school was important for future success; therefore they might not have understood the value of mathematics. Success extends beyond the college mathematics classroom, and mathematics is important for careers beyond STEM majors. In an open-ended response, one student stated "I'm an artist and a writer, neither of which require advance knowledge in math." Clearly, this student did not understand the importance of geometry in art and how improved analytic skills could transfer to writing. Updegraff et al. (1996) stated, based on their findings, that "given the importance of the perceived utility of math as the critical attitudinal predictor of the math courses taken, interventions need to focus directly on increasing the perceived utility of high school math courses" (pp. 256–257). The perceived supportiveness of teachers has been shown to have an impact on junior high students' value of mathematics (Midgley et al., 1989). Therefore, the teacher is important in transforming students' value of and an interest in mathematics.

Advisement

The second leading reason, after lack of interest, for not taking a mathematics course in the senior year of high school was not being advised to do so. Although no statistically significant differences existed between genders on advisement, comparisons of responses in Table 2 suggest that slightly fewer females in the present sample are being advised to take a mathematics course during their senior year. These results indicate that some students need further encouragement to seek advisement to make academic choices that will help them become college ready. Conley (2005) illustrated the influence people have on academic choices students make and how these choices help determine the academic options afforded to students upon entering college. Because counselors and teachers face the responsibility of reaching out to all parents and students, other innovative sources of student encouragement may need to be implemented. One way of encouraging students to complete a mathematics course may be to have current developmental mathematics students speak to high school students about the need to build better mathematics skills during high school and their senior year so as to avoid being placed in developmental mathematics courses.

Meeting Graduation Requirements

Setting graduation requirements is essential; however, some students will take the minimal requirements. As illustrated in Table 1, 6.5% of the students under the age of 26 who had not completed a mathematics course during their senior year of high school did so because they had met the graduation requirements. This was a theme that was established from developmental mathematics students' responses and was not a choice provided. Responses from some of these students suggested these same students did not understand the importance that completing these courses would have on their college- and career-readiness. The following are some of the students' statements that illustrate this point: (a) "already completed my math," (b) "already took what was required," (c) "already had my credits," (d) "because I finished math before my senior year of high school," (e) "had already met entrance requirements," (f) "had taken all that was required of me at the time", (g) "wasn't required and I had ample credits," and (h) "because I took 4 years already."

Other Reasons

Other reasons for not completing a mathematics course during their senior year are provided in Figure 1. The list of other reasons varied. Four students reported they were not good at mathematics or they disliked mathematics. In a previous study, Abedi, Hofstetter, Baker, and Lord (2001) found that on the NAEP survey "more than half (54%)" of the eighth grade students in the sample from southern California "agreed or strongly agreed with the statement, 'I am good at mathematics.'" (p. 27). Abedi et al. also found that *how good someone believed they were at mathematics* was a predictor of performance in mathematics.

Some students indicated extra-curricular activities ($n = 6$) as reasons for not choosing a mathematics course. One student noted that after "taking AP classes since my freshmen year [I]wanted a break especially since I was involved in clubs and worked over 40 hours per week."

High School Dropout Rates

As seen in Table 1, some of the students over the age of 25 did not complete a mathematics course during their senior year because they had dropped out of high school. Although Greene and Winters (2005) indicated the graduation rate remained flat from 1991 to 2002, students dropping out of high school remains to be a concern of educators. Encouraging a fourth year of mathematics during students' senior year of high school will not impact the students who leave. Therefore, educators need to continue to implement initiatives to encourage students to complete high school.

Work

Economic factors are not something within the control of the schools. In the present sample, approximately 12% of the traditional age students indicated they needed to work as a reason for not completing a mathematics course during their senior year. The majority of these students were attending a CC. As noted by Provasnik and Planty (2008),

> the population of immediate enrollees going to community colleges includes seniors from a wide spectrum of family backgrounds. However, at the same time, these immediate enrollees consist disproportionately of seniors who are among the least likely to attend a 4-year college or university right out of high school—Hispanics and those from families in the lowest quarter of SES. (p. 15)

Some students in the present study were making academic choices that may hinder their college-readiness based on decisions to work. Only a small percentage of this group self-reported they were receiving financial aid when they entered a college or university. As noted by King (2004), low-income students often miss opportunities to apply and receive financial aid. While we cannot eliminate students' need to work, we can attempt to alleviate added financial stress by reaching these students when they register and are enrolled in classrooms. Explaining to students, prior to their high school graduation, that all students should apply for financial aid is essential to lessening outside burdens that may hinder student success. High school, community college, and university instructors can provide financial aid packets to their students in the classroom. High schools can encourage and institutions of higher education can require students to complete online questionnaires that help inform students of possible financial opportunities. To help students who were working, King (2006) included the following recommendations: additional grant aid, personal financial management education, and making work experiences, academic goals, and career goals more supportive of each other.

CONCLUSION

The more traditional-age developmental mathematics students in the present study were not completing a mathematics course during their senior year of high school because they were not interested followed by not being advised, concern over lowering their GPA, and a need to work; however the students concerned about lowering their GPA were primarily the students who attended a state university. These results suggest that for this population of developmental mathematics students more needs to be conducted on developing and successfully implementing a curriculum that will help students value mathematics, and that these curriculum changes need to occur both in K-12 and in developmental mathematics. Our findings also suggest that subsets of high school students are not being advised to complete a mathematics course during their senior year of high school.

Limitations and Future Research

The sample was a convenience sample and consisted primarily of developmental mathematics students who were passing the course. In addition, student participation was contingent on both teacher's and student's willingness to participate. The sample was also primarily comprised of students who passed the mathematics course, which was a consequence of either the students' willingness to participate or teachers' disseminating the survey prior to the drop date. However, the community college sample was comparable to the national population on the number of hours per week they were working although a slightly larger sample of females was represented.

In future studies, all developmental mathematics students should be surveyed regarding their mathematics academic choices. For example, students who did complete a course may have completed a course because the course was required for graduation, yet these students may still lack an interest in mathematics. In addition, more survey options should be provided for students to select from, based on the themes developed and students' open-ended responses in the current study. Besides the themes provided in Table 1, some additional reasons to consider adding based on open-ended responses include (a) dislike mathematics, (b) not good at mathematics, (c) teachers, (d) extra-curricular activities, (e) no mathematics courses were offered at a level I believed I could be successful, (f) no mathematics courses were offered beyond the courses I had completed, and (g) was not planning to attend college. We recommend that the impact of completing a senior-year mathematics course on placement and college success be investigated.

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What is Given in the Problem? Looking through the Lens of Constructions and Dragging in DGE

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ABSTRACT: *This paper provides an analysis of geometry tasks in light of associated investigation procedures in DGE. It demonstrates that construction and dragging are task dependent and reveals relationships between the order of the givens in the problem, the construction of the given geometrical object, and corresponding dragging procedures. The paper claims that soft constructions may be used effectively in the investigation process without use of robust constructions.*

Key words: *Geometrical construction, Dragging, Investigation, Geometry tasks.*

BACKGROUND

Raising the Quality of School Mathematics by Using Investigation Tasks

This paper is based on the assumption that using inquiry tasks raises the quality of mathematics lessons and intensifies mathematical knowledge construction by students. In their comprehensive analysis of mathematics lessons in the US, Germany, and Japan, Stigler and Hiebert (1999) pointed out the importance of the type of mathematics that is being taught: "If the content is rich and challenging, it is more likely that the students have the opportunity to learn more mathematics and to learn it more deeply" (p. 57). Researchers have considered the *quality of school mathematics* as a function of content elaboration, content coherence, and making connections. The quality of mathematics at each lesson contributes to the development of the students' mathematical understanding. The dialog taking place during mathematical inquiry (Wells, 1999) has been seen as a means of increasing the quality of school mathematics. Inquiry tasks are usually challenging, cognitively demanding, and enable highly motivated work by the students. In environments of this nature student conjecture, debate the conjectures, search for explanations and proofs, and discuss their preferences regarding different paths leading to a solution. Yerushalmy, Chazan and Gordon (1990) emphasized the importance of inquiry-based instructional activities:

> In using "inquiry," we want to bring to the readers' mind the process of learning employed by creative people at the forefront of the field – people interested in a particular area and continuously motivated to learn more

> about it, who set themselves problems; design methods to explore them; and then try to create the solutions (p 219).

Facilitative and inquiry-based teaching are central to many reform-oriented curricula that establish, strengthen, and broaden conjecturing, exploration, and investigation procedures in the mathematics classroom (e.g., Jaworski, 2003; Yerushalmy et al., 1990). Inquiry-based mathematics classes are consistent with a constructivist view of learning because they offer challenges to stimulate mathematical thinking and create opportunities for critical reflection on mathematical understanding (von Glasersfeld, 1996; Cobb, Wood, & Yackel, 1990; Wells, 1999).

Borba and Villarreal (2005) integrate different meanings of "experimentation" in education and regard experimentation (inquiry) as a "different way of learning." Inquiry includes (p. 75) the use of tentative procedures and educated trials that support the generation of mathematical conjectures; the discovery of mathematical results previously unknown to the experimenter; the possibility of testing alternative ways of obtaining a result; and the opportunity to propose new experiments. The researchers add (*ibid.*, pp. 75-76) that the "experimentation approach gains more power with the use of technological tools" by providing learners with the opportunity to propose and test conjectures using multiple examples, obtain quick feedback, use multiple representations, and be involved in the modeling process.

Inquiry and Dragging in Dynamic Geometry Environments (DGEs)

Inquiry in geometry is naturally associated with use of DGEs. Many studies have explored the role of DGEs in the instructional process, specifically in concept acquisition, geometrical constructions, proofs, and measurements (e.g., Mariotti, 2002; Jones, 2000; Hölzl, 1996, Yerushalmy & Chazan, 1993; Chazan & Yerushalmy, 1998). Other studies have concentrated on the theoretical foundations of working with dynamic geometry tools, e.g., the differences between drawing and figures (Laborde, 1993), differences between soft and robust constructions (Hoyles; 2000), the modalities of dragging (Arzarello, et al., 2002), and understanding the dynamic behavior of geometrical objects in DGEs (Talmon & Yerushalmy, 2004). Dragging is usually considered as a problem solving tool that can include construction, searching for commonalities, conjecturing, and proving and refuting conjectures (Talmon & Yerushalmy, 2004; Hölzl, 1996; Healy & Hoyles; 2001).

Dragging is a critical feature of DGEs, which makes experimentation possible. The two main functions of dragging are *testing* and *searching* (Hölzl, 2001):

1. Testing verifies that the figure constructed in the process of experimentation satisfies all the conditions given in the task.
2. Searching is aimed at finding new properties of the given figure and recognizing unforeseen regularities, relationships, and invariants.

The research literature describes several types of dragging that can be performed in DGEs as part of conjecturing tasks. Following Arzarello et al. (2002) and Smith (2002), Talmon and Yerushalmy (2004) summarize dragging modalities as follows:

Wandering dragging, random dragging used when exploring a construction; the *drag test*, random dragging used to diagnose the correctness of the construction; and *lieu muet dragging*, dragging a point in a way that preserves some features [These three are based on works by Arzarello].... *Bounded dragging*, dragging a point along a perceived trajectory.

In light of dragging geometrical constructions, another distinction is also important for the analysis of mathematical problems provided in this paper. In this paper I suggest an analysis of geometry problems through the lens of investigations that include dragging in DGE. For this purpose I transform geometry problems into conjecture-type tasks that are intended to help students discover mathematical regularities of different types (Yerushalmy et al., 1990, p 224). When coping with conjecture-type tasks students are supposed to (a) perform a *construction* without guidance; (b) *drag* the construction to *test* it and *search* for new and previously unknown properties; and (c) prove or refute their conjectures.

Testing a Robust Construction vs. Correcting a Soft Construction

The distinction within DGEs between *drawing* and *figure* was introduced by Parzysz (1988) and further developed by Laborde (e.g., 1992). Drawings and figures are visual images of geometric objects. Figures (robust constructions) are images of geometric objects constructed in such a way that all the necessary properties of the object are present. For example, if users drag any corner of a figure representing a square, the figure changes in size but remains a square. In this sense, a "figure does not refer to one object but to an infinity of objects" (Laborde, 1992, p. 128), which continuously preserve all critical properties under dragging. By contrast, drawings resemble the indented geometric object, with all its properties, but in a DGE they do not pass the drag test. In this way a corrected soft construction in DGE is a drawing. Soft constructions have only part of the properties of a given object, and naturally -- when corrected -- do not pass the drag test. For example, when a drawing of a square is dragged it loses some of its properties and becomes some type of quadrilateral, i.e. a rectangle.

In most cases, when constructing a given object in a DGE for the purpose of further investigation, learners perform a construction exactly according to the order of givens in the task. I call this a *consecutive construction* (see the Examples section below). In some cases an object obtained by a consecutive construction is a figure (e.g., Task 1 in the examples); in other cases a consecutive construction produces a soft construction (Tasks 2 - 4). Soft constructions can lead to two main scenarios:

1. learners are dissatisfied with the soft construction they have obtained and reorder or elaborate the construction (e.g. Tasks 2 and 3 sf. Healy, 2000, Holzl, 1996); or

2. learners correct the soft construction until they obtain a drawing of the object.

Corrected soft construction –drawing in DGE) satisfies (*approximately* in most cases) the given conditions, and enables (approximate) observation of new properties of the object (Tasks 2, 3, and 4). In these cases, investigation of the object is performed by iterative correction of the soft construction, as dragging breaks the given properties of the drawings (Leikin, 2004). As learners observe a property repeatedly, each time the object is corrected they formulate a conjecture and prove it. The process is repeated until the learners prove their conjecture and no longer need to construct a formal figure that passes the dragging test.

This course of reaching a solution using a correction strategy is consistent with the observation of *abduction for conjecturing* by Healy and Holes (2001). They claimed that this procedure is less mathematical and that "it seems almost paradoxical that of the two strategies, the one that might be described as less mathematical was associated with better results (*ibid*, p.254). Leikin (2004) also showed that drawing the figure and using a "break and fix" strategy was naturally less mathematical at the stage of investigation than dragging a figure constructed according to the given sufficient conditions and searching for the necessary conditions, but that those who used this strategy arrived at the formal proof of the property earlier than their peers. Leikin (2004) found that the investigation strategy chosen by learners may depend not only on their knowledge of geometry and DGE-related skills but on the structure of the task. In this paper I analyzed the structures of geometrical tasks through the lens of the construction and dragging strategy that learners may use in the process of inquiry.

I call investigations performed with a constructed figure a *figure investigation,* and the dragging in this investigation is called *figure dragging.* By contrast, when learners experiment in the DGE by correcting a sketch of a given object I call this problem solving strategy *correction investigation.* A component of correction investigation is *correction dragging,* a deliberate dragging aimed at obtaining (rather than preserving) the given properties.

EXAMPLES

The four types of tasks presented in this section differ with respect to the complexity of the construction that allows figure investigation of the given object when the task is transformed from a proof task to an inquiry-based task.

Task 1

I start with a well known theorem: "The median in a right triangle equals half of the hypotenuse." Naïve learners who are unaware of the theorem may be presented with a corresponding inquiry task:

What can you say about the ratio between the median and the hypotenuse in a right triangle?

A consecutive construction of the given object (right triangle given in the problem → a median) results in a figure. There are different ways of constructing a right triangle: (a) as "half" of a rectangle (Figure 1a); (b) as a triangle inscribed in a half-circle (Figure 1b); (c) as half of an isosceles triangle. Learners can use wandering dragging to find new properties of the figure and to conjecture that the median equals half of the hypotenuse, that the two small triangles are isosceles, or that the areas of the triangles are equal. This conjecture may be proved in different ways that correspond to the construction of a right triangle: (a) diagonals of the rectangle bisect each other and are equal; the median is half of the diagonal and the hypotenuse is the second diagonal; (b) the median is a radius of the circle in which the rectangle is inscribed, and the hypotenuse is its diameter; (c) the median is the midline of the initial isosceles triangle, and the hypotenuse is its lateral side. Other constructions of a right triangle are possible as well. But no matter how the right triangle is (correctly) constructed, it is possible to search for new (not given) properties of the median by figure dragging because random dragging preserves properties of the given object, the measurement is exact, and the ratio between the hypotenuse and the median is continuously constant (Figure 1c).

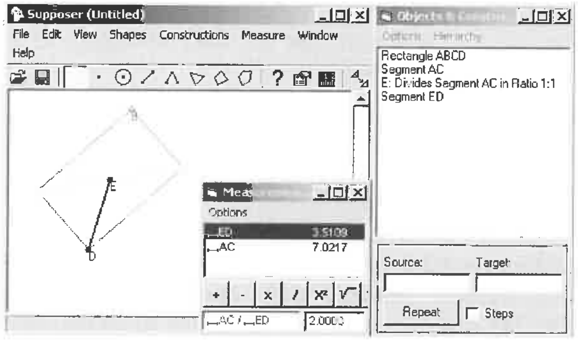

*Figure 1*a. Right triangle is "half" of a rectangle.

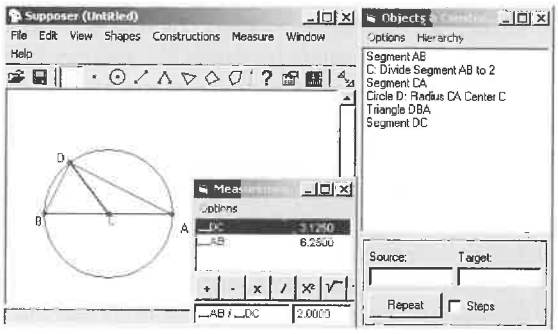

Figure 1b. Right triangle is a triangle inscribed in a half-circle.

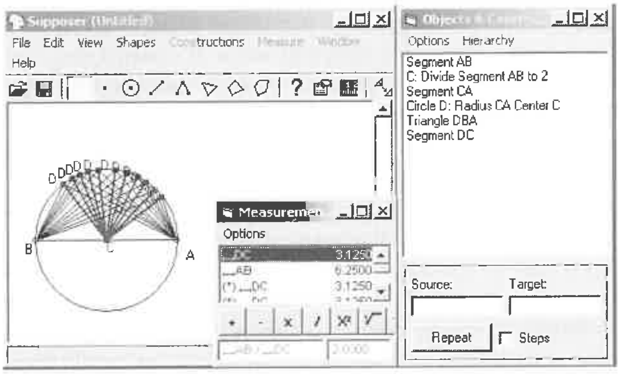

Figure 1c. The ratio between the hypotenuse and the median is continuously constant.

Task 2

Let us consider an inverse theorem: "If the median to one of the sides of the triangle equals half of this side, this is a right triangle." The proof is simple: we know that the two small triangles are isosceles and that the sum of angles in a triangle is 180°, therefore one of the angles of the triangle is 90°.

The inquiry problem presented to naïve learners unfamiliar with the theorem appears to be unexpectedly problematic:

Find the value of the angle in a triangle if the median from its vertex to the opposite side equals half the side.

A *consecutive construction* (random triangle → median of the triangle) results in a soft construction. Some learners proceed with correction investigation of the soft construction (Figure 2a). Usually, the modified object (drawing) does not allow exact measurement: when the median equals *approximately* half the corresponding side, the angle value is *about 90 degrees*. Moreover, the measurement procedure is discrete (non-continuous): the property of the object collapses under dragging and the object needs to be corrected again. Using this correction strategy, the learner observes that when the median is *almost* half of the side the angle is *almost* 90° and can formulate a conjecture that this is a right triangle. Other learners may conclude that no conjecture can be produced since the measurement is imprecise.

Alternatively, learners dissatisfied with the approximate measurement can perform a construction according to a *changed order of givens* (segment → another segment from its midpoint with a length that equals half of its length). Regardless of how this segment is constructed, the triangle is defined by the three endpoints of the two segments. Under random dragging it becomes clear that the constructed object is a figure: the median equals exactly half the side (according to the construction) and the angle *continuously equals exactly* 90°. At this point the proof may be based on the way in which the segment is constructed. For example, the angle equals 90° because it is inscribed in a circle based on the diameter (Figure 2b).

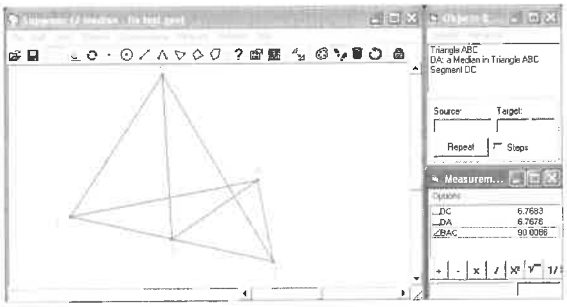

*Figure 2*a. Correcting a soft construction "Median equals half of the corresponding side".

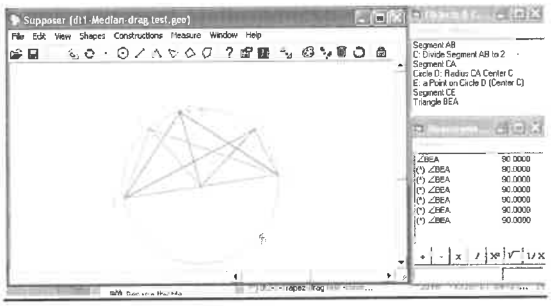

Figure 2b. Robust construction "Median equals half of the corresponding side".

Task 3

In the isosceles trapezoid ABCD the diagonals are perpendicular (AC⊥BD). Find possible relationships between two midlines of the trapezoid that join mid-points of the opposite sides of the trapezoid.

A consecutive construction of the given object (a random or equilateral trapezoid → perpendicular diagonals) leads to soft construction and its correction in order to investigate the properties of the given object (Figure 3a). For a figure investigation an *elaborated construction* must be performed, one according to the properties of the object that are not given in the problem.

Learners would have to start such a construction with perpendicular and equal diagonals, so that the intersection point of the diagonals divides them into parts as follows: AC⊥BD, AC=BD, AC∩BD=O, AO=OD, BO=OC, E, F, G, H midpoints on AB, BC, CD, AD correspondingly. Then the relationship between EF and GH may be found by figure dragging (Figure 3b): they are equal and perpendicular. One possible proof of this property is related to the construction performed: the two segments are diagonals in the quadrilateral EGFH, which is a square because the diagonals of the given trapezoid are equal and perpendicular.

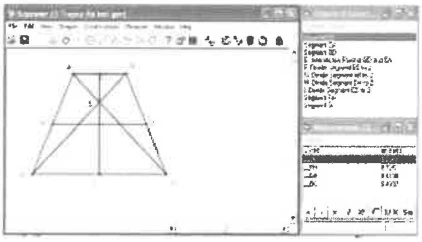

*Figure 3*a. Correcting a soft construction of an isosceles trapezoid with perpendicular diagonals.

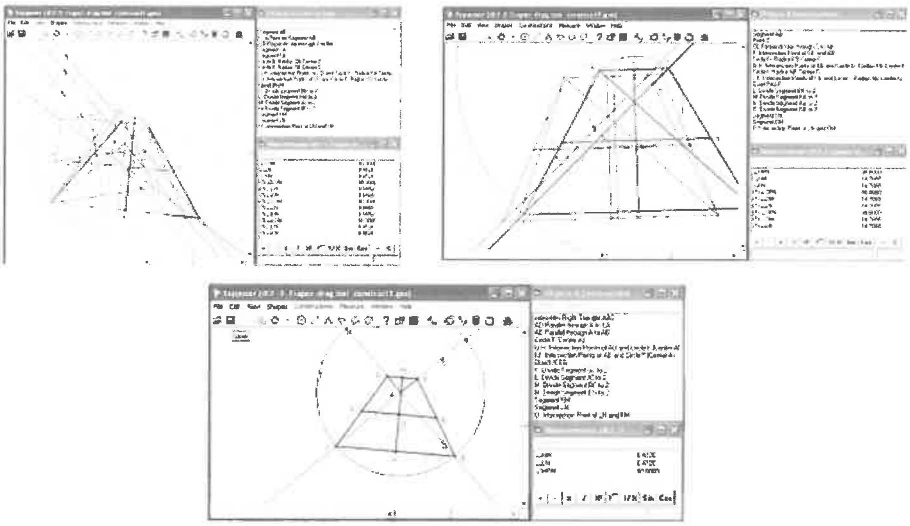

Figure 3b. Three different robust constructions of an isosceles trapezoid with perpendicular diagonals.

Task 4

Find angles of a quadrilateral inscribed in a square if the quadrilateral is a rhombus.
As in tasks 2 and 3, the consecutive construction (square → inscribed quadrilateral) can lead only to a soft construction that requires implementing correction strategy: for example, the inscribed quadrilateral must be corrected to become a rhombus. But to perform figure investigation [constructing a figure (robust construction) that is immune to dragging] one must first prove that the inscribed object is a square; once this property

has been proved learners no longer need to perform an investigation. It follows that it is sufficient for learners to follow a correction strategy to perform an investigation of a given object. This task requires correction dragging and is not appropriate for figure investigation (Figure 4).

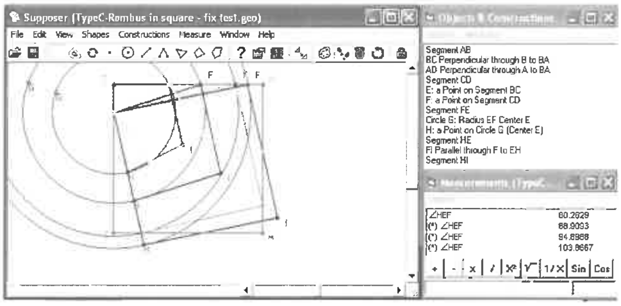

Figure 4. Soft construction only is possible for rhombus inscribed in a square.

DISCUSSION

How do These Tasks Differ for Naïve Learners?

In Task 1 the consecutive construction performed in the order of the givens (right triangle → median) results in a figure representing the given object, on which learners can perform *figure investigation.* The figure is *immune to dragging* and allows *exact measurement.* Learners can observe the givens and the new properties of the object *continuously* as they *randomly drag* the figure.

In *Task 2* the consecutive construction (triangle → median, which is half the side) results in soft construction of the given object. Learners can either perform a *correction investigation on the soft construction* or carry out a figure investigation of a figure constructed according to a *changed order of givens*. The correction strategy includes repairing the soft construction and making it into a *drawing,* which is *not-immune to dragging* and allows *approximate measurements* only.

In *Task 3,* as in Task 2, the straightforward construction (equilateral (random) trapezoid → diagonals) demands a correction investigation of the obtained soft construction. But unlike the case of Task 2, a reordered construction of the object is not sufficient for a figure investigation. Learners must analyze which additional (non-given) properties of the object are sufficient for the construction of a figure.

Contrary to all previous tasks, Task 4 requires correction investigation only. Figure investigation is impossible because one would have to prove the unknown property of the object to construct a figure.

Tasks 1-4 are representative of the internal structure of four types of geometrical tasks that we can observe through the lens of geometry investigations with dragging (Table 1).

Table 1
Different types of geometric investigation tasks.

Type of construction / Type of task	Consecutive construction	Reordered construction	Elaborated "no-proof" construction	Elaborated proof-based construction
Type 1: Straightforward inquiry task	Allows *figure dragging*	Irrelevant		
Type 2: Reordered inquiry task		Allows *figure dragging*		
Type 3: Elaborated inquiry task	Requires *correction dragging*		Allows *figure dragging*	
Type 4: Proof task		Requires *correction dragging*	Requires *correction dragging*	Allows *figure dragging* — ***irrelevant***

Differences between Figure and Correction Investigations

The differences between figure and correction investigation procedures are substantial. When a consecutive construction of a given object results in a soft construction, those who prefer to use a figure strategy search for the internal structure of the problem, reorder the givens (e.g., Task 2), or perform an elaborated construction according to the not-given properties of the object (e.g., Task 3). In some cases the construction of a figure requires proving the properties, which prevents the possibility of a figure investigation since the properties have been proved (e.g., Task 4). In these cases only a correction strategy is applicable to the investigation.

When constructing a figure, learners search for new properties that are immune to dragging. Those who employ a correction strategy observe new repeating properties of a corrected soft construction (as it repeatedly satisfies the givens). In a figure investigation, measurements are always exact, whether testing or searching, but when using a correction strategy, measurements are approximate. Correspondingly, when using a figure dragging strategy, learners perform a continuous search of additional properties (to those given) that are immune to dragging. Table 2 summarizes these distinctions.

Reformulation of Proof Tasks and Didactical Applications of the Findings

Yerushalmy et al. (1990) pointed out that it is not easy to create problems worth exploring, and that the task must strike a balance between providing clear guidance and leaving room for creative investigation. The need for guidance in the construction of a figure is one of the central didactic questions that must be addressed by designers and teachers: too much or too little guidance is relative to every teacher's and each student's experience and expertise.

One way of designing inquiry tasks is to open the proof tasks. For example, Task 4 is an open version of the task "Prove that a rhombus inscribed in a square is a square." Proof tasks that are appropriate for inquiry-based activities must be carefully chosen. Based on analysis of the tasks performed in this paper, it is clear that suitability of tasks depends also on perceiving figure dragging as a necessary part of the inquiry procedure, contrary to viewing both figure and correction strategies as legitimate investigations.

Table 2
Distinctions between figure and correction investigations.

Dragging strategy / Features	Figure investigation	Correction investigation
Testing given properties	Dragging tests for immunity to dragging of the constructed object: the figure *preserves the given properties*	Dragging test demonstrates that the object is not immune to dragging: the form does *not preserve the given properties*
Correcting the form	Not appropriate	Correction dragging aimed at *obtaining given properties* for the object: transforms a form into a drawing
Searching for the new (unknown for learners) properties	Wandering dragging: searching for figure *properties that are immune to dragging*	Correction dragging: searching for object *properties that repeat in corrected constructions*
Dragging	Random	Purposeful, to obtain a drawing
Search	Continuous	Discrete
Measurement	Exact	Approximate

Analysis of the tasks described in this paper demonstrates that such a reformulation is not trivial from a didactic point of view. Different types of tasks may be used in the learning process for different purposes: Tasks of type 1 enable the consecutive construction of geometric figures, so that guidance in construction is less needed than in tasks of type 2 and 3. Teachers can use tasks of type 1 to evaluate their students' ability to perform a geometrical construction with compass and ruler. At the same learning stage, tasks 2 and 3 can be given with more detailed guidance about the order in which the constructions should be performed. If such guidance is not provided, tasks of types 2 and 3 require students to formulate an equivalent problem in order to perform a figure investigation. Those tasks can be used at more advanced stages of geometrical investigations than tasks of type 1. Tasks of type 4 cannot be reformulated into investigation problems if we consider figure dragging to be one of the objectives of geometry investigations. But if we accept correction investigation as a legitimate strategy, we can include these tasks as well in the inquiry-based lessons. The suitability of correction investigation depends on the teachers' instructional objectives and on the solvers' expertise in geometry and the use of DGEs. The gradual introduction of different types of problems may develop awareness of the internal structure of geometry problems.

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Mathematical Modeling and Reflective Thought in Mathematics Teacher Education

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ABSTRACT: *This paper tries to articulate mathematical modeling and reflective thought in courses of Mathematics Teachers' Education. Considering the characterization of the mathematical modeling and of the reflective thought that we presented, the mathematical modeling can be viewed as a reflective practice or as a means of unchaining the reflective thought. Our study indicates that the road of the mathematical modeling and the road of the reflective thought, as characterized by John Dewey, are concordant. To illustrate that agreement, we present an activity of mathematical modeling developed by a group of students from a teachers' education course. Therefore, with the creative and interactive potentiality of the mathematical modeling configured; with the road of the reflective thought and the road of the mathematical modeling seen as concordant; signaled that situations of the mathematical modeling can be included in a course for preservice teachers and also tacked the idea that the development of the reflective thought in the future teacher has positive influence on their education, this article defends a situation based on contextualization, on creativity, on the possibility to raise higher cognitive levels.*

Key words: *Mathematics education; Mathematical modeling; Mathematics teacher education; Reflective thought.*

INTRODUCTION

Mathematics teacher education concerns developing teachers' knowledge and practice. In this context, a central challenge of professional education for teachers is to prepare novices for skilful beginning practice. Doing this depends on theories regarding the relationship between teaching practice and teacher education.

Literature presents several researches that aim at understanding how the courses of Mathematics Teacher' Education must be thought and organized from a perspective that takes into consideration this relationship.

Llinares (1998) argued that future teachers have knowledge and perceptions on aspects of their professional future, generated by a determined school culture which, possibly, will determine the form in which they construct the meaning and the type of activity that they accomplish as teachers.

In this context the development of reflective practices in didactic actions is one of the referrals indicated (Alarcão, 1996; Darsie and Carvalho, 1998). Though, how to incorporate in the courses of teacher education activities and procedures with potential to unchain these reflexive practices is one of the research subjects still emerging.

On the other hand, it seems important to admit that the activities developed by the students in their course of initial education can unchain, with greater or minor intensity, the process of reflection.

Thus, this paper pointed out the introduction of Mathematical Modeling activities in these courses as an alternative to develop the students' reflection.

For this purpose initially, from a theoretical framework, we define 'reflective thinking' and present a characterization for Mathematical Modeling activities. Considering these conceptualizations, we looked for approaches between the road of the reflexive thought and the set of actions taken by students when they are involved with Mathematical Modeling.

To illustrate this potential of the Mathematical Modeling we present an activity performed by students of preservice Mathematics Teacher Education course.

THEORETICAL FRAMEWORK

The Reflection and the Mathematics Teacher Education

The term reflection has been frequently used in the researches of teacher education and a wide variety of approaches has been employed in attempts to foster reflection in pre service teachers.

Donald Schön is a researcher who marked the form as we understand the reflection in the scope of teachers' education. Schön (1987) writes about reflection is intimately bound up with teachers' action and he has discussed in his research about the importance of seeing the teacher as somebody who decides and finds pleasure in the learning and in the investigation of the process of teaching and learning. The ideas of the author on the development of the professional knowledge are based on the notions such as the research and experimentation and on the reflections that may happen from this point.

Freire (1980, p. 35), affirmed that "the more the subject reflects on the reality, on their concrete situation, the more they emerge, plainly aware, fully engaged, ready to intervene in the reality to change it". The reflection, in this perspective, can lever the indignation and the mobilization for the act of teaching. In this sense we cannot think the reflection and the action as separate bodies. According to Barbosa (2001), the reflection modifies the way to see the things and it moves us for an action in the world. In this sense, reflection is a metacognitive strategy that helps teachers think critically upon their experience, actions and decisions during their teaching practices. In a preservice Mathematics Teacher Education course the actions executed by students, while future teachers, can unchain reflections on the complexity developed in the educational process. But, as we can characterize this reflection the one what did we refer?

What Characterizes the Reflection?

When accepting the reflection as an important component in the initial education of teachers of Mathematics, we initially, try to understand it and to use it from the characterization presented by Ross (1989, p.22): "The reflection is defined as a way of thought on educational questions that involves the ability to rationally make choices and to assume the responsibilities for these choices".

According to Alarcão (1996), in general, the reflection implies an active, voluntary, persistent and rigorous attitude regarding what we judge to believe or what we habitually practice. Thus, the reflection evidences the reasons that justify our actions or certainties and illuminate the consequences to which we are conducted. In this way, the reflection would be more based on the will of the person to think than on momentary impulses.

Dewey (1979) believed the reflection starts when the person is surprised by something, when he/she begins to feel restless and uncertain about the occurrence of phenomena and, later, when he/she orients his/her own conceptions to the achievement of a goal. According to this author, we reflect on a set of things when we think about them; but the analytical thought only has its place when there is the recognition of a real problem to be solved; thus, the capacity to reflect emerges when there is the recognition of a problem, of a quandary, and when the acceptance of the uncertainty happens. Consequently, the idea of reflection seems associated to the way we deal with problems, in the professional as well as in the educational context. The possibility of accepting a state of uncertainty and of being open to new hypotheses may make the future professor able to give form to these problems, discovering new ways and materializing solutions.

Darsie and Carvalho (1998), argued that

> The reflection can contribute for the awareness, for the conceptual evolution, the consolidation of conceptions, as well as for the overcoming of the beliefs and negative feelings; because such reflection puts in evidence the previous conflicts, the cognitive conflicts and the knowledge generated by the new learning (p. 61).

The reflection is unchained, it is incorporated and it is developed in the subject as a form of thought, which is characterized by Dewey as reflective thought.

Reflective Thought

The concept reflective thought was introduced by John Dewey (1933) in his How We Think, a work designed for teachers. Dewey's most basic assumption was that learning improves to the degree that it arises out of the process of reflection.

The idea of reflective thought as proposed by John Dewey, represents a systematic dialogue that is established by the person with him/herself, when he/she faces a problem, a conflict, having a continuous evaluation of values, beliefs, assumptions, principles and hypotheses underlying in the presence of a set of information and possible interpretations (Dewey, 1979). According to the author, the intentionality of a reflective thought consists always in discovering the truth, even though this truth is provisional. When distinguishing the reflective thought from other forms of thought, Dewey (1933) claimed that the reflective thought involves "a state of doubt, hesitation, perplexity, in which the thought originates an act of searching, inquiry, investigation, to find something that can solve the doubt, clarify and undo the perplexity" (p.12).

In this sense it seems clear that the reflective thought, as an inquiry booster, directs itself to the solution of a problem and the nature of this problem influences the process of thinking. It seems, therefore, that to think more or less reflectively, depends on the willingness for the search, for the investigation. The way of the reflective thought is a continuous that goes from the situation of doubt to the situation of solution, of stability.

Bairral (2003) considers the reflective thought as a chain of ideas and consequences, in which the observation and the possibility to accept or to share suggestions also become necessary. The author, based on the ideas of Dewey, ponders that the reflective activity has essential functions that may be grouped in five phases:

> Survey of suggestions (recognition of possible solutions); intellectualization (problematization); conducting idea (construction of hypotheses); thinking on a restricted sense (reasoning as part of the deduction, not its totality); confirmation of hypotheses (by means of real or imaginary actions) (p.10).

For the last decades, consensus thinking is that reflective activities in a classroom can take place only when a questioning strategy promotes it. Paradigms and models of questioning have proliferated endlessly. In this text we are interested in approaching the mathematical modeling as a possibility to introduce reflective activities and, consequently, reflective thought in courses of teachers' education.

Mathematical Modeling and Reflective Thought

According to Erickson (2009), one of the essential principles for instruction focused on building understanding in mathematics is: Make the subject problematic. Instruction ought to allow "students to wonder why things are, to inquire, to search for solutions, and to resolve incongruities". (p. 1). In this article we present the mathematical modeling as a possibility to make the subject problematic.

The understanding we have in mind in this text, based on Almeida and Ferruzzi (2009), is that, in general, a Mathematical Modeling activity may be described in terms of an initial unfamiliar situation (problem), of a desired end (which represents a solution to the initial situation) and a set of procedures and concepts needed to pass from the baseline to the final situation.

In this perspective it is configured as an activity that is developed as a scheme - a modeling cycle - in which the situation to be investigated represents a problem for those involved in the development of the activity.

Although this cycle is not necessarily expressed by means of explanative schemes, developing an activity of mathematical modeling implies in a set of actions that involve the identification and election of the variables, the elaboration of hypotheses, the acquisition of a mathematical model, the resolution of the problem by means of adequate procedures and the analysis of the solution, identifying its acceptability or not. These actions are associated to the involvement with:

a) a problem-setting: the students involved with the modeling activity need to own a problem and to define goals for the resolution of the problem; to understand the problem-situation through the Mathematics implicates in seeking answers for the problem raised by this situation; in this context, Mendonça (1999) argued that:

> the formulation of a problem front a situation presupposes some lack of understanding, a partial lack of meaning about such situation, but it also supposes that something is already understood and there is curiosity on the subject, because otherwise it would be impossible to arouse a problematization, an almost spontaneous process in the direction of the understanding with meaning (p. 24);

b) a process of investigation: it refers to the action of investigating; to investigate, according to Ferreira (1986), it means to follow the track', to do diligences to find, to research; actions on how to look for information, to identify and to select variables, to define hypotheses, to do simplifications, they constitute, therefore, elements of that process and they require an appropriate interpretation and a certain degree of intuition to overcome the lack of understanding that Mendonça (1999) refers to;

c) The search for a mathematical representation (or mathematical model[1]): in general, the problematic situation to be investigated comes in natural language and it doesn't seem directly associated to a mathematical language; it generates, therefore, the need of the transformation of a representation (natural language) for another (mathematical language); this mathematical language evidences the mathematical problem to be solved; the search and elaboration of a mathematical representation are mediated by

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[1] Mathematical model is, according to Lesh et al. (2006), a conceptual system, descriptive or explanatory, expressed through a language or a mathematical structure, in order to describe the behavior of another system and allow predictions on this other system.

relationships among the characteristics of the situation and the concepts, techniques and appropriate mathematical procedures to represent these characteristics mathematically; Kehle and Lester (2003) claimed that the mathematical problem acquires a sense "only yours", and becomes a private mathematical problem, but it doesn't mean that the found results and the analyses accomplished will be only useful for those that problematize the situation and that solve the problem;

d) the analysis of an answer to the problem: the analysis of the answer constitutes an evaluation process accomplished by the ones involved in the activity and it implicates in a validation of the mathematical representation associated to the problem, considering as much the mathematical procedures as the adaptation of the representation for the situation

e) the communication of results for others: this communication implicates essentially in developing an argument that can convince your own students and those to which the results are presented.

By identifying these actions of the students, we can agree with Niss (2003), when he arguments that Modeling is inseparably linked with other mathematical competencies such as reading and communicating, designing and applying problem solving strategies. In this context, Blum et al. (2007), concisely define "Modeling competency" as the ability to construct models by carrying out those various steps appropriately as well as to analyze or compare given models.

This characterization of the mathematical modeling bases the argument that it constitutes an investigation practice that provides unexpected approaches, or even original approaches, and it develops the students' creative expression.

Considering this characterization of mathematical modeling and the organization of the reflective thought as presented in section 2.2, we have indications that the mathematical modeling activities thus structuralized combines with a perspective that searches for the way that the reflective thought defines: from the situation of doubt to the situation of stability.

So, if on the one hand we can say that the instability, the problem as a starting point; the intentionality in the search; the problematization, the hypotheses as factors that are placed on the way to indicate directions; the possibility to accept and or to share suggestions; the destitute perplexity; the established truth even though it is provisional; the necessity of validating, constitute aspects placed on the road of the reflective thought, on the other hand, this characteristics are also placed on the road of the development of a mathematical modeling activity. Therefore, once its interactional, creative and scientific characteristic is respected, the Mathematical Modeling can unchain the development of the reflective thought.

Mathematical Modeling and Reflective Thought in Mathematics Teacher Education: the Structuring of a Relationship

The introduction of Mathematical Modeling activities in preservice Mathematics Teacher Education courses, either in a specific discipline, or in activities of other disciplines of the curriculum, still comes strongly based on utilitarian arguments for the scholar mathematics.

The perspective of thinking about the situations of Mathematical Modeling as a way to stimulate in the future teacher the capacity to confront situations which are uncertain and subjected to the unexpected, especially their future teaching practice, is also perceived in some researches. In this text, particularly, we intend to establish an articulation between Mathematical Modeling and the development of the reflective thought.

Initially we defend the idea that the situations of Mathematical Modeling with future professors are carried through, essentially, in two perspectives: the personal education and the professional education. We do not consider them disjoined, but, on the other hand, we perceive them likely of interaction, conjunction; interaction that is consolidated and that goes deeper while the reflection is developed in the pupil and incorporated in his/her as a form of thought.

The pupil, a future teacher who is involved with modeling situations while being somebody that has a problem to solve (in this context it is in the scope of the personal education), in some opportunities extends his/her reflection to the future professional activity while being a mathematics teacher that could introduce problematizing situations as the modeling in his/her teaching practice. However, we need to make possible deep discussions on the influences and contributions of the mathematical modeling for the professional education.

Dewey (1933), when emphasizing that the programs of teachers' education must stimulate teachers and even future teachers to reflect about the practice, was one of the precursors of the reflective thought. In this sense, we argue that the reflective thought represents a link of interaction between the personal education and the professional education to which we refer.

In the understanding of Barrial (2003), the reflection is a strategy of self-education once it can provide the pupil, future teacher, with a questioning regarding problems and principles related to his/her future teaching practice.

In the previous section we perceive the mathematical modeling as a reflective practice or as a way to unchain the reflective thought. Therefore, the introduction of situations of mathematical modeling in the courses of initial teacher education is in tune with the idea that it is necessary to develop a reflective thinking in future teachers.

In the scope of a course of teachers' education, according to Mewborn (1999), the reflective thought is constituted of personal knowledge (beliefs, previous experiences, etc), of the knowledge of learning-teaching (abilities to teach) and of the propositional knowledge (deriving from combining research, theory and practice). This way, the reflective thought comprehends the teaching action in its different aspects. In this context, Dewey (1979) considers that the development of the reflective thought should constitute an educational objective, which is searched during the entire education trajectory, once, according to the author; it is a process that can be continuously improved.

The reflective thought mediated by all this different knowledge, constituted as an educational objective and stimulated in future teachers, represents, therefore, an expectation of improvement to the teacher's education.

So, considered the interactional and creative potentiality of the Mathematical Modeling, perceived that the road of the reflective thought and the road of the mathematical modeling are concordant, established that situations of mathematical modeling can be enclosed in the teacher education courses, instituted the idea that the development of the reflective thought by the future teacher has positive influence on his/her education, it seems to be configured a situation based on the contextualization, on the creativity, on the possibility of elevating higher cognitive levels with potential to unchain the reflexive practices.

METHODOLOGICAL DESIGN

To investigate the potential of Mathematical Modeling activities for development of the students' reflexive thought, we look for the relationship among the 'road' taken by the reflexive thought and the 'road' taken by students during the development of Mathematical Modeling activities. So we developed modeling activities with students of the last year of a preservice Mathematics Teacher Education course. The activities were developed in the discipline of Introduction to the Mathematical Modeling whose teacher was the first author of this text.

The results here presented elapse of the analysis of one of these activities developed by a group of those students. The information that subsidize our arguments on the indications of relationships between the development of Modeling activities and the development of the reflexive thought elapse of data predominantly descriptive that we collected during the classes in that the activity was developed. For the collection of data instruments of collection of data were used as, annotations in field diary, recordings in audio and video, questionnaires, glimpsed as well as the analysis of the report given by the students. The analysis of the collected data, although subsidized by the theoretical framework that base the research, it comes enclosed of the researchers' understanding. In this sense, according to Lüdke and André (1986), the research can be characterized as a 'qualitative research'.

An Example to Illustrate the Roads

Rescuing the considerations presented in elapsing of the text, we enlarged the discussions in this section, looking for reflections about the possibility that the Mathematical Modeling can unchain the reflexive thought.

The students' interest for the situation - "the relationship between the speed and the braking distance" emerged of the own students. One of the students of this group had suffered a motorcycle accident. A car had hit the motorcycle that he was riding. In the Police Report registered at DENATRAN (National Department of Transport) an 8-meter mark of tires was established and the speed of the car that caused the accident, according to these findings, was 40 km per hour.

The student's idea was that the speed of the car should have been superior to the 40 km/hour due to the intensity of the accident. So, they were interested in investigating the subjects:

1. A car is at a certain speed. When the brake is pressed it will still run for a certain distance before stopping (braking distance). How to determine this distance?
2. Once we know the distance to which we are referring to in question 1, how to determine the speed of the car at the moment immediately before braking?

By choosing this theme, engaging in actions that relate to a Mathematical Modeling activity, the group of students who studied the problem made a clear statement of intent - that is, had the intention of seeking answers to a conjecture and problems experienced by the group. The problem as a starting point is, therefore, the driving force of intentionality. In order to find answers of them problem, considering the cases as indicative of the way forward, students made efforts in finding the solution.

To accomplish the data collection the group used a car model Volkswagen Gol year 1995. The part of the street was straight, dry, with little or no inclination and of good asphalt pavement. The following procedures were adopted for that collection: they marked a line on the asphalt and signalized it with a flag; the driver of the car drove for approximately 500 m to stabilize the speed; when the front wheel crossed the line on the asphalt an acoustic and luminous signal was given to the driver who immediately initiated braking; they waited for the car to stop and the distance from the line to the place where the front wheel stopped was verified. In figure 1 we present one of the pictures taken by the students during data collection.

Figure1. One of the pictures taken by the students.

This procedure was repeated three times at the same speed (v_i) in order to obtain the braking distance S_1, S_2 and S_3, as shown in Table 1. To solve the problem, the auxiliary variable x_i, defined in column 5 of Table 1, was introduced. In figure 2 there are the braking distances for the 3 experiments.

Table 1
The Collected Data

Speed (km/h) (v_i)	Braking Distance 1 S_1	Braking Distance 2 S_2	Braking Distance 3 S_3	$x_i = \frac{v_i - 15}{10}$
25	4,14	4,21	3,7	1
35	6,31	6,8	7,3	2
45	10,18	10,68	11,9	3
55	14,24	15,9	17,2	4
65	19,20	19,94	23,3	5
75	28,9	30,7	31,1	6

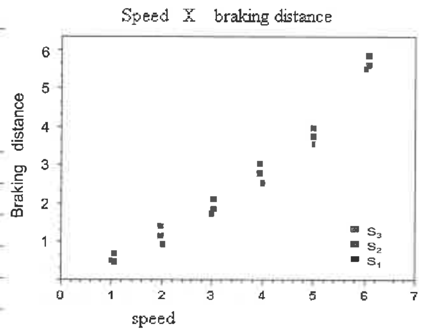

Figure 2. The Collected Data.

Taking into account the tendency of the data of the three experiments (as in the Figure 2), the students opted for a quadratic function for the relationship between the braking distance and the speed and fit a quadratic function as:

$$R(x_i) = a(x_i)^2 + b \qquad (1)$$

The procedure for finding the best-fitting curve was to use the Least Squares Fitting method:

$$E(a,b) = \sum_{j=1}^{m}\sum_{i=1}^{n}(S_{ij} - R(x_i))^2 \qquad (2)$$

where S_{ij} is the measured value, and $R(x_i)$ is the predicted value, and m = nº of tests (j=1,2,3)

and n= nº of speed (i=1,2,3,4,5,6)

Considering that $x_i = \frac{v_i - 15}{10}$, the braking distance S(v) and the speed using the braking distance V(s) are given by

$$S(v) = 0{,}723556\left(\frac{v-15}{10}\right)^2 + 3{,}787175 \quad \text{for} \quad v \geq 25 \qquad (3)$$

$$v(S) = 10\sqrt{\frac{(S - 3{,}787175)}{0{,}7235556}} + 15 \qquad (4)$$

those are the answers of the two questions formulated by the students.

To analyze the case of the student's accident, whose braking distance was of 8m according to the Police Report, the students used the equation (4) and they obtained that: $v(8) \cong 39{,}13\, km/h$.

To the students' surprise, the found value is very close to the value mentioned in the Police Report (40km/h) and, therefore, the found model validated this information.

The Road Taken by Students in the Development of Activity

In general, a problem exists when a person is curious, puzzled, confused, or unable to resolve an issue. In this case, a situation that needs to be clear and imperturbable is still cloudy or obstructed.

But, this is the condition in which the students that developed that activity found themselves in - it was necessary to explain a situation that was obscure, confusing: are the findings of the Police Report accurate or not?

In the activity, imbued by the creativity and by the search for a solution to original problem situation, students were dismissing the perplexity of the problem and were established truths, even if provisional.

To clarify the situation, the students had to involve themselves with a problem-setting, a process of investigation, the construction of a mathematical representation and the analysis of an answer to the problem. The students' conclusion, presented in the report that they gave, was that the information from DENATRAN is correct - the car was really attn 40 km/hour the informed by the findings and the Police Report:

> *"Made the delimitation of the problem, the collection of data, the construction of the model and the validation of the same, we admitted that it was not that we waited. We believed that the car that reached our friend was to a higher speed. However we were forced to accept the opposite, because for our study it was corroborated that the car was to 40 km/hour". (report given by the students).*

So we can observe that in the activity the students, pre-service teachers, pass from the situation of doubt (the findings of the Police Report) to the situation of solution, of stability (the confirmation: the Police Report is correct). In this sense, the road of reflective thought is to take shape.

What can be seen, at first is that it is a problem that arose from concerns of the students. Thus, the activity represented a 'real' problem for the group and genuine in the sense that comes from a 'no mathematics' situation with some relevance and also in the sense that it was necessary to obtain a 'new' mathematical knowledge as well as to use cotents and concepts already known to solve the problem.

Moreover, given this description of the activity presented in the report of the students, we can argue that:

1. a situation of instability exists: (are the data from DENATRAN correct?);
2. the problem is the starting point: (the claim of the information from DENATRAN);
3. there is intentionality in the search: (the data collection simulates the situation of the accident);
4. there is a problematization: (the definite questions lead to the answers);
5. the destitute perplexity: the established truth (the mathematical model obtained leads to the answers to the questions and represents the solution of the problematic situation).

Thus, considering the characterization that we presented in the previous sections of this paper, considering these five aspects as elements that it composes the road of the reflexive thought, we can infer that that the road of the mathematical modeling and the road of the reflective thought are concordant.

During the development of the activity it was possible to see a large group involvement in obtaining data and developing the model to answer the questions they have elaborated. We believe that this involvement was influenced by the context in which the group was inserted. If the student that suffered the accident could prove that the car's speed was greater than that declared in the police report, he could make a claim to assist in medical expenses.

These arguments indicate that the activity affected the personal, social and cultural aspects, developing a reflective attitude. The understanding that the arguments used by the agencies responsible for issuing the reports of the accident were interwoven in mathematics made the students write in the introduction of the report that describes the activity " we will make use of mathematics to act critically in society.

To structure and solve the problem they themselves have proposed from a real situation, the students had to obtain experimental data, making hypotheses, establish questions and find answers, featuring a wide range of mathematical experiences on solving problems and conducting investigative work and to establish relationships of mathematics with other areas of human knowledge as matters of transit, in order to obtain physical data appropriate to the problem, statistics and computer science.

In this context scientific training with regard to theoretical and conceptual aspects of mathematics was strongly influenced by the study of the students. Parts of the report given by the students reveal this aspect:

> *"We can use various mathematical contents learned in the classroom and had the opportunity to study other as yet unknown".*

"Looking at the mathematical competences, we reached some such ones as delimiting problems, to do conjectures and you refute them. We were the center of the education scene. In the film of preservice Mathematics Teacher Education course, we were the main actors" (report given by the students).

With regard to the context of scientific and technological knowledge, a more critical and reflective vision about the mathematical concepts seems to have become apparent. The students' statement signals that idea:

"We've also opened the door for future studies in this area since our model using only one car (Gol) and a type of brake (drum brake)".

"In short, we act in a critical and reflective in society by means of mathematics" (students' statements in the interview).

According to Chevallard (2001), a great problem of the mathematics classes is that in general, the activities are presented ready and ended, depriving the student of the participation of a "mathematical thinking". Thus, the activity, being built by the students themselves with guidance from the teacher, allows developing the thought in the direction of Stewart's (1996) saying: "Mathematics is not about symbols and accounts. Mathematics is about ideas, in particular on ways in which different ideas relate to each other. "

In this forward, it is important to consider the sequence of procedures to resolve the problem effectively was not exactly planned; the mathematical concepts that were used were not predefined. It is exactly in this sense, the activity contributes to a teacher's education that has capacity to create atmospheres and learning situations mathematically rich, and in the possibility to give answer to the unexpected and of drawing models to adapt the uncertain ones and no expected learning conditions that can happen in the classes of Mathematics. This signals the contribution of the student's formation for formation addressed to the teacher's work, for the constitution of the future teacher's knowledge.

Considering that put students, future teachers, in contact with the activity of teaching and prepare them for the exercise of this activity is the goal of the degree in Mathematics, also discussions about the inclusion of such activities in future teaching practice were launched during the presentation of students' activity.

In this context we can consider, from the response of one student, that the road of reflective thinking was being trailed during the development of the activity:

"In relation to our activity as a whole, we see that at its height! We left the classroom, the chalkboard, the sit, listen and pretend to learn, the old conception of teaching is still present in our day to day and, in fact, learned. We investigate, doubt, seek and act, that is, produce knowledge".(the students' statement in the interview).

Therefore, once it's interactional, creative, and scientific characteristic is respected, the Mathematical Modeling can unchain the development of the reflective thought.

TOWARDS THE END

Although the introduction of mathematical modeling is widely distributed in the field of mathematics education, there is a *gap* between the educational debate (and even official curricula) and classroom practice. According to Blum and Ferri 92009), the main reason is that Modeling is difficult also for teachers, for real world knowledge is needed, and teaching becomes more open and less predictable. These aspects require the teacher reflexivity.

Considering this issue, the purpose of this study is to articulate Mathematical Modeling and reflective thought in courses of Mathematics Teacher Education.

Questions which promote thought begin with the assumption that students do not think unless they have something to think about. Dewey (1979) establishes that this something must be a problem. It is better if the problem is a real problem – a problem in fact for those involved in the development of the activity.

So, in classrooms the low-level questioning activities must be replaced by a more fruitful approach that stimulates students to reflect on problems, real problems.

To introduce this kind of problems in courses of Mathematics Teacher Education in this text we suggest the Mathematical Modeling with focus on student thinking and reasoning.

Considering the characterization of mathematical modeling and the characterization of the reflective thought as presented in the text we have indications that the activity of mathematical modeling, thus structuralized, combines with a perspective that searches for the way that the reflective thought defines: from the situation of doubt to the situation of stability. So the roads of the mathematical modeling and the reflective thought should proceed being concordant.

This is the idea of our paper: the mathematical modeling, which is opposed to the ideas of the fragmented education, introduced in the courses of pre-service teachers' education in the parameters enunciated in this text, will be able to unchain the reflective thought and can have positive influence on the teacher's knowledge and practice once reflective practice helps teachers to have a deeper understanding of their own teaching styles, teaching beliefs and teaching identities.

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The Relationship between Students' Spatial Ability and Geometrical Figure Apprehension

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ABSTRACT: *The aim of this paper is to present and describe a research program designed to study the relationship between spatial ability and the geometrical figure apprehension development of pupils aged 10-13 years. The program developed after an extensive review of the relevant literature in mathematics education and cognitive psychology in relation to the spatial ability and geometrical figure apprehension. For the implementation of the research we developed a test with two parts: a spatial ability test which consists of 11 sub-tests and a geometrical figure apprehension test which consists of 13 tasks. The expected results of this study give a first insight into the relationship between spatial ability and geometrical figure apprehension as well the performance differences between the different ages of students.*

Key words: *Spatial ability, Geometrical figure apprehension, Relationship, Theoretical frameworks, Spatial factor, Tests.*

INTRODUCTION AND THEORETICAL FRAMEWORK

Spatial ability has been the focus of much research ever since it was documented as a distinct dimension by factor analytic studies dating from the 1920s (Goldberg & Meredith, 1974). It is generally accepted that this ability is related to skills involving the retrieval, retention and transformation of visual information in a spatial context (Lohman, 1993). More specifically, spatial abilities relate to individuals' abilities to search the visual field, to apprehend forms, shapes, and positions of objects, to construct mental representations of these forms, shapes, and positions, and to manipulate such representations mentally (Carroll, 1993).

According to D'Oliveira, (2004) four main areas have been identified in which conflicting perspectives have been proposed.

1. Definitions of spatial ability. Spatial abilities and skills have been defined in a variety of ways. Some abilities have similar descriptions, but different denominations whereas others with identical terms have different definitions.

2. Number of abilities. The number of abilities that have been identified in the spatial domain varies considerably and ranges from 2 to 10.
3. Factor names. Factor names are also a source of controversies in the spatial literature, as they seem to vary across authors and even within work of the same author.
4. Tests used to measure each factor. The tests that have been used to measure or evaluate spatial abilities often give rise to disputes. There is quite a variety of spatial ability tests and confusion exists regarding their names and content.

Lean and Clements (1981) defined spatial ability as "the ability to formulate mental images and to manipulate these images in the mind" (p. 267). Linn and Petersen (1985) stated that spatial ability generally refers to skill in representing, transforming, generating, and recalling symbolic, nonlinguistic information. Carroll (1993) stated that spatial abilities relate to individuals' abilities to search the visual field, to apprehend forms, shapes, and positions of objects, to construct mental representations of these forms, shapes, and positions, and to manipulate such representations mentally.

Spatial intelligence, which also has been referred to as spatial ability, involves the manipulation of information presented in a visual, diagrammatic or symbolic form in contrast to verbal, language based modality (Lohman, Pellegrino, Alderton, & Regian, 1987). Spatial intelligence can be inferred from the ability to invoke and use particular representations and reasoning. Spatial intelligence includes an ability to perceive and represent the visual-spatial world accurately and to form and manipulate mental images (Gardner, 1983).

Conflicting perspectives have been identified also on number of factor or components. Johnson and Meade, (1987) considered spatial ability to be a unitary trait (i.e., a single spatial ability rather than two or more relatively independent components). They found three lines of support their position. First is the work of a number of investigators (e.g., Adkins & Lyerly, 1952; French, 1965; Horn, 1978) who have concluded that the most parsimonious factor-analytic characterization of spatial ability is that it consists of a single trait. Second, Meade (1986) has found a single spatial factor to account for the performance of a college-student sample using a design that gave full opportunity for two factors to emerge. Third, a factor analysis of the tenth-grade portion of their sample has shown performance to best be described by a single factor (Carroll, Meade, & Johnson, 1991).

On the other side psychometric studies of spatial ability (e.g., Carroll, 1993; Eliot & Smith, 1983; Lohman, 1988; McGee, 1979) have identified several different spatial ability factors. Researchers have broken down the concept of spatial ability into specific factors that are believed to contribute to spatial comprehension. Unfortunately, factor labelling and definitions vary across researchers. As Linn and Petersen (1985) indicated spatial ability is combination of sub-skills.

For example McGee (1979), supports that spatial ability consists of two major components: spatial relations and spatial visualization. Spatial Relations are described as comprehension of the arrangement of elements within a visual stimulus pattern (McGee, 1979). According to D'Oliveira (2004), spatial relations refer to the ability to solve simple rotation problems or to identify reflected versions of the target. Spatial visualization is described as the ability to imagine rotations of objects or their parts in 3-D space by folding and unfolding, for example (McGee, 1979).

As a result of a meta-analysis of studies carried out between 1974 and 1982, Linn and Petersen (1985) made a classification of spatial tests into three distinct categories and labeled these categories spatial perception, spatial visualization and mental rotation. Spatial perception was defined as the ability to determine spatial relations despite distracting information and can be done efficiently using a gravitational/kinesthetic process. Spatial visualisation is the ability to manipulate complex spatial information when several stages are needed to produce the correct solution and can be done efficiently using an analytic process (Linn and Petersen, 1985). Mental rotation was defined by Linn & Petersen (1985) as the ability to rotate, in imagination, quickly and accurately two- or three-dimensional figures.

In another review, Lohman (1988) argues that there are three major spatial ability factors: Spatial Visualization, Spatial Orientation and Speeded Rotation. As Hegarty and Waller (2005) stated, the most comprehensive review of factor analytic studies of spatial ability was conducted by John Carroll in 1993. Carroll (1993) analyzed more than 140 datasets and detected five major clusters: Visualization, Spatial Relations, Closure Speed, Flexibility of Closure, and Perceptual Speed. Carroll's (1993) definition of Visualization factor does not differ from than that of other researchers cited above. Spatial Relations factor reflects the ability to perceive an object from different positions. It is usually defined by speeded tests involving rotations and/or reflections (Lohman, 1988). Closure Speed factor concerns individual differences in ability to access spatial representations in long-term memory when incomplete or obscured cues to those representations are presented. The subjects are not told what to look for in a given representation. Flexibility of Closure factor involves finding hidden patterns or figures in a bigger complex pattern when the subjects are informed about what to look for. Flexibility of Closure factor is sometimes called Field Independence or Disembedding by other researchers (Velez, Silver, and Tremaine, 2005). Perceptual Speed factor is characterized by the speed in finding a given configuration in a mess of distracting material. The task may include comparing pairs of items, locating a unique item in a group of identical items, or locating a visual pattern in an extended visual field.

Beyond the different definitions, spatial ability attracted the attention of many researchers because is routinely implicated in accounts of creative and higher-order thinking in science and mathematics (Shepard, 1978; West, 1991). There is extensive research in mathematics showing a relationship between spatial ability and mathematical performance (e.g., Battista, 1990; McGee, 1979) and documenting the central role that spatial ability has in learning within many areas of mathematics.

However, the role of spatial ability is elusive and, even in geometry, complex (Clements, 1997). Several studies have shown that spatial ability is positively related to success in geometry and mathematics. Battista (1990), found that spatial visualization and logical reasoning were significantly related to both geometry achievement and geometry problem solving.

Geometry is typically regarded as a difficult branch of mathematics to many students. Importantly, students' ability to recognize plane shapes seems to be influenced by a number of factors, including visual perception. According to English and Warren (1995), the way an individual visualizes a shape is one of the most important factors that affect the development of spatial ability. Yet many people experience difficulty with this process.

During the past twenty years, several mathematics educators have investigated students' geometrical reasoning based on different theoretical frames. For example, van Hiele developed a model referring to levels of geometric thinking (van Hiele, 1986), Fischbein introduced the theory of figural concepts (Fischbein, 1993) and Duval reported the cognitive analysis of geometrical thinking (Duval, 1998). In particular, Duval (1995, 1999) distinguishes four apprehensions for a "geometrical figure": perceptual, sequential, discursive and operative. To function as a geometrical figure, a drawing must evoke perceptual apprehension and at least one of the other three apprehensions.

Deliyianni, Elia, Gagatsis, Monoyiou and Panaoura (2009) have confirmed a three level hierarchy about the role of perceptual, operative and discursive apprehension in geometrical figure understanding. However, the relation between spatial ability and the four apprehensions for a "geometrical figure" proposed by Duval (1995, 1999), have not been empirically verified yet.

The National Council of Teachers of Mathematics (NCTM, 2000) has recognized the importance of introducing spatial reasoning as an approach to mathematics right at the point at which children are being introduced to mathematics learning. The timing and the content are two crucial elements to be considered in designing instructional tasks to improve spatial ability. Piaget and Inhelder (1971), suggest that an individual's cognitive development determines the potential of what one could achieve. At the initial stage a child exhibits a purely egocentric view of the world that continues to the second stage. A child develops spatial thoughts independent of images but still requires the actual presence of the object being manipulated at the third stage in the range of 7 to 12 years old. This stage of development concurs with Ben-Chaim's et. al (1988) proposition that the optimal age for acquiring spatial skills through instruction is between 11 and 12 years old. The capability of exploring mental manipulation involving infinite spatial possibilities and complex mathematical concepts is attainable at the formal operational stage from the age of 13 years old onwards. The maximum potential is reached at the age of 17 implying that students in high schools or post-secondary education are formal operational thinkers.

Although there are somewhat conflicting results in the literature regarding whether spatial ability can improve, numerous studies (e.g., Ben-Chaim, et. al., 1988) have indicated that it improves through training if appropriate materials are provided.

Lohman (1993), reports that spatial abilities may improve with practice and training. Studies showed that activities with solids can increase students' abilities to visualize objects (Ben-Chaim, et. al., 1988). Other studies noted that children in primary schools where there was a lot of use of manipulatives tended to perform better on spatial tasks than children in schools where no manipulative materials were used (e.g. Bishop, 1980). As Greabell (1978) mentioned, using a greater variety of manipulatives is beneficial. Because of the perceptual-conceptual interplay in spatial intelligence, appropriate perceptual experiences are necessary to generate representations. Children need opportunities to interact with information presented in a spatial format in order to construct knowledge representations, which are reconstructions of experience, and to engage in spatial reasoning with these representations (Diezmann & Watters, 2000). Through the present program we will investigate if teaching with manipulatives can improve students' spatial ability and influence their geometrical figure understanding.

RESEARCH GOALS AND METHOD

The main aim of this study is to investigate the development of spatial ability in relation to students' geometrical figure understanding. Specifically, the research goals of the program are:

1. To investigate students' spatial ability development in the last grades of elementary school and in the early grades of secondary school.
2. To examine the influence of students' perceptual, discursive and operative apprehension of geometrical figures in the last grades of elementary school and students in the early grades of secondary school, to their geometrical figure understanding.
3. To explore the relationship between spatial ability and perceptual, discursive and operative apprehension of students.
4. To assess the improvement of spatial abilities through intervention and its implications for geometrical figure understanding.
5. To draw useful implications for a curriculum program, teaching methods and materials, regarding the use of spatial ability and geometrical figures apprehension in the teaching of geometry in the last grades of elementary school and the early grades of secondary school.

The study will be conducted among students, aged 10 to 14, of elementary (Grade 5 and 6) and secondary (Grade 7 and 8) schools in Cyprus.

Data were collected through a two-part test that was constructed for the purposes of the present study: a spatial ability test and a geometrical figure apprehensions test. A total of 11 marker tests were included in the battery for the domain of spatial ability. The following reference tests are from the Ekstrom, French, Harman, and Dermen (1976) kit of factor-referenced cognitive tests, except where indicated. We made some minor changes to the tests in order to be solved by the students of our research.

Paper Form Board (FB). Each item presents 5 shaded drawings of pieces some or all of which can be put together to form a figure presented in outline form. The task is to indicate which of the pieces when fitted together would form the outline (see Eliot & Smith, 1983, test 127, p.149).

Paper Folding (PF). For each item successive drawings illustrate two or three folds made in a square sheet of paper. A drawing of the folded paper shows where a hole is punched in it. The subject selects one of 5 drawings to show how the sheet would appear when fully opened (see Eliot & Smith, 1983, test 312, p.337).

Surface Development (SD). In this test, drawings are presented of solid forms that could be made with paper and or sheer metal. With each drawing there is a diagram showing how a piece of paper might be cut and folded so as to make the solid form. Dotted lines show where the paper is folded. One part of the diagram is marked to correspond to a marked surface in drawings. The subject is to indicate which lettered edges in the drawing correspond to numbered edges or dotted lines in the diagram (see Eliot & Smith, 1983, test 315, p.341).

Perspective Taking/Spatial Orientation (P; Hegarty, Kozhevnikov, Waller, 2008). On each page subject see a picture of an array of objects and an 'arrow circle' with a question about the direction between some of the objects. For the question on each page, the subject should imagine that he or she is standing at one object in the array (which will be named in the center of the circle) and facing another object, named at the top of the circle. Subject task is to draw an arrow from the center object showing the direction to a third object from this facing orientation.

Card Rotations (CR). Each item gives a drawing of a card cut into an irregular shape. To its right are six other drawings of the same card sometimes merely rotated by different amounts and sometimes turned over onto its other side. The subject indicates which ones show the card not turned over (see Eliot & Smith, 1983, test 176, p.198).

Cube Comparisons (CC). Each item presents two drawings of a cube. Assuming no cube can have two faces alike, the subject is to indicate which items present drawings that can be of the same cube and which ones present drawings that cannot be of the same cube (see Eliot & Smith, 1983, test 266, p.290).

Hands (H; Thurstone's Primary Mental Abilities Tests, 1937). Each item presents a picture of hand. The subject indicates if the picture represents a right or a left hand (see Eliot & Smith, 1983, test 211, p.234).

Hidden Figures (HF). The task is to decide which of 3 geometrical figures is embedded in a complex pattern. The difficulty level of the test is high (see Eliot & Smith, 1983, test 053, p.71).

Hidden Patterns (HP). In the instructions, the subject is shown a single geometrical configuration. Each item presents a geometrical pattern. Some of the items contain the given configuration, embedded. The task is to mark each pattern in which the configuration occurs. These are easy items given under speeded conditions (see Eliot & Smith, 1983, test 054, p.72).

Overlapping figures (OF; Schrammel-Brannan Army Group Examination Alpha: Part 1, 1936). In the instructions, the subject is shown a diagram consists of different figures, overlapping. Each part of the complete geometrical figure has been numbered. The task is to answer several questions about the number belong to some figures and not belong to some other (see Eliot & Smith, 1983, test 057, p.76). For research purpose we construct another similar task with overlapping fruits instead of overlapping figures.

Water Level (WL; Linn & Petersen, 1985) A task that requires subjects to draw or identify a horizontal line in a tilted bottle (e.g., DeAvila, Havassy, & Pascual- Leone, 1976; Harris, Hanley, & Best, 1978; Inhelder & Piaget, 1958).

A total of 13 tasks were included in the test for the following first-order factors from the domain of geometrical figure apprehension test: Perceptual apprehension (PE), Operative apprehension (OP) and Discursive apprehension (DI). Most of the following reference tasks are used by researches of geometrical figure apprehension in previous studies (see Deliyianni E., Elia I., Gagatsis A., Monoyiou A., & Panaoura A., 2009; Elia, I., Gagatsis, A., Deliyianni, E., Monoyiou, A. & Michael, P., 2009; Michael, P., Gagatsis, A., Deliyianni, E., Elia, I. & Monoyiou, A., 2009). Particularly the test included:

a) The first group of tasks includes Perceptual apprehension (PE) of a geometrical figure. Perceptual apprehension refers to the recognition of a shape in a plane or in depth. In fact, one's perception about what the figure shows is determined by figural organization laws and pictorial cues. Perceptual apprehension indicates the ability to name figures and the ability to recognize in the perceived figure several sub-figures.

1. Task 1 examines the ability to discriminate and recognize in the perceived figures several subfigures (PE1).
2. Task 2 examines students' ability to identify and name the squares in a complex figure (PE2).
3. Task 3 examines the ability to identify side CD as a side of the triangle and of the square simultaneously (PE3).

b) The second group consists of Operative apprehension (OP) tasks. It is through *operative apprehension* that we can get an insight to a problem solution when looking at a figure. Operative apprehension depends on the various ways of modifying a given figure: the mereologic, the optic and the place way.

4. The tasks 4 and 5 require an optic way of modification. Task 4 asks students to find the thumbnail of a geometrical figure, whereas task 5 asks for the enlarged figure (OPop1 and OPop2).
5. Tasks 6, 7, 8 demand the place way of modifying figures (OPpw1, OPpw2 and OPpw3).
6. Tasks 9, 10 and 11 require a mereologic modification of the geometrical figure and particularly a reconfiguration of the given figures (OPme2 and OPme3).

c) The third group includes discursive apprehension tasks. *Discursive apprehension* is related with the fact that mathematical properties represented in a drawing cannot be determined through perceptual apprehension. In any geometrical representation the perceptual recognition of geometrical properties must remain under the control of statements (e.g. denomination, definition, primitive commands in a menu). The epistemological function of the discursive apprehension is the proof.

7. Tasks 12 (DI1) and 13 (DI2) require inferences based on definitions and properties and on procedures for proof for their solution.

Representative samples of the tasks used in the test appear in the Appendix.

EXPECTED RESULTS

The results of the present project will enable us to trace the development of spatial ability but in conjunction with performance in geometry skills. The collection of data derived from three measurements at distinct points in a period of time for the same selected sample will give valuable information about the rate of change of geometrical figure comprehension regarding the spatial ability development. The results of the study will contribute to the ongoing research on the relationship between spatial ability and geometry for different childhood ages and mainly for primary and secondary education. At the same time we will be able to answer the important question whether spatial ability influence students' learning outcomes and geometrical figure comprehension.

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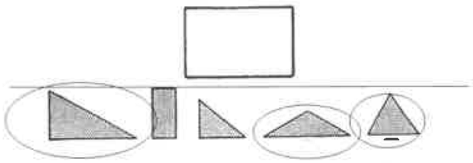
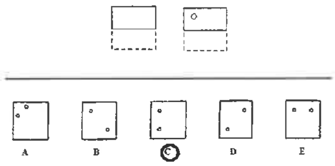
APPENDIX

SPATIAL ABILITY TESTS

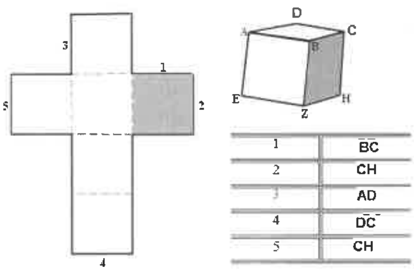
Decide which of the five shaded pieces can be put together to form the rectangle (**FB**).

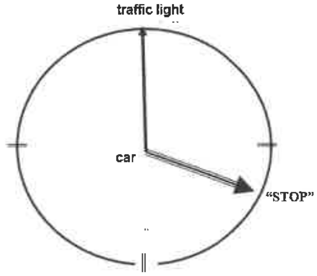


Try to find which of the five figures below the line shows where the holes will be when the paper is completely unfolded (**PF**).

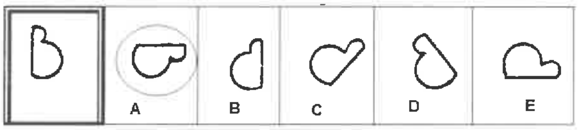


You are to imagine the folding and are to figure out which of the lettered edges on the object are the same as the numbered edges on the piece of paper at the left. Write the letters of the answers in the numbered spaces at the far right (**SD**).



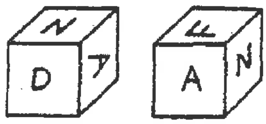


Imagine you are standing at the car and facing the traffic lights. Point to the STOP sight (**P**).



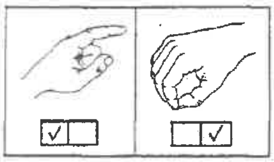
You are to decide which of the five cards on the right is same as the card in the box is (**CR**).

Compare the two cubes in each pair below (**CC**).

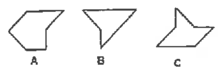


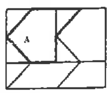
Same **Different**

If the picture represents a right hand put a check mark in the right square; if it represents a left hand put a check mark in the left square (**HA**).



Write the letter of the figure which you find in the pattern (**HF**).

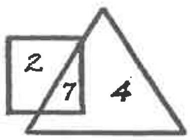




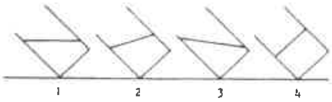
This test contains many rows of patterns. In each pattern you are to look for the model shown below (**HP**):

The diagram below consists of overlapping figures. Answer the following questions about the diagram (**OF**).

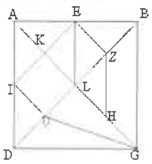
In figure, what number is in the square but not in the triangle?



Indicate which tilted bottle has a horizontal water line (**WL**).



GEOMETRICAL FIGURE APPREHENSION TASKS



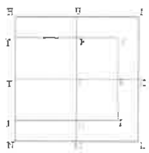
PE1

Recognize the figures:

KEZL: ____________

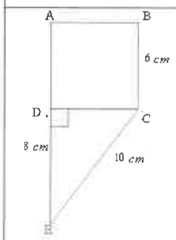
EZHL: ____________

BEL: ____________



PE2

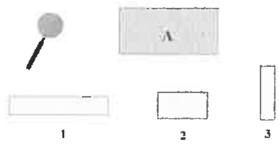
Name the squares



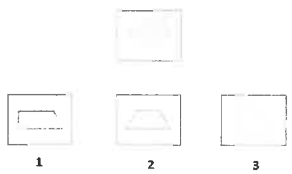
Find the length of the side CD

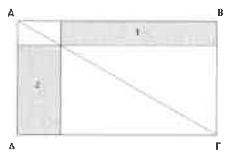
PE3

Vasilis constructed a rectangle in his writing book. Shape A is the rectangle as it looks through a magnifier. Circle the picture that shows the rectangle, as it is in Vasilis writing book. **OPop1**



Maria must match the cards with the same shape. Circle the yellow card that has exactly the same shape with Maria's card. **OPpw2**





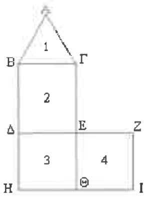
OPme1

Underline the right sentence:

a) Rectangle 1 has bigger area than rectangle 2

b) Rectangle 1 has equal area with rectangle 2

c) Rectangle 1 has smaller area than rectangle 2



DI1

In the figure:

- 1 is equilateral triangle
- 2 is a rectangle
- 3 and 4 are squares

Show that the length of ΑΓ and ΘΙ are equal.

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The Operative Apprehension of a Geometrical Figure through Students' Transition to a Next Educational Level[1]

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ABSTRACT: *This paper focuses in the examination of the operative apprehension of a geometrical figure, which constitutes one of the four dimensions of the conceptual apprehension of a geometrical figure, under the light of students' transition from one educational level to a next one. In specific the results of two research studies that approach the operative apprehension of a geometrical figure in different educational levels are presented. The first study investigated the role of various aspects of figure modification proposed by Duval (1995), i.e., mereologic, optic and place way, on the operative apprehension of geometrical figures in 125 6th graders and 120 7th and 8th graders whereas the second study dealt with 616 secondary (grade 9) and high school students (grade 10). The findings of both studies revealed interesting differences but similarities as well in the way primary, secondary and high school students behave in the application of the different types of figure modification.*

Key words: *Operative apprehension, Geometrical figure modifications, **Primary-**secondary-high school, Transition.*

INTRODUCTION AND THEORETICAL FRAMEWORK

Geometry occupies a particular place within mathematics; it appears as a model of physical space, and it follows that the objects it deals with (e.g. lines, planes, points) are directly taken from sensory experience, unlike in the other areas of mathematics (Parzysz, 1991). In geometry, a very important factor that influences the understanding is that there will be no confusion between the mathematical objects and their representation (Duval, 1999). A figure constitutes the external and iconical representation of a concept or a situation in geometry. According to English and Warren (1995), the way an individual visualizes a figure is one of the most important factors that affect the development of spatial ability.

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[1] A part of the results was published in the *Proceedings of the 2nd Conference of five cities in Mathematics Education* (Michael, Gagatsis, Avgerinos & Kuzniak, 2011, in press).

During the past twenty years, several mathematics educators have investigated students' geometrical reasoning based on different theoretical frames. For example, van Hiele (1986) developed a model referring to levels of geometrical thinking. Fischbein (1993) introduced the theory of figural concepts and Duval (1995, 1999) reported the cognitive analysis of geometrical thinking. Duval (1995) discriminated four apprehensions for a geometrical figure: perceptual, sequential, discursive and operative. Each has its specific laws of organization and processing of the visual stimulus array.

Previous research studies investigated extensively the role of external representations in geometry (e.g. Duval, 1998; Mesquita, 1996). Recent studies aimed at verifying empirically some of the cognitive processes underlying the geometrical figure apprehension as proposed by Duval (1995, 1999). Deliyianni, Elia, Gagatsis, Monoyiou and Panaoura (2009) affirmed the existence of a third-order model about the role of perceptual, operative and discursive apprehension in geometrical figure apprehension. From the particular research study the second order factor concerning the operative apprehension occurred as the most important for the conceptual apprehension of the geometrical figure. Consequently further studies (Elia, Gagatsis, Deliyianni, Monoyiou, & Michael, 2009; Michael, Gagatsis, Deliyianni, Elia, & Monoyiou, 2009; Michael, Elia, Gagatsis & Kalogirou, 2010) gave emphasis on the role the mereologic, the optic and the place way modifications exert on the operative figure apprehension and they verified a second-order model. The aforementioned studies also showed the invariance of the structure concerning the geometrical figure apprehension across elementary and secondary school students.

In order to have a deeper and a more comprehensive understanding of students' operative apprehension a second research study concerning students operative apprehension was performed, based on the design of the previous studies of Elia et al. (2009) and Michael et al. (2009, 2010). Taking into account the changes that occur during students' transition from one educational level to another universally (Mullins & Irvin, 2000), the second study examined secondary and high school students' operative geometrical figure apprehension, according to the three types of a figure's modification comprising it. This paper thus presents the results of the first study about the operative apprehension of geometrical figures conducted by Michael et al. (2009) in combination to the results of the second effort to examine the particular dimension of the apprehension of the geometrical figure.

The aforementioned studies were based on the following research questions:

1. Does students' performance differ for each type of geometrical figure modification?
2. Are there any differences between students of different educational levels concerning their performance in each of the three types of geometrical figure modification?
3. How consistently do students of different educational levels apply each of the three types of geometrical figure modification?

THEORETICAL BACKGROUND

The Geometrical Figure

In geometry three registers are used: the register of natural language, the register of symbolic language and the figurative register. A figure belongs to a specific semiotic system, which is linked to the perceptual visual system, following internal organization laws. As a representation, it becomes more economically perceptible compared to the corresponding verbal one, because in a figure various relations of an object with other objects are depicted (Mesquita, 1996).

Fischbein (1993) called geometrical figures "figural concepts" since these entities are simultaneously concepts and spatial representations. Generality, abstractness, lack of material substance and ideality reflect conceptual characteristics. But a geometrical figure also possesses spatial properties like shape, location and magnitude. In this symbiosis, it is the figural facet that is the source of invention, while the conceptual side guarantees the logical consistency of the operations (Fischbein & Nachlieli, 1998).

According to Duval (1995), the usefulness of the geometrical shape in the analysis of a geometrical problem is considered to be unquestionable, since it provides an intuitive presentation of the components and relationships in a geometrical situation. However, very often a figure does not help students to reach a solution to a problem. In fact the double status of external representation in geometry often causes difficulties to students when dealing with geometrical problems due to the interactions between concepts and images in geometrical reasoning (Mesquita, 1998).

Discriminating the Apprehensions of Geometrical Figures

Duval (1995) distinguishes four apprehensions for a geometrical figure: perceptual, sequential, discursive and operative. To function as a geometrical figure, a drawing must evoke perceptual apprehension and at least one of the other three.

Particularly, *perceptual* apprehension refers to the recognition of a shape in a plane or in depth. In fact, one's perception about what the figure shows is determined by figural organization laws and pictorial cues. Perceptual apprehension indicates the ability to name figures and the ability to recognize in the perceived figure several sub-figures.

Sequential apprehension is required whenever one must construct a figure or describe its construction. The organization of the elementary figural units does not depend on perceptual laws and cues, but on technical constraints and on mathematical properties.

Discursive apprehension is related with the fact that mathematical properties represented in a drawing cannot be determined through perceptual apprehension. In any geometrical representation the perceptual recognition of geometrical properties must remain under the control of statements (e.g. denomination, definition, primitive commands in a menu).

However, it is through *operative* apprehension that we can get an insight to a problem solution when looking at a figure.

Operative Apprehension: Visualization and Figural Processing

Duval (1999) considers representation and visualization to be at the core of understanding in mathematics. Visualization in mathematics is needed because it displays organization of relations, but it is not primitive, because it is not mere visual perception. Operative apprehension is different from perceptual apprehension because perception fixes at the first glance the vision of some shapes and this evidence makes them steady. In fact, there are differences between visual perception and visualization.

The way of watching is not the same in vision than in visualization. The use of visualization requires a specific training, specific to visualize each register. Geometrical figures are not directly available as iconic representations can be. Visualization consists in grasping directly the whole configuration of relations and in discriminating what is relevant in it (Duval, 1999). A fundamental component of visualization is visual processing. Visual processing includes the following functions-processes of mental images: change in the position of the represented object (e.g. object rotation), change in the structure of the represented object, combination of the above changes (Yakimanskaya, 1991). Operative apprehension is a form of visual processing that concerns geometrical figures.

It depends on the various ways of modifying a given figure: the *mereologic* way refers to the division of the whole given figure into parts of various shapes and the combination of them in another figure or sub-figures (reconfiguration), the *optic* way is when one makes the figure larger, narrower or slant, while the *place* way refers to its position or orientation variation. Each of these different modifications can be performed mentally or physically, through various operations. These operations constitute a specific figural processing which provides figures with a heuristic function. In operative apprehension the given figure becomes a starting point to explore other configurations that stem from the applications of these visual operations. A configuration may give insight into the solution of a problem. The ability to draw some units on a given figure is an indication of the operative apprehension (Duval, 1999).

The Transition between Two Educational Levels

The transition problem from one educational level to another is universally (Mullins & Irvin, 2000). In the recent years the research community was occupied with the difficulties that students seem to face during their transition from one educational level to another. When students move from one learning environment (i.e., classroom, grade, or school) to another there is less stability, although achievement goals are somewhat stable. This is compatible with a social-cognitive perspective that proposes that as contexts change, individuals reevaluate and reconstruct their goals and actions (Gutman, 2006). Research findings suggest that there is a negative change in students' perceptions about school, their attitudes towards various school subjects, their motives, their self-confidence beliefs and their competence, but also a decrease in their performance during the transition from primary to secondary education (Middleton, Kaplan, & Midlgley, 2004; Mullins & Irvin, 2000; Anderson, Jacobs, Schramm, & Splittgerber, 2000; Anderman & Midgley, 1997). During the transition to high school, students experience a multitude of changes in their academic and social worlds that often clash with their developmental needs (Gutman, 2006).

METHOD

Study 1 – Primary and Secondary Students' Operative Apprehension of Geometrical Figures

The study was conducted among 125 students, aged 11 to 12, from primary schools (grade 6) and 120 students, aged 12-14, from grades 7 and 8 from secondary schools in Cyprus. The a priori analysis of the test that was constructed in order to examine the research questions of this study is the following:

1. The first group of tasks includes task 1 (M1), 2 (M2) and 3 (M3) concerning students' mereologic way of modifying a given figure.
2. The second group of tasks includes task 4 (O4), 5 (O5) and 6 (O6). These tasks examine students' optic way of modifying a given figure.
3. The third group of tasks includes task 7 (P7), 8 (P8), 9 (P9) and 10 (P10) that correspond to the place way of modifying a given figure.

Representative samples of the tasks used in the test appear in the Appendix 1. Right and wrong or no answers to the tasks were scored as 1 and 0 respectively. The results concerning students' answers to the tasks were codified with M, O and P corresponding to mereologic, optic and place way modification respectively, followed by a number indicating the exercise number.

Students' performance was examined through the use of the t-criterion for paired and independent samples. The paired sample t-tests were used to examine whether there were statistically significant differences in students' performance in each of the three operative apprehension dimensions. The independent sample t test analysis was also performed to examine if there were statistically significant differences between primary and secondary school students' performance in the different types of modifying a geometrical figure.

For the analysis of the collected data, the hierarchical clustering of variables and Gras' implicative statistical method were also conducted, using the computer software C.H.I.C. (Classification Hiérarchique, Implicative et Cohésitive) (Bodin, Coutourier, & Gras, 2000). These methods of analysis determine the hierarchical similarity connections and the implicative relations of the variables respectively (Gras 1992). Thus, a hierarchical similarity diagram of the primary and secondary students' responses to the tasks of the test respectively was constructed. The similarity diagram allows for the arrangement of the tasks into clusters according to the homogeneity by which they were handled by the students. The implicative diagrams contain implicative relations, which indicate whether success to a specific task implies success to another task related to the former one.

Study 2 – Secondary and High School Students' Operative Apprehension of Geometrical Figures

The participants of the research were 312 secondary school students (grade 9) and 304 high school students (grade 10) in Cyprus. The test used in the research comprised of three groups of tasks:

1. The first group included tasks examining students' mereologic (me) way of modifying a given figure (OPme1, OPme2, OPme3).
2. The second group consisted of tasks concerning the optic (op) way of modifying a given figure (OPop1, OPop2, OPop3).
3. The third group was comprised of tasks corresponding to the place way (pw) of modifying a given figure (OPpw1, OPpw2, OPpw3).

Representative samples of the tasks used in the test appear in the Appendix 2. Right and wrong or no answers were scored as 1 and 0 respectively. As in study 1, students' performance was examined through the use of the t-criterion for paired and independent samples. For the examination of students' consistency in their behaviour in the solution of the tasks was examined by the performance of the hierarchical clustering of variables and Gras' implicative statistical method (Bodin, Coutourier, & Gras, 2000). Therefore two similarity and two implicative diagrams of the 9^{th} and 10^{th} graders' responses to the tasks of the test were produced.

RESULTS

Study 1

In order to answer the two first research questions, students' success was examined using the means and standard deviations of their performance in each type of modification of geometrical figures. Table 1 indicates that primary students' performance in the place modification tasks was higher than their performance in the optic modification tasks, but the use of the t-criterion for paired samples revealed that this difference was not statistically significant ($p<0.05$). In contrast, students' performance was significantly lower in the mereologic modification tasks than their performance in the other two types of modification tasks ($p<0.05$).

As secondary school students are concerned, although their performance in the place modification tasks was higher than their performance in the optic modification tasks, this difference was not statistically significant ($p<0.05$). On the other hand, their performance in the mereologic modification tasks was significantly lower than their performance on the other two types of modification tasks ($p<0.05$). Furthermore, the use of the t criterion for independent samples indicated that primary school students' performance is significantly higher in the optic and the place modification of the geometrical figure compared to primary school students' performance ($p<0.05$).

Table 1
Primary and Secondary Schools Students' Mean Scores and Standard Deviations for Each Type of Figure Modification

Level	Optic		Place		Mereologic	
	$\overline{X}$	SD	$\overline{X}$	SD	$\overline{X}$	SD
Primary (Grade 6)	0.61	0.26	0.66	0.24	0.28	0.29
Secondary (Grade 7,8)	0.55	0.23	0.57	0.26	0.35	0.33

Figures 1 and 2 present the similarity diagrams of the primary and secondary school students' responses to the tasks of the test respectively. Particularly in Figure 1 two similarity clusters can be identified. The first cluster involves students' responses to all the mereologic modification tasks (M1, M2, M3) and two of the place modification tasks (P9, P10). The second cluster is comprised of students' responses to all the optic modification tasks (O4, O5, O6) and the other two place modification tasks (P7, P8). In Figure 2 three similarity clusters can be distinguished. The first cluster includes students' responses to all the mereologic modification tasks (M1, M2, M3), the rest of the place modification tasks (P9, P10) and one optic modification task (O6). The first and second similarity clusters are formed by variables corresponding to an optic modification task and a place modification task (O5 – P8 and O4 – P7 respectively).

Concerning the third research question, the comparison of the two diagrams indicates that there are similarities but also differences between primary and secondary school students' behaviour regarding the different types of modifying a geometrical figure. Specifically, consistency was displayed by primary and secondary school students when applying the mereologic modification of the geometrical figure. Consistency appears also in the solution of the optic modification tasks but only from primary school students. For primary school students all the three operative modification tasks are found in the same similarity cluster (Figure 1), whereas for secondary school students these tasks are separated into the three different similarity clusters (Figure 2).

The application of the place modification cannot be characterised by consistency neither for primary nor for secondary school students, since the variables of these type of figure modification are separated into different similarity clusters. Primary and secondary school students' behaviour during the application of the place modification of the geometrical figure is similar to the mereologic modification, while weaker similarity relations with the application of the optic modification are also found. Actually there is a relation concerning the two of the operative apprehension tasks (P9 and P10) which remains constant in primary and secondary school students' similarity diagram respectively. In fact these two place modification tasks are significantly related with the mereologic modification tasks in both diagrams. Despite the invariance of this relation, a greater consistency is displayed by secondary school students regarding the solution of two place modification tasks (P9, P10), as the similarity relation between them appears to be more significant, compared to the relation between them in the similarity diagram for primary school students.

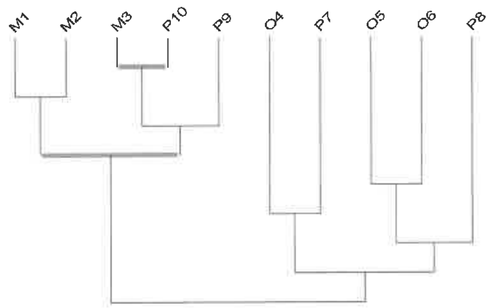

Figure 1. Similarity diagram of primary school students' responses to the tasks of the test (grade 6).

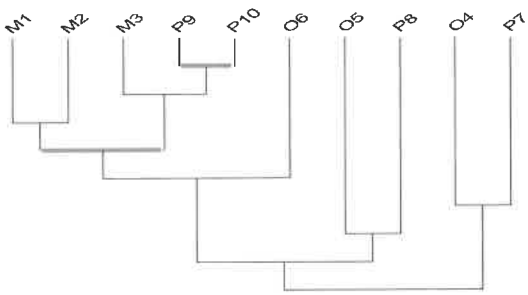

Figure 2. Similarity diagram of secondary school students' responses to the tasks of the test (grades 7 – 8).

The implications between the variables according to students' behaviour to the tasks of the test are presented in Figures 3 and 4. Concerning figure 3, two implicative chains are discriminated. The first implicative chain concerns the implicative relation between two optic modification tasks. This implicative relation shows that the primary school students who are able to solve the task O5 are also able to solve the task O4. The second chain is formed by the implicative relations that occurred between the tasks P10, P9 and M3. What is indicated by this chain is that students who accomplish the solution of task P10 also solve tasks P9 and M3. Regarding figure 4, two implicative chains can also be distinguished. The task P10 is situated at the top of the first implicative chain, meaning that the students who succeed in this task can also solve successfully the task M3. Similarly, when secondary school students succeed the solution of task M3 they can also achieve a correct solution in the task P9. The second implicative chain concerns two mereologic modification tasks. What is indicated is that the success in the task M2 implies a success in the tasks M1 and P9 respectively.

Comparing the two implicative diagrams (Figures 3 and 4) some common implicative relations between variables can be observed. Specifically the implicative relations between the tasks P10, M3 and P9 are found in the implicative diagrams of both groups of students. What differentiates the two diagrams is the implicative relation between the optic modification tasks concerning primary school students and the implicative relation between the mereologic modification tasks in secondary school. Thus primary school students' behaviour appears to be more coherent during the solution of the optic modification tasks, as also occurred from the similarity diagram. On the other hand secondary school students' behaviour concerning the performance of the mereologic modification seems to be more coherent than primary school students.

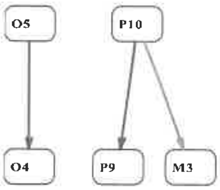

Figure 3. Implicative graph for primary school students (grade 6).

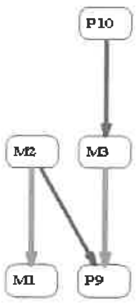

Figure 4. Implicative graph for secondary school students (grades 7 – 8).

Study 2

The means and standard deviations of secondary and high school students' performance in each modification type are presented in Table 2. As observed, both 9th and 10th graders' performance is higher in the optic modification tasks than in the place modification task), although the paired samples t test revealed that this difference was not statistically significant (p<0.05).

On the contrary, students' performance in the mereologic tasks was significantly lower (p<0.05) compared to the other two types of modification tasks. In addition, the findings indicate that differences exist in the performance of secondary and high school students, since the t criterion for independent samples showed that high school students' performance is significantly higher in all the types of geometrical figure modification (p<0.05). Thus this difference reveals a progress in students' ability in modifying a geometrical figure after their transition from secondary to high school.

Table 2
Secondary and High Schools Students' Means and Standard Deviations for Each Type of Figure Modification

Level	Optic		Place		Mereologic	
	$\overline{X}$	SD	$\overline{X}$	SD	$\overline{X}$	SD
Secondary (Grade 9)	0.63	0.24	0.61	0.27	0.36	0.25
High (Grade 10)	0.68	0.22	0.66	0.28	0.43	0.23

Figures 5 and 6 constitute the similarity diagrams of the 9th and 10th graders responses to the tasks of the test respectively. In both diagrams two similarity clusters are identified. In Figure 5 the first similarity cluster is comprised of all the mereologic modification tasks (OPme1, OPme2, OPme3), a place modification task (OPpw3) and an optic modification task (OPop1). The second cluster consists of two optic modification tasks (OPop2 and OPop3) and two place modification tasks (OPpw1 and OPpw2). In Figure 6 the first similarity cluster is formed by students' responses to all the mereologic modification tasks (OPme1, OPme2, OPme3) and one of the place modification task (OPpw3). In the second similarity cluster the optic modification tasks (OPop1, OPop2, OPop3) and the rest of the place modification tasks (OPpw1, OPpw2) can be found.

Thus mainly common features appear between secondary and high school students' behavior in the operative apprehension tasks. The similarity diagrams suggest that students of both educational levels displayed consistency in applying the mereologic way of modifying geometrical figures. This is not the case, though, for the place way of modifying geometrical figures, as the responses of the students of both grades in the three corresponding tasks are split into the two different similarity clusters. A number of the place way modification tasks were approached similarly to the optic modification tasks, while the rest of the place modification tasks were tackled similarly to the mereologic modification tasks. However, there is a difference between the two groups of students concerning the optic modification tasks. Grade 10 students displayed consistency in this type of modification, since the corresponding variables are found in the same similarity cluster. In contrast, 9th graders' responses to these tasks are split into different clusters and are linked with tasks of the other two types of modification. Furthermore secondary school students' behavior during the solution of the three different types of tasks seems to be more coordinated compared to primary school students, since the two similarity clusters (Figure 6) are significantly linked.

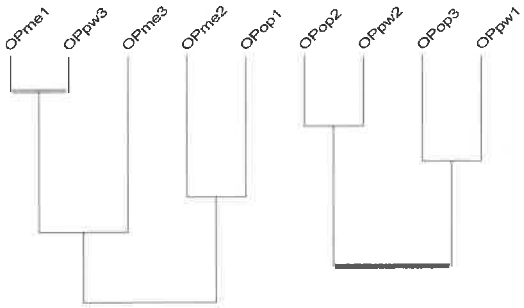

Figure 5. Similarity diagram of secondary school students' (grade 9) responses to the tasks of the test.

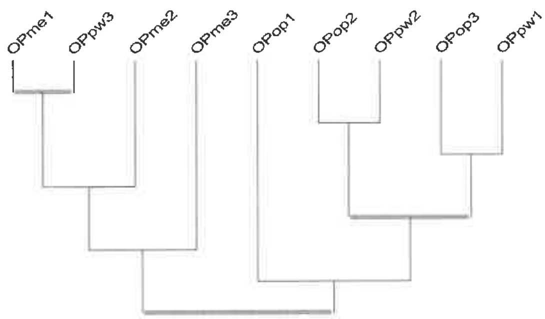

Figure 6. Similarity diagram of high school students' (grade 10) responses to the tasks of the test.

The implicative relations between the tasks of the test are presented in Figures 7 and 8 for the 9^{th} and 10^{th} graders respectively. In Figure 7 implicative relations appear between the three mereologic modification tasks (OPme1, OPme2, OPme3) and the place modification tasks (OPpw1 and OPpw3). Also implicative relations occur between the place modification tasks and the optic modification tasks (OPpw1 – OPop2 and OPpw2 – OPop3). On the other hand, there are no implicative relations between the mereologic modification tasks and the optic modification tasks. This is also the case for high school students' implicative diagram (Figure 8). In this implicative diagram only one optic modification task (OPop2) is involved in an implicative relation with a place modification task (OPpw2). Actually this relation remains constant related to the implicative diagram of secondary school students. The rest implicative relations in high school students' implicative diagram are formed between the mereologic and the place modification tasks. What is interesting to be mentioned is the fact that stronger implicative relations appear between these tasks in the implicative diagram for high school students, revealing a more coherent behavior of these students regarding the solution of these types of tasks.

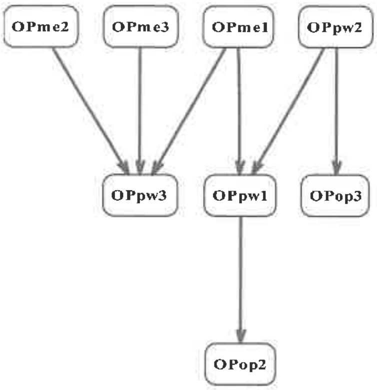

Figure 7. Implicative diagram of secondary school students' responses (grade 9).

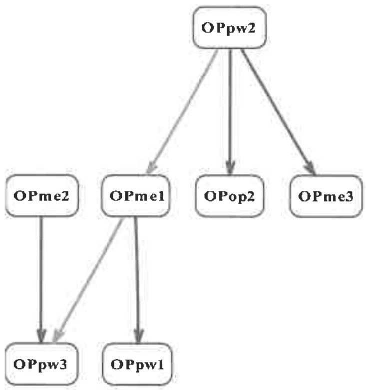

Figure 8. Implicative diagram of high school students' responses (grade 10).

The results of the present project will enable us to trace the development of spatial ability but in conjunction with performance in geometry skills. The collection of data derived from three measurements at distinct points in a period of time for the same selected sample will give valuable information about the rate of change of geometrical figure comprehension regarding the spatial ability development. The results of the study will contribute to the ongoing research on the relationship between spatial ability and geometry for different childhood ages and mainly for primary and secondary education. At the same time we will be able to answer the important question whether spatial ability influence students' learning outcomes and geometrical figure comprehension.

CONCLUSIONS AND DISCUSSION

This paper presents a combination of the results of two research studies that focused in the examination of students' operative apprehension of the geometrical figure. In these studies the different types of modifying a given figure, i.e., mereologic, optic and place way, were investigated in primary (grades 6), secondary (grades 7, 8 and 9) and high school students' (grades 9 and 10) respectively. The combination of the results of the two research studies allow the formation of a more comprehensive idea about students' operative apprehension, since this significant dimension of the geometrical figure apprehension (Deliyianni et al., 2009) is approached through different educational levels and from the perspective of students' transition from one educational level to a next one.

The results of the first research study (Elia et al., 2009; Michael et al., 2009) showed that similarities existed in the primary and secondary students' performance in the mereologic, optic and place modification tasks. Particularly, the 6^{th}, 7^{th} and 8^{th} graders' performance in the place modification tasks was close to their performance in the optic modification tasks. On the contrary, their performance in the mereologic modification tasks was significantly lower than in the other two types of modification tasks. The weak performance on these tasks could be linked to the fact that they required more complex figural processes. That is, the students needed to understand the division of the given figure into parts and their combination in another figure and proceed to calculations of specific areas (e.g. M3) or estimations of the figures' perimeter (e.g. M2) in order to provide a solution to the corresponding tasks. As Duval (1999) explains, the recognition of a figure is independent from its magnitude or its perimeter. Therefore it is possible that a conflict occurs to students while trying to perform measurements and modifications in a figure simultaneously. In the case which students form a hypothesis based on measurements, operative apprehension is neutralised and the figure stands only as a picture. On the other hand the comparison between primary and secondary (7^{th} and 8^{th} graders) school students' performance in the three types of figure modification revealed a change after students' transition from primary to secondary education. The change concerns the reduction of secondary school students' performance in the application of the optic and the place modifications of the geometrical figure. This may be linked to fact that the move from sixth to seventh grade may be accompanied by an increase in the saliency of performance differences (Gutman, 2006).

The similarity diagrams provided indications for primary and secondary students' consistency across the three types of geometrical figure modification. Although primary and secondary school students exhibited consistency in the mereologic modification tasks, they applied the place way of modifying geometrical figures in a rather fragmentary way. A number of the place modification tasks (P9, P10) were handled similarly to the mereologic modification tasks, whereas students' behaviour during the solution of the rest of the place modification tasks (P7, P8) appeared to be similar to the optic modification tasks. This finding suggests that although it is the place modification that gives insight to the solution of the corresponding tasks (Duval, 1995), some additional operations need to take place so that students successfully reach the ultimate solution. These additional operations may have common characteristics with the figural processing which is required in either the mereologic modification tasks or the optic modification tasks. Specifically, in the first case, the place modification tasks P9 and P10 did not require only the understanding of the position or orientation variation of the figures, but also the combination of figures in another figure (reconfiguration), which is a characteristic of the mereologic type of geometric figure modification. Moreover, both mereologic and place modification tasks (P9, P10), involved measurement or estimation concepts (e.g. perimeter) and processes in combination to the spatial processes. In the second case, both optic and place modification tasks entailed principally spatial skills and specifically the comparison of figures of the same form which differed either in their position and orientation because of rotation (P7, P8), or in their magnitude (Fischbein & Nachlieli, 1998), because of enlargement (O4) or variation of distance from a reference point (O5, O6).

Despite the similarities revealed between 6^{th}, 7^{th} and 8^{th} graders' consistency in the application of each type of figure modification, it appears that the 6^{th} graders' solutions in the optic modification tasks were more coherent than those of the 7^{th} and 8^{th} graders'. On the other hand, secondary school students display a greater consistency regarding their responses in two of the place modification tasks modification tasks (P9, P10). In addition, according to the implicative diagrams, secondary school students' behaviour concerning the performance of the mereologic modification seems to be more coherent than primary school students. These could be attributed to a development of students' measurement or estimation skills and to the development of their skills in performing spatial processes, due to their age and to their teaching experience.

Based on the first research study, the second study (Michael, Gagatsis, Avgerinos, & Kuzniak, 2011) aimed at examining the operative apprehension of secondary (9^{th} graders) and high school students (10^{th} graders), regarding their performance and behaviour in the mereologic, the optic and the place way of modifying a geometrical figure. The results revealed similarities, but differences as well concerning students' performance in the different kinds of figure modification. Particularly, both 9^{th} and 10^{th} graders' performance is higher in the optic modification tasks, in relation to their performance in the mereologic and the place way modification tasks. In fact, their lowest performance was in the mereologic modification tasks. Therefore the mereologic modification tasks are the most difficult for students, in contrast to the optic modification tasks, which appear to be the easiest type of modification for students to handle. Consequently, it seems that students of this age are more able to perform modifications relating to changes in the size of the shape, than in its form and structure.

The realization of the structural relations of a figure and of its subfigures appears to cause more difficulties to students, than the variation in its size or its orientation.

However, the 10^{th} graders outperformed the 9^{th} graders in all the three types of figure modification. Thus, students' abilities to manipulate and modify a given figure in order to reach a solution in a task seem to be improved after their transition from secondary to high education. This might be an indication of the improvement of operative apprehension of the geometrical figure, as students grow up and move to a next educational level. With the potential of improving students' abilities on the operative apprehension, which is considered to be crucial for problem solving as it provides a heuristic function to a geometrical figure (Duval, 1995), teachers should try to provide their students the chance to strengthen their skills in performing the three types of figure modifications through relative tasks.

Regarding the third research question examined in the second study, the similarity diagrams indicated that there are mainly commonalities in secondary and high school students' behaviour in the operative apprehension tasks. Students' behaviour during the solution of the mereologic modification tasks can be characterized by consistency, in contrast to their behaviour while performing the place way of modifying geometrical figures that can be characterized as unsystematic. Similarity was traced in the solution of a number of the place modification tasks with the mereologic modification tasks and the optic modification tasks as well. This finding is in line with the result which occurred from the first study regarding the necessary additional operations that students need to perform in order to solve a geometrical task successfully (Duval, 1995).

Since the main idea of this paper is to give an overall sense of the main indications that occur from the two research studies that examined primary, secondary and high school students operative figure apprehension, the main conclusions of each study are summarized in Figure 9. Thus the figure allows the comparison of students' performance and the consistency in their behaviour during the solution of the operative apprehension tasks. To begin with students' performance, it is observed that the lower and higher performance in the figure modification tasks is constant during students' transition from primary to secondary education. Similarly these results remain invariant during students' transition from secondary to higher education.

	1ST RESEARCH		2ND RESEARCH	
	Primary	Secondary		Higher
	Grade 6	Grade 7-8	Grade 9	Grade 10
Higher performance	place	place	optic	optic
Lower performance	mereologic	mereologic	mereologic	mereologic
Consistency	mereologic	mereologic	mereologic	mereologic
Compartmentalization	optic	optic	optic	optic

Figure 9. Summary of the results about students' performance and consistency in the different types of figure modification.

As regards students' consistency in their behaviour during the solution of the tasks, the results indicate that the transition from primary to secondary school partially affects students' consistency regarding the type of the figure modification. However the results also show that the consistency in the application of particular types of figure modification increases in secondary school compared to primary school level. Similarly a shift is noticed after students' transition from secondary to high school, concerning the type of figure modification which they apply with consistency, as a second type occurs for the students of the high school. On the other hand the type of figure modification which secondary school students apply with coherence occurs to be the same from both researches. Overall, it seems that the changes occur in students' behaviour during their transition from one educational level to the next one. Thereafter it seems that the structure of the cognitive abilities regarding the operative apprehension of the geometrical figure is similar enough in all the educational levels examined in this study, whereas the differences that occur concern students' degree of success. Despite though the differences in students' success in the tasks, a greater coordination is noticed in the cognitive processes involved in the different types of figure modifications during the students' transition to the next educational level.
Another remark extracted from figure 9 is that the lower performance appears in the mereologic modification tasks for the three educational levels. Consequently it is essential for teachers to emphasise on the specific type of figure modification, as it is related to visualization, figural processing (Owens, 1990) and dynamic imagery. Imagery influences attendance during problem solving (Owens & Clements, 1998). Furthermore the type of figure modification which causes the higher performance remains the same during the transition from primary to secondary education (place way) and from secondary to high education (optic way). Duval (1999) argues that the place type of figure modification constitutes the weakest transformation, compared to the other two types of figure modification, which seems to be enhanced by the results about primary and secondary school students.

It is also important to mention that from the implicative diagrams of all the four groups of students occur more and stronger relations between the mereologic and the place modification tasks. In primary and secondary school there are no implicative relations between the optic modification tasks and the tasks of the other two types of figure modification, whereas in the last grade of secondary school (grade 9) and the high school these relations appear but are limited. This conclusion is also enhanced by the similarity diagrams. In all groups of students the phenomenon of compartmentalization appears regarding the solution of the optic modification tasks, especially in relation to the mereologic modification tasks. In addition, in all the four groups of students, the stronger similarity relations are formed between the mereologic and the place modification tasks. Consequently, it seems that the cognitive processes during the application of the mereologic and the place modifications have more commonalities. On the other hand different kind of cognitive processes seem to be involved during the application of the optic modification on a geometrical figure. These findings provide support to Duval's (1995) conceptualization of the cognitive processes underlying operative figure apprehension and suggest that, in order to develop operative apprehension during mathematics instruction in school, emphasis should be given on each of the three types of figure modification. What is important is that these observations occur from all the four different groups of students, showing that this fact is common for students of different age and educational level.

To sum up, the structure and the nature of the cognitive processes underlying the different types of modifications of a geometrical figure which comprise the operative apprehension of the geometrical figure, does not seem to be influenced by any transition to a next educational level. What appears to change is the degree of success, the stability and the coordination in their behaviour during the use of the different types of figure modification.

Consequently, it is obvious that the dimension of the operative apprehension of a geometrical figure should be further investigated. Firstly, it would be interesting to examine deeper whether the difficulties of students and their limited consistency when applying specific types of figure modifications remain invariant in respect to their development and learning at school. The identification and examination of the factors than inhibit, trigger or improve the operative apprehension of geometrical figures is considered important to be studied as well, in order to form a more comprehensive understanding of the cognitive processes underlying this kind of apprehension of the geometrical figure. The effects of intervention programs, which aim to develop students' abilities in modifying a figure, on the operative and other apprehensions for geometrical figures or in geometry problem solving, could also be investigated in future studies. Finally it would also be interesting to compare the strategies primary, secondary and high school students use in order to solve tasks of the three types of modifying a geometrical figure.

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The Effect of the Use of SRA Real Math Building Blocks Prek Software on Kindergartners' Geometrical Competencies[1]

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ABSTRACT: *The aim of the present study is to investigate the contribution of using SRA Real Math Building Blocks PreK software on kindergartners' geometry understanding. A total of 42 kindergartners were administered a pre-test and a post-test involving five geometrical tasks. After pretesting all the children received an intervention program on geometry which involved shape identification, shape composition and decomposition, congruency, shape construction and shape rotation with the use of SRA Real Math Building Blocks PreK. The geometry software was found to improve children's performance and strategies in the tasks of the test. Furthermore, the intervention enhanced children's operative apprehension of geometrical figures such that it provided support to other types of geometrical figure understanding, i.e. the sequential apprehension, which refers to shape construction. Finally, the instructional program helped students solve the test immediately, systematically and confidently.*

Key words: *Computers, Geometry understanding, Mathematics learning, Operative apprehension, Perceptual apprehension, SRA Real math building blocks PreK, Kindergarten.*

INTRODUCTION

Most schools have some computer technology, with the ratio of computers to students ranging from 1:125 in 1984 and 1:22 in 1990 to 1:10 in 1997 (Clements & Nastasi, 1993; Coley et al, 1997). However, this increased interest in bringing computers into schools, does not imply that children use computers. In one study, only 9% of fourth graders (they did not collect data on younger children) said they used a computer for schoolwork almost every day while 60% said they never used one. A study of preschool and kindergarten classrooms indicated low use by most teachers (Cuban, 2001).

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Nevertheless, there seems to be an increasing *potential* for children to use computers in early childhood settings. Is such use appropriate?

An old concern is that children must reach the stage of concrete operations before they are ready to work with computers. Research, however, has shown that preschoolers are more competent than has been thought and can, under certain conditions, exhibit thinking traditionally considered 'concrete' (Gelman & Baillargeon, 1983).

Amante (2007) highlighted the great significance of the use of computers in mathematics. Specifically Amante (2007) noted that the use of computers has shown to encourage the acquisition of some mathematical concepts as the recognition of shapes, counting and classifying (Amante,2007). Furthermore, Clements and Nastasi (2002) stated that children who can make associations between the direct manipulative experiences and the use of a computer program have been more successful in classification operations and logical thinking than those that only had experiences with concrete manipulatives.

It is well documented that computers' main contribution seems to be at the level of the development of geometrical and spatial thinking, and specifically in the development of concepts of symmetry, patterns, and spatial organization, among others (Clements & Swaminthan, 1995). Children are able to use the graphic programs to create objects and act on them, enlarge or reduce them in size, add shapes that lead to other new shapes, colour closed areas, and reflect on the topological characteristics of those areas (Clements & Nastasi, 2002).

THEORETICAL FRAMEWORK

Learning in Geometry

Geometry is the study of shapes and space and provides a powerful system for representing, describing and understanding the spaces and objects around us. Studying geometry enables children to further develop their mathematical reasoning abilities (National Research Council, 2009). Two-dimensional shapes, even very simple ones, have various aspects that can be addressed.

Duval (1995, 1999), distinguishes four apprehensions for a "geometrical figure": perceptual, sequential, discursive and operative. To function as a geometrical figure, a drawing must evoke perceptual apprehension and at least one of the other three. Each has its specific laws of organization and processing of the visual stimulus array. Particularly, *perceptual apprehension* refers to the recognition of a shape in a plane or in depth. In fact, one's perception about what the figure shows is determined by figural organization laws and pictorial cues. Perceptual apprehension indicates the ability to name figures and the ability to recognize in the perceived figure several sub-figures. *Sequential apprehension* is required whenever one must construct a figure or describe its construction. The organization of the elementary figural units does not depend on perceptual laws and cues, but on technical constraints and on mathematical properties.

Discursive apprehension is related with the fact that mathematical properties represented in a drawing cannot be determined through perceptual apprehension. In any geometrical representation the perceptual recognition of geometrical properties must remain under the control of statements (e.g. denomination, definition, primitive commands in a menu). However, it is through *operative apprehension* that we can get an insight to a problem solution when looking at a figure. Operative apprehension depends on the various ways of modifying a given figure: the mereologic, the optic and the place way. The mereologic way refers to the division of the whole given figure into parts of various shapes and the combination of them in another figure or sub-figures (reconfiguration), the optic way is when one makes the figure larger or narrower, or slant, while the place way refers to its position or orientation variation. Each of these different modifications can be performed mentally or physically, through various operations. These operations constitute a specific figural processing which provides figures with a heuristic function. In a problem of geometry, one or more of these operations can highlight a figural modification that gives an insight to the solution of a problem.

A basic skill included in the mereologic aspect of geometrical figure's operative apprehension is shape composition. The composition of two-dimensional geometric figures is considered significant for students in two ways. First, it is a fundamental geometric competence, growing from preschoolers' building with shapes to sophisticated interpretation and analysis of geometric situations in high school mathematics and above. Second, the concepts and actions of creating and then iterating units and higher–order units in the context of constructing patterns, measuring and computing are established bases for mathematical understanding and analysis (Clements et al., 1997; Reynolds & Wheatley, 1996; Steffe & Cobb, 1988).

In the kindergarten it is more feasible to teach mainly two basic apprehensions of figures: the perceptual apprehension, i.e., the recognition and naming of geometrical figures, and the operative apprehension with emphasis on reconfiguration, i.e., investigating and predicting the results of putting together and taking apart geometrical figures.The importance of the development of these abilities for promoting young children's geometrical understanding has been highlighted by the joint position statement of the National Association for the Education of Young Children and the National Council of Teachers of Mathematics (2002).

SRA Real Math Building Blocks PreK

Building Blocks is a National Science Foundation-funded project designed to enable all young children to build a solid foundation for mathematics. Based on theory and research on early childhood learning and teaching (Bowman et al, 2001; Clements, 2001), it was determined that the basic approach of *Building Blocks* would be *finding the mathematics in, and developing mathematics from, children's activity*. The materials are designed to help children extend and mathematize their everyday activities, from building blocks to art to songs and stories to puzzles. Activities are designed based on children's experiences and interests, with an emphasis on supporting the development of *mathematical* activity. So, the materials do not rely on technology alone, but integrate three types of media: computers, manipulatives (and everyday objects), and print. The materials were organized into two areas: spatial and geometric competencies and concepts, and numeric and quantitative concepts. Three mathematical themes are woven through both these main areas: patterns, data, and sorting and sequencing.

Moreover, all components of the building blocks project are based on learning trajectories for each core topic. First, empirically based models of children's thinking and learning are synthesized to create a developmental progression of levels of thinking in the goal domain (Clements & Sarama, 2004; Cobb & McClain, 2002; Gravemeijer 1999; Simon, 1995). Second, sets of activities are designed to engender those mental processes or actions hypothesized to move children through a developmental progression.

Clements and Sarama (2006) in a previous study analyzed group differences between the Building Blocks for Math intervention group and the business-as-usual comparison group on one math outcome measure. The difference between groups was statistically significant and favored children in the Building Block for Math group.

METHOD

Purpose and Research Questions of the Study

The purpose of this study is to investigate the effects of Building Blocks for Math in kindergartners' geometrical understanding. More specifically the present study addresses the following research questions:

1. Are there any differences in students' errors before and after the intervention program?
2. Do children's strategies change after the intervention program?
3. Are there any differences in children's behavior when taking the test before and after the participation in the instructional program?
4. How does children's general performance in geometry change after their involvement in the intervention program using the Building Blocks project?
5. How does children's performance on each geometrical competence change after their involvement in the intervention program?
6. Are there any differences in the interrelations of the various geometric competences of the children before and after the participation in the instructional program?

To answer the research questions, we collected data about kindergartners' geometry performance. These children followed an intervention program with a pretest–posttest design. The intervention program involved activities with the use of computers from Building Blocks for Math that addressed various geometry skills.

Participants

The participating children were 42 students of 4-6 years of age from two kindergarten classes in different pre-primary schools in Cyprus.

Intervention

Several activities based on the Building Blocks for Math project (Clements and Sarama, 2003) were employed over three consecutive weeks. Specifically, the time devoted to this type of instruction was approximately 45 minutes a day.

The particular curriculum includes 9 activities on recognition shapes, 10 activities on composition of shapes, 10 activities on rotation of shapes, 9 activities on matching congruent shapes and 9 activities on construction. Each activity of the program has several levels, often containing quite different tasks. For example, *Shape Puzzles* invites children to solve outline puzzles by putting shapes together. They use the tools to move the shapes into place. They move through research-based levels, from puzzles that are simple and 'obvious' to puzzles that are 'open' and challenging and require children mentally to combine shapes. In the computer activity *Mystery Toys*, the computer pronounces each shape name as children match the shapes to build a surprise toy. At another level, the children are asked to click on the correct shape when the computer pronounces its name. The activity is popular among the children and they enjoy imitating the computer voice when they name shapes. In the computer activity *Memory Geometry* children are asked to place two sets of memory geometry cards face down each in an array. Players take turns exposing one card from each array. Cards that do not match are replaced face down and cards that match are kept by that player.

The kindergartners that received the intervention program were engaged in a number of selected computer activities that are described above. In particular, they were involved first in the five levels of Mystery Toys, then in the first level of Memory Geometry and finally in the five levels of Shape Puzzles. Postgraduate students, who had been previously trained, guided children by discussing the task, eliciting children's strategies, and when necessary modeling successful strategies.

Assessment Instruments

To assess children's ability in geometry and specifically their perceptual and operative aprehension, a number of test items (see Appendix) were developed. These items were used for pretesting as well as for posttesting. The six tasks of the test asked students to recognize shapes (task 1), compose (task 3) and decompose shapes (task 2), match congruent shapes (task 4), classify rotated shapes (task 5) and construct shapes (task 6). Children's correct and incorrect answers were coded as 1 and 0 respectively.

Check lists and semi-structured interviews with children who participated in the program were also conducted in order to test their strategies and attitudes before and after the intervention. Table 1 shows the encoding of the variables which were used for the analysis of the data about students' answers on the test items, their strategies on task 3 and their attitudes about the test.

Table 1
Encoding of the Variables

Responses to the tasks of the test
A1: identifying squares in the first task
A2: identifying rectangles in the first task
A3: identifying triangles in th\e first task
A4: identifying rhombus in the first task
B3: right solution in the second task
C: right solution in the third task
D1: identifying congruent right triangles in the fourth task
D2: identifying congruent acute triangles in the fourth task
E1: identifying all acute triangles rotated in the fifth task
E2: identifying all right triangles rotated in the fifth task
E3:identifying all parallelograms rotated in the fifth task
E4: identifying all rectangles rotated in the fifth task
E5: identifying all squares rotated in the fifth task
E6: identifying all trapezoids rotated in the fifth task
F1: construction of square in the sixth task
F2: construction of triangle in the sixth task

Strategies in task 3
S1A: Placing pieces randomly that are not connected
S1B: Putting shapes together without leaving gaps
S1C: No response
S2A: Turning shapes after placing them on outline figure in an attempt to get them to fit
S2B: Turning them into correct orientation prior to placing them
S2C: No response
S3A: Trying out shapes by picking them seemingly at random, then putting them back if they do not look right
S3B: Appearing to search for just the right shape that they "know will fit" and then finding and placing it
S3C: No response

Attitudes about the test
S4A: Hesitant and not systematic
S4B: Overall, solving the test immediately, systematically and confidently.
S4C: No response

It must be noted that we coded only the strategies which were intrinsically related to the level of the thinking that the task 3 was designed to measure. Specifically the task 3 was designed to measure childrens' ability in filling a puzzle considering the angles and the length of sides and also their ability in geometrical transformations (mental rotations).

RESULTS

In this section we present the results of the study, according to the stated research questions, and we provide a discussion of the main findings.

Performance Before and After the Intervention

Children's percentage of success on the posttest (55%) was much higher than their success rate on the pretest (32%). This result suggests that students benefited from the intervention program.

Figure 1 illustrates children's success rates on each task of the test before and after the intervention as well as the gains in children's performance from the pretest to the posttest.

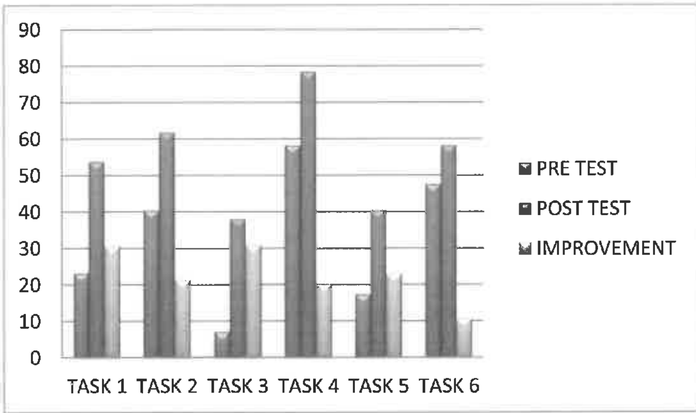

Figure 1. Students' performance in specific task before and after the intervention program.

In all the tasks there are gains in children's success from the pretest to the post test. The greatest differences (more than 30%) were found in the recognition of shapes (task 1) and in the composition of shapes (task 3). Lower gains but similar to each other (21-24%) appeared in the decomposition of shapes (task 2), in the identification of congruent shapes (task 4) and in the classification of rotated shapes (task 5). The lowest gain (11%) was identified in the construction of shapes (task 6). This could be explained by the fact that children didn't receive enough attention in the construction of shapes in the prevention program.

The pattern of the relative difficulty levels of the tasks did not remain invariant before and after the intervention. Identifying congruent shapes (task 4) was the easiest task in both measurements. However, in the pretest, constructing shapes (task 6) was easier than decomposing shapes (task 2), while in the posttest this was not the case; because of the much greater gain in task 2 relatively to task 6, constructing shapes was slightly more difficult than decomposing shapes. Recognizing shapes (task 1) in the pretest was much more difficult for the children than constructing (task 6) or decomposing figures (task 2). Due to the great improvement in children's ability to recognize shapes, in the posttest the difference in the difficulty level between these tasks was significantly decreased. The most difficult task in the pretest was clearly task 3, which involved the composition of figures, but in the posttest, because of children's great progress in this task, this was not so clear. Composing figures became almost as easy as task 5, which asked children to classify rotated shapes.

Structure of the Geometrical Competences Before and After the Intervention

To identify the similarities and congruencies between the various geometry tasks based on children's responses and the variation of these associations before and after the intervention, a similarity analysis was undertaken on the pretest and the posttest data separately, using the statistical software CHIC (Classification Hierarchique, Implicative et Cohesitive) (Gras, Suzuki, Guillet, & Spagnolo, 2008).

Figure 2 shows the similarity relations based on children's responses in the pretest. Three groups were identified by the similarity analysis. The first similarity group involves the identification of rectangles (A2) and triangles (A3) in task 1, two strategies used in the composition of the figure in task 3, namely, turning the shapes into correct orientation prior to placing them (S2B) and searching for the right shape that they "know will fit" and then finding and placing it (S3B) and the construction of a square (F1) and a triangle (F2) in task 6. What is common in all these responses is the use of perceptual apprehension of geometrical shapes. This is clearer in the identification of shapes and the shape composition strategy of turning the figures into the correct orientation before putting them in the right place. However, it appears that children were based on perceptual processes and holistic visual recognition also in giving the other responses, that is, in constructing a shape or in searching for the shape that they assume it fits in a particular place.

The responses of the second similarity group are distinguished into two subgroups. The first subgroup includes the decomposition of shape in task 2 (B3), the composition of shape in task 3 (C), the shape composition strategy of putting shapes in the figure without leaving gaps in task 3 (S1B), identifying squares (E5) and trapezoids (E6) in different orientations in task 5. The responses of the first subgroup involve the mereologic type of modifying (composing and decomposing) a geometric figure and the ability to identify geometric figures in various positions and orientations, namely, the place way of modifying a figure. The second subgroup consists of the recognition of congruent right triangles (D1) and congruent acute triangles (D2) in task 4 and identifying all the rotated acute triangles (E1), right triangles (E2) and rectangles (E3) in task 5. The immediate, systematic and confident completion of the test (S4B) was also included in this subgroup. Comparable to the first subgroup, the responses of the second subgroup involve the application of the place way of modifying different figures as well. In addition they involve the optic way of dealing with a figure, as in recognizing congruent figures, the students had to compare figures with respect to their shape and size, e.g. reject figures that had the same shape but were larger or smaller compared to each other and select the figures that had the same shape and size. Hence, what the responses in the second similarity group as a whole have in common is, in one way or another, the application of the operative apprehension of a geometrical figure. It is noteworthy that the application of this ability is linked to the correct composition of the figure in task 3 and the positive stance towards the test.

The third similarity group involves three shape composition strategies in task 3, placing pieces randomly not connected to each other (S1A), turning shapes after placing them on the outline figure in an attempt to get them fit (S2A) and trying out shapes by picking them seemingly at random, then putting them back if they do not look right (S3A). Two other variables that were included in the third group were not giving a response in task 3 (S3C) and hesitant and not systematic behavior with the test items (S4A). This group is characterized by children's difficulty in putting shapes together to compose a complex figure. This is justified in two ways. First, they used the trial and error strategy rather than systematic strategies based on spatial visualization or on the characteristics of the shapes. Second, this group of strategies is linked to children's inclination not to give a response and to the rather hesitant stance towards the test.

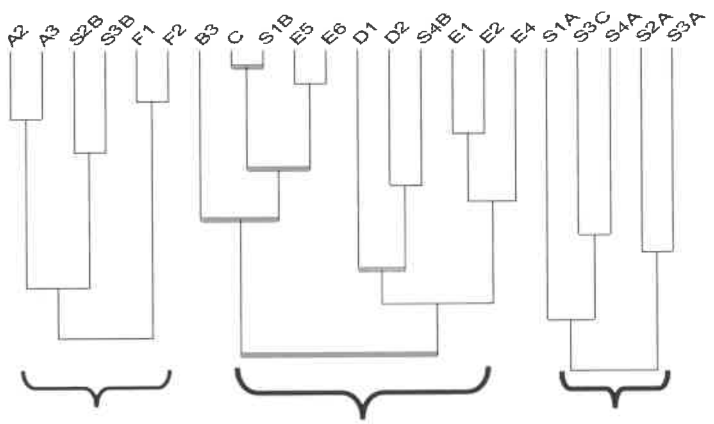

Figure 2. Similarity diagram of students' responses on the pretest.

Carrying out the similarity analysis for the responses of the children in the posttest, revealed that the similarity groups of children's responses in the geometrical tasks differed significantly from the similarity groups identified in the pretest. The results for the posttest are shown in Figure 3.

Three similarity groups were established by the analysis of the posttest data. The first similarity group is comprised of two subgroups which are weakly linked to each other. The first subgroup includes the identification of squares in different orientations in task 1 (MA1), the identification of all the rotated acute triangles (ME1), right triangles (ME2), parallelograms (ME3) and rectangles (ME4) in task 5 and the shape composition strategy of turning the shapes into the correct orientation before placing them on the figure (MS2B) in task 3. This subgroup can be considered as reflecting children's operative understanding of geometrical figures related to the variation in their position and orientation. The second subgroup consists of the trial and error shape composition strategies in task 3, that is, placing pieces randomly not connected to each other (MS1A) and trying out shapes by picking them seemingly at random, then putting them back if they did not look right (MS3A) and the hesitant and not systematic approach towards the test (MS4A). The link between the two subgroups shows that the students who encountered difficulties in the composition of figures have started to treat geometrical figures in an operative way mainly by recognizing the various positions and orientations of a geometrical shape. This was not found in the similarity diagram for the pretest, since the deficient shape composition strategies were not associated to any response including operative apprehension of figures.

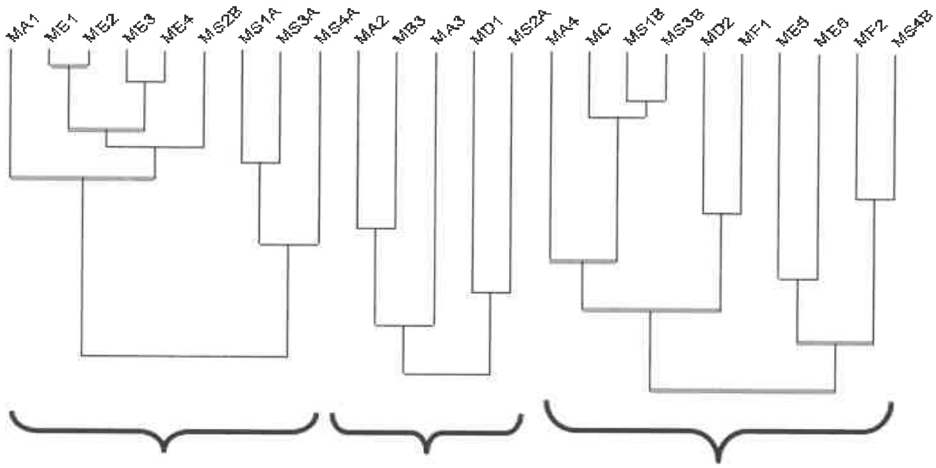

Figure 3. Similarity diagram of students' responses on the posttest.

The second similarity group involves the identification of rectangles in task 1 (MA2), decomposition of shape in task 2 (MB3), identification of triangles in task 1 (MA3), recognition of congruent right triangles in task 4 (MD1) and the shape composition strategy of turning shapes after placing them on the figure in an attempt to get them fit in task 3 (MS2A). In this similarity group links are formed between responses involving perceptual apprehension of a figure and responses requiring mereologic and place types of modifying a figure. This indicates that children's geometrical thinking in the particular tasks was based not only on perceptual processes but also on the mereologic and place aspects of modifying a figure.

The responses that comprise the third similarity group involve the abilities referring to the operative apprehension and the construction of geometrical figures. Particularly, this group consists of responses involving the place way of modifying a figure, that is, identification of all the rectangles (ME4), squares (ME5) and trapezoids (ME6) in different orientations in task 5. It also involves responses that reflect the mereologic aspect of modifying a figure, i.e. correct shape composition of a figure in task 3 (MC), the shape composition strategies of putting together the shapes without leaving gaps (MS1B) and of searching for the right shape that they know will fit, then finding and placing it (MS3B). The recognition of congruent acute triangles (MD2), which involves the optic way of dealing with a figure, is also included. Within the third similarity group, the above operative apprehension variables are linked with the correct construction of a square (MF1) and a triangle (MF2) as well as the immediate, systematic and confident completion of the test (MS4B). These similarity connections suggest that children's ability to construct a figure is not anymore based on the perceptual apprehension of the figure as shown in the similarity results of the pretest, but on the operative apprehension, meaning that they have started to understand the different parts, characteristics and properties of the geometrical figures and they use this understanding in constructing shapes. Furthermore, it is suggested that after the intervention the abilities to treat geometrical figures in an operative way and to construct geometrical figures are interconnected to a positive stance towards the solution of geometrical tasks.

Children's Errors, Strategies and Behavior Before and After the Intervention

Table 2 provides an overview of the mistakes which appeared on the task 4, in which children were asked to recognize the congruent figures, before and after the intervention program.

From Table 2 it seems that a number of children could not understand that geometrical figures can have the same size and shape at the same time. They instead concentrated on only one attribute, either size or shape, and ignored the other one in comparing geometrical figures. This deficiency may explain the response of a small proportion of children who matched a big trapezoid to a big square (17%) or a small trapezoid to a small square (12%) in the pretest. Furthermore, a greater proportion of children could not recognize that geometric figures can be similar or belong to the same shape class, but have different proportions and measures. Particularly 31% of the children in the pretest matched squares which did not have the same measures, while the same pattern of mistake appeared on acute triangles (26% and 10%) and on trapezoids (24%). A closer look at the data revealed that students benefited from the instructional program, as these mistakes were less frequently found.

Table 2
Mistakes in Pretest and Posttest on Task 4

	Match big trapezoid to big square	Match small trapezoid to small square	Match small acute triangle to big acute triangle	Match acute triangle to right triangle	Match squares with different measures	Match trapezoids with different measures
Pre Test	17%	12%	26%	10%	31%	24%
Post Test	10%	7%	7%	2%	19%	12%

In the task that required shape composition of a figure (task 3) children used a variety of strategies both in the pretest and the posttest. The percentages of each strategy use (Table 3) support the scored results and provide additional description of the group before and after the intervention course. In the posttest the children were more likely to combine shapes without leaving gaps (43%), turn shapes into correct orientation after placing them on the puzzle (81%) and search for the correct shape (29%) compared to the pretest. This increased use of more sophisticated shape composition strategies suggests the development of mental imagery, which could be a result of children's involvement in activities with the Shape Puzzle.

Concerning children's overall behavior towards the test, the proportion of children who solved the test immediately, systematically and confidently increased from the pretest (45%) to the posttest (62%), indicating the positive effect of the intervention also on children's attitudes about geometry and geometrical tasks.

Table 3
Percentages of Children's Strategy Use Before and After the Intervention Program

	Pre Test	Post Test
Placing pieces randomly on task 3 that are not	81%	52%
Put shapes on the task 3 together without leaving gap	14%	43%
No response	5%	5%
Turning shapes after placing them on the task 3 in an	71%	81%
Turning them into correct orientation prior to placing	24%	12%
No response	5%	7%
Trying out shapes by picking them seemingly at	62%	62%
Appearing to search for just the right shape that they "know will fit" and then finding and placing it	24%	29%
No response	14%	7%

DISCUSSION

The present research aimed to investigate the benefits of utilizing Building Blocks program on kindergartners' geometrical abilities and confidence. The study revealed that the program was sufficient to enhance children's performance. Specifically, the use of the program was found to increase children's learning outcomes on the topics of shape identification, composition and decomposition of shapes, congruence and construction of shapes. This result is in accord with previous research which found that Building Blocks curriculum seems to have made a special contribution, with quite large relative gains to children's learning of the topics of shape identification and composition of shapes (Clements & Sarama, 2007).

The results of the similarity analysis showed that the intervention program triggered the operative apprehension abilities of the children, that is, the mereologic, optic and place ways of modifying geometrical figures, such that they supported the solution of various geometrical problems requiring different geometrical competences, i.e. recognition (perceptual apprehension) and construction of geometrical shapes (sequential apprehension). This phenomenon was not found in the structure of children's geometrical competences before the intervention. An explanation that can be given for this positive effect of the intervention program is that the emphasis of the computer activities on different types of figural processing (shape composition, shape comparison, shape recognition in various orientations etc) helped children to move beyond the visual perception of geometrical figures and use the different types of operating on a figure (operative apprehension). Thus, on the one hand, as Duval (1995, 1999) argues, operating on a figure in an optic, mereologic or place way could have served as a valuable heuristic method for the children in solving different geometry problems. On the other hand, it is likely that the emphasis on figural processing enabled children to develop a deeper understanding of the various components, characteristics and properties of shapes, which in turn provided support to other types of geometrical figure understanding, such as the construction of figures. These findings are in line with previous research suggesting that the use of technology in mathematics teaching supports and facilitates conceptual development, exploration, reasoning and problem solving (Papert, 1996). It is also noteworthy that the operative way of treating a geometrical figure was associated with children's positive attitudes and confidence with respect to the solution of the test both before and after the intervention. This relationship indicates that children's ability to operate on geometrical figures in various ways which go beyond visual perception, can endorse their confidence and self-regulatory strategies in geometrical tasks.

With respect to children's strategies, the results showed that the intervention program had a positive effect in the use of more sophisticated strategies and the development of spatial imagery. Children's participation in computer activities involving the combination of shapes to construct outline puzzles such as Shape Puzzles could provide an interpretation for this improvement. Furthermore, the findings showed an improvement in students' conceptions of congruency between shapes. Before the intervention program, a significant proportion of children were identifying similar shapes as congruent. A smaller proportion focused only on the size and not on the shape to identify congruent figures. A considerable decrease of these mistakes was found after the intervention. These findings could be explained by the involvement of the children in computer activities in the Mystery Toys and Memory Geometry that required matching of congruent shapes. In general, it can be deduced that the Building Blocks activities linked children's informal knowledge to more formal school mathematics and were quite efficient in helping preschoolers learn fundamental mathematics concepts and skills (Clements & Sarama, 2007).

The results indicated that providing students with the opportunity to engage in Building Blocks activities is one possible way to enhance children's self-representation about the self-regulatory strategies they use in geometry. It seems that the program created a powerful learning environment in which students were inspired in their own experiences. The findings showed that most of the students after the intervention program dealt with the test immediately, systematically and confidently. This result is in accord with previous research, which found that the use of technology enabled students to engage in active learning strategies, verify conjectures, have positive attitudes, and build confidence in their ability to do mathematics (Huang & Waxman, 1996; Lawrenz, Grayelu & Ooms, 2006).

To place the findings of the present study in the right perspective, it is necessary to take into account the fact that our study was carried out without a comparison group and with a limited number of students and that the geometry performance was measured with a small number of tasks covering to a limited extent the different geometric competences. It is important to design methods that cancel out the limitations of this study. Further research on the contribution of the use of computer software on the development of children's geometrical understanding could be carried out with larger and different samples of learners (high and low achievers), including a control and an experimental group and using a richer collection of items for each geometric competence.

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