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Athanasios Gagatsis

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The First Crucial Point in Geometry Learning: Visualization

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ABSTRACT: *There is a wide discrepancy between the teaching of geometry and the constant troubles of students with figures and visualization for solving problems. The teaching of geometry is organized as if concepts would guide perception for "reading" figures. But students always confine themselves to the only perceptive recognition of obvious forms. This cognitive gap between perception and concepts raises two reverse questions, from an epistemological and a cognitive point of view. How can geometrical properties and objects be visualized through figures, even if it is a single "drawing"? How can visualization be a powerful heuristic way for seeing how to solve problems, without any reasoning from given properties? In this paper we analyze the cognitive processes that underlie the mathematical way to see figures in geometry. They are independent of any concept acquisition or perceptual recognition of shapes. They consist in the dimensional deconstruction of all shapes 2D or 3D. To describe them we introduce the notion of figural unit characterized by the number of its dimensions. And we show how and why the relationship between visualization and language is done at the level of figural units and not at that of figures. Finally we explain why we cannot move directly from reality to geometry or apply directly any geometrical knowledge to solve real problems. An intermediary diagram is required for being able to connect a real situation to a geometric figure.*

Key words: *Visualization, Shape recognition, Object recognition, Figural unit, Dimensional deconstruction, Figure, Configuration, Reconfiguration, Qualitative calculation, Language, Reasoning, Problem solving, Diagrammatic model.*

INTRODUCTION

Since the seventies, big changes have occurred in the objectives of the teaching of geometry and issues about it. We can distinguish three main trends that have become prevailing in turn. From 1970 to 1995 objectives and issues were mainly focused on proof. At first it was within an axiomatic and logical framework, but soon issues have changed towards an approach that emphasized on arguing and the students' interactions. From 1990 to 2000, with the increasing importance of new technologies, the "dynamic geometry" has been extensively developed. The widespread use of Cabri Geometry is a typical example. Nowadays, with the globalization of Education policy and assessment the objectives are the acquisition of a more "practical geometry" instead of a "theoretical geometry" that seems to discourage most of the students. But the students' troubles with figures and visualization for solving problems had never been considere(and yet, what the students can do or understand in geometry depends entirely on wh: they are able to recognize at the first glance on a figure.

Figures in geometry are paradoxically used as perceptual objects and at the same tim as visual representations of ideal objects, which however do not look like them. A perceptual objects they can be constructed with specific tools (software, rule compasses), but they can also be empirically measured, changed or matched up wit material shapes. As visual representations they refer to properties that can only b indicated by words or coding. In fact, this double use requires two quite different way of looking at figures and mainly recognizing what they stand for. The one is th spontaneous perceptual recognition of what is seen as for any visual representation c material objects or spatial organizations (images, diagrams, plans, maps, etc.). The othe is the mathematical requirement of confining this recognition in the frame set by th given properties, from which other properties can only be inferred to solve a problem c apply geometrical knowledge to real situations. This paradoxical double practice raise two important issues for the organization of the teaching of geometry. Are these tw ways of looking at figures mutually exclusive? How can students learn to move fror one way to the other? These issues can be considered from two points of view.

From the mathematical point of view, a mutual exclusion would not be really problem, since the mathematical way of seeing must prevail over the perceptual one Anyway, the given properties are always associated to the figure in the form of words c integrated into it by coding. And in mathematics education it is assumed that th construction of a figure and the use of computer would allow overcoming the cognitiv conflict, because in these activities any figure has to be seen according to th geometrical properties that make its construction with specific tools possible. Thus, th recurrent students' difficulties, when they face any given figure to solve a problem would be caused by a lack of concept acquisition or some misleading perception. On question though remains unresolved. If the importance of reasoning in the way of seein; the figures is mainly put forward, how can we analyze the heuristic role of visualizatio: in searching a geometric solution to a problem?

From a cognitive point of view, on the contrary, perception and mathematical way o seeing are mutually exclusive. On the one hand, the perception is an obstacle that make incomprehensible the mathematical way of looking at figures. On the other hand th mathematical way of seeing at figures according to the given properties amounts to ge rid of visualization or to reduce it to reasoning and mathematical language. The rea problem about visualization is the opposite question, which is at the same tim cognitive and epistemological: How can geometrical properties and objects be showe through a figure, even if it is reduced to a single drawing? This question is far fron being theoretical. This is the prior issue for any basic geometry education. It directl concerns the cause for the recurring students' difficulties throughout the curriculum when they face the figures. It allows identifying the decisive factors to take up th challenge of a basic education in geometry: How can all students learn to see figure mathematically and not just perceptually for becoming able to solve problems b *themselves*? Here for a student "seeing" means to see with his own eyes and not through the eyes of the teacher or of other students.

In this paper an overview of the cognitive analysis of the mathematical way to see figures is presented. It is based on four key ideas. (1) The processes of visualization are cognitively independent of any concept acquisition or perceptual recognition of shapes, that are typical for representing basic geometric objects, and even of their construction procedure. (2) To analyze these processes, it is necessary to introduce the notion of figural unity. (3) The number of dimensions characterizes figural unit. The visualization focuses on the recognition of all the possible figural units in a given figure and not their magnitudes. (4) It involves the dimensional deconstruction of the forms that can be perceptually recognized and therefore goes against perception. From a mathematical point of view this means that we must separate the affine properties from any estimation or magnitude calculation.

In the first part, preliminary distinctions about what is called a figure and the interactions between the shape recognition and the object recognition were introduced. In the second part, the two key ideas of figural unit and visualization as dimensional deconstruction of shapes is explained. The confusion that always occurs under the word "figure" between shapes, basic geometrical figure and configuration is also highlighted. In the third part, it is shown that the coordination between visualization, language and reasoning requires the dimensional deconstruction of shapes. And in particular, we must not confuse this cognitive coordination with the mathematical association between any configuration and some properties given in the context of a problem to be solved. In the fourth part, it is shown that we cannot move directly from reality to geometry or apply directly any geometrical knowledge to solve real problems. We must produce an intermediary diagram. Finally some perspectives for a different approach of teaching geometry for all the students up to fifteen years are presented.

I. TWO PRELIMINARY DISTINCTIONS ABOUT FIGURES AND VISUALIZATION OF GEOMETRICAL OBJECTS AND PROPERTIES

1.1 What is Called a "Figure" in Geometry Merges Two Things and it Overwrites a Third by Coding

First of all, a figure in geometry is a visual representation, which is constructed, or can be constructed, with specific tools (ruler and compasses, software, etc.) The construction process depends on the primitives of the tools that are used. These primitives can be either affine or metric properties, or some basic geometrical figure such a circle. The instructional sequence of construction changes according to the tools that are used[1].

[1] Duval R. Godin M. (2005). Les changements de regard nécessaires sur les figures. Grand N 76, 7-27.

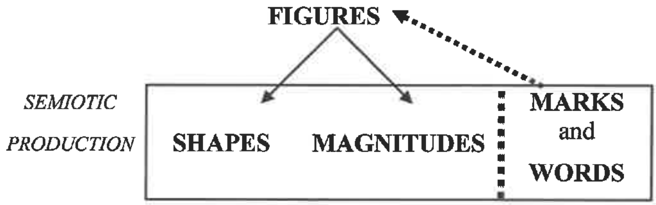


Figure 1. The three faces of a figure in geometry.

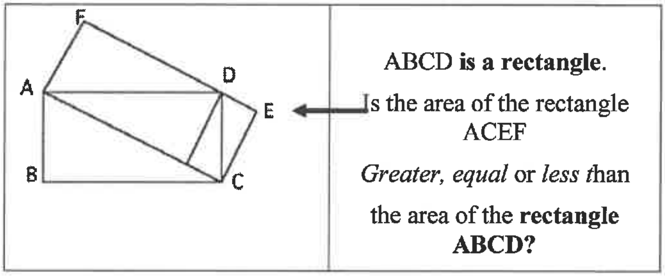


Figure 2. Kind of classical problems in elementary geometry.

That means a figure merges semiotic representations depending on three differen registers of semiotic representation. And each register makes *the mind able to develo and run a specific cognitive activity and only one, independently of the others.*

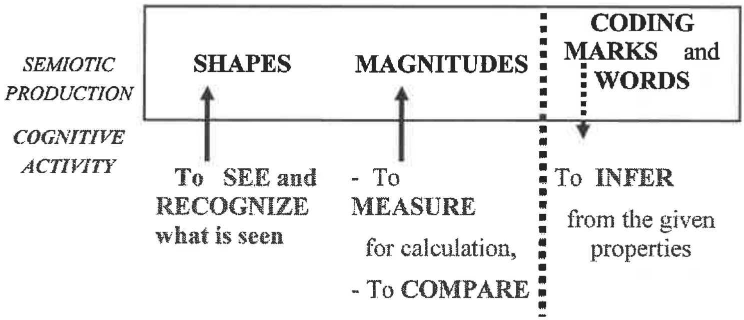


Figure 3. The three kinds of cognitive activity required in geometry problem solving.

Visualization in geometry is specifically related to the cognitive activity of SEEIN SHAPES within a figure and not of estimating or measuring their magnitudes. In order t analyze visualization in geometry and the everlasting conflict between the perceptua way and the mathematical way of looking at figures, we must first focus only on seein independently from any magnitude consideration, from given properties and from thei construction process. What matters is what is seen and what is not seen in the figure, anyway the given properties or the construction process, of which the result is the watched figure.

1.2 Seeing as Double Recognition at a Glance

Seeing is recognizing at a glance what appears from sensory perception or what an image represents. In fact, seeing involves two levels of recognition:

— ***Shape*** recognition

— ***Object*** recognition.

Hence the first question is whether this double recognition runs, or does not, in the same way for the figures in geometry, which are figures constructed with specific tools, and for pictures, maps, graphics, diagrams, etc outside geometry. This issue is crucial for a real learning.

1.2.1 Out of Mathematics Shape and Object Recognition Merges Into One at First Glance

We must distinguish two processes of recognition corresponding respectively to iconic recognition and to non-iconic recognition.

The iconic recognition is the spontaneous way, the one that is always predominant: the visual representation *looks like* some real objects or some model. The recognition process is on the fact that the *topological relations* between the parts of the whole configuration are the same as the ones between the parts of the object represented. So there are degrees in iconicity of images according to whether the parts are iconic or not: sketch, cartoon, painting, maps, and diagrams. So the similarity to real objects is not based on comparison or association processes but on the conservation of topological relations (Figure 4, right column).

On the contrary, the non-iconic recognition of shapes is intrinsic, without any reference to real object or memory from the outline of any real object previously seen. Focusing on the perceptual recognition of shapes constructed with tools, the Geltalttheory has shown how any set of lines or of points are automatically organized into shapes according to *the principles of closure, symmetry and economy.* It is the reason why the non-iconic recognition of shapes is subjected to visual illusions that always remain the same for adults having geometric knowledge. So a single shape or several ones stand out against a background set of lines or points. In the example below, even if there are five other shapes possible according the principle of closure, only two are recognized at a glance according to the two other principles (Figure 4, left column). These two symmetrical shapes are also recognized as objects.

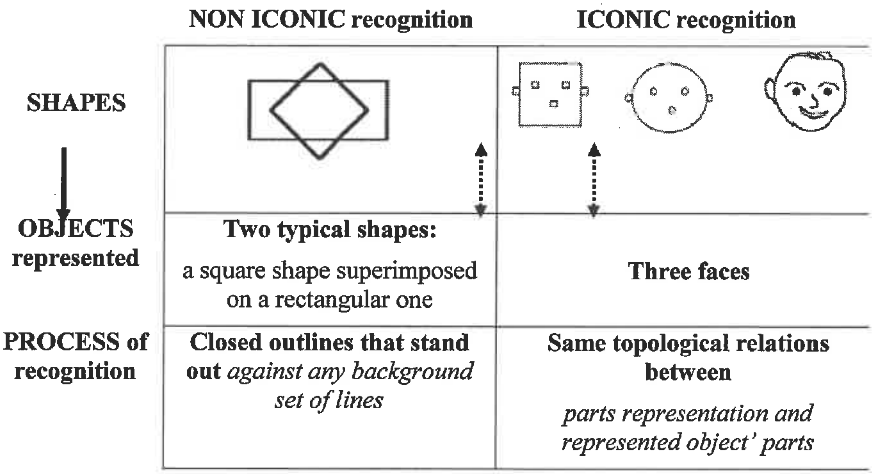
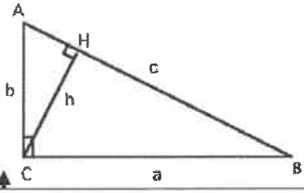


Figure 4. Two kinds of perceptual recognition of objects represented outside geometry

1.2.2 In Geometry, the Two Levels of Recognition Have to be Reversed

In Geometry the two levels of visual recognition must be disconnected in order to solv problems. The mathematical recognition of objects represented in figures depends onl; on the given properties and not on the perceptual recognition of shapes. One has to star from the given properties in order to recognize the relevant shapes for solving th problem.

<table>
<tr>
<td>Non iconic recognition of the shapes is the same as for the spontaneous perceptual way

On right :one triangular divided into two other)</td>
<td>
</td>
</tr>
<tr>
<td>Recognition of the objects represented is only from properties given by words and coding marks:

square opposite: representations from two discursive registers</td>
<td>The problem wording

two rectangle angles marked on the figure and the tree following equalities:

AC = b, BC = b, and AB= c

Show that $c^2 = a^2 + b^2$</td>
</tr>
</table>

Figure 5. The required order of recognition of objects represented by a figure.

This disconnection between the two levels of recognition and the reversal of their priority can easily be explained from the mathematical point of view. Properties (parallelism, equality, symmetry, etc.) *can never be inferred from a perceptual estimation nor from physical measurements.* Perceptual estimation is very often deceitful, as the well-known visual illusion experiments have showed it, and there are thresholds in the perceptual estimation of magnitudes. About physical measurements, there is always a margin of error dependent on the tool used, and students cannot measure with rulers beyond the first decimal place. Perceptual recognition is misleading for the recognition of geometrical properties and, therefore, for the recognition of the geometrical objects that are represented.

This mathematical requirement in the way of seeing figures in geometry creates a strong cognitive conflict for learning and becoming able to solve problems, because it runs against or off the spontaneous and predominant perceptual recognition of shapes and objects. And this cognitive conflict is the first challenge for teaching geometry. How to make students reverse the spontaneous order of priority between the two levels of visual recognition? Words can do nothing against any obvious perception at first glance. And, anyway the wording of problems, this perceptual recognition of shape/objects keeps steadily predominant over the curriculum for most students. Every teacher knows that.

On the contrary, this mathematical requirement in the way of seeing figures raises a paradox about the heuristic role of figures for solving problems. If they must be seen from the properties necessarily given through words or symbolic notations, how can looking at a figure given or constructed be useful for finding out the idea of the mathematical solution? Here we cope with another cognitive conflict, the one between reasoning and seeing. Reasoning in a mathematically relevant way from the given properties is *a linear process,* which is carried out step by step. Visualization is the *immediate grasp of a configuration* as a whole. The heuristic use of figures is based on the range of the various configurations that can be immediately recognized within given figure. But most of the students cannot go beyond the perceptual recognition («shapes-objects». And, even at a higher level of curriculum, reasoning and seeing a not linked at all. So figures do not provide anything for solving any problem.

1.3 Misunderstanding in Geometry Teaching and Educational Research abo Figures and Visualization

There are two blind spots in the organization of teaching. One concerns the relatio between visualization and figures. The other concerns the relation between figures an concepts. In both cases, activities about shapes and those about magnitudes are neith distinguished nor separated as quite different tasks for learning and understanding. O the contrary, they are always intertwined.

So tasks about figures are introduced without separating the recognition of shapes fro the estimation or measurement of the magnitudes of their sides. Figures are considere as a visual whole given to perception. Seeing means perceiving both shapes and the magnitudes as if it would depend on the same cognitive activity and not two quit different (above, Figure 3).

In this way, affine properties (parallelism, intersection), which are only related t shapes, and metric properties that are related to magnitudes (equality of magnitudes) a not conceptually separated in the organization of learning situations. The problem designed for enabling students to acquire knowledge confuse them in the given that a needed in order to know mathematically what a figure represents. So two kinds of task are asked, which do not single out the same kind of cognitive activity:

— Qualitative comparison tasks about lengths or areas related to triangle parallelograms, or circles. These tasks require that students infer in natur language or substitute equalities each other, only by using the given propertie without using any numerical values (above, Figure 2; Figure 5, col. 2).

— Numerical calculation tasks about lengths or areas. These tasks require that student think of a formula or a specific relation between magnitudes for calculating a unknown numerical value from numerical given data. When they refer to concret situations or to figures, numerical data are closely associated to empiric measurements made with tools.

These two kinds of tasks involve that students could recognize the relevant shape according to the given properties or formulas to be used. This visual recognition at glance is the first cognitive condition for solving problems in elementary geometr Without such recognition students are unable to solve the classical problems of area comparison as in the above problem (Figure 2). They measure the sides with a ruler t calculate and conclude that the area of the slant parallelogram is greater than th horizontal parallelogram or they give up. Solving geometric problems about relation between magnitudes does not depend primarily on the difficulty of the mathematic properties to be used, but on the cognitive complexity of visualization that is involved.

The opposition between the "drawing", and the "figure" is often rehearsed in mathematics education as the key distinction. The "drawing", which is always particular, is the given visual whole to be seen. The "figure" corresponds to the given properties, anyway the possible variations of the drawing. This opposition is explained by the following fact: geometrical properties correspond to what remains invariant when the drawing is changed, by moving either one of its point or its segment. This opposition, which means nothing else than the mathematical process of object recognition (above, Figure 5), is symptomatic of the double misunderstanding about visualization. So the drawing perception merges the processes of shape recognition and those of magnitude estimation or measurement. And the figure merges the affine properties and the metric ones, without taking into account the gap between their conceptualization processes. *Within the framework of this epistemological or mathematical opposition, only two types of activities can be taken into account for analyzing the cognitive process of problem solving*: either perception of the drawing, but is often misleading and does not really help the student's visual investigation, or inferences from the affine and/or metric properties given by the hypotheses, but there are no more links with the drawn figure.

That is why the cognitive gap between perception and concepts raises *two reverse questions*, from an epistemological and a cognitive point of view:

— How can geometrical properties and objects be visualized through figures, even if it is a single "drawing"?

— How can visualization be a powerful heuristic way for seeing how to solve problems, without any reasoning from given properties?

II. THE MATHEMATICAL WAY OF SEEING FIGURES: DIMENSIONAL DECONSTRUCTION OF SHAPES

2.1 Figural Units, Basic Geometrical Figures and Configurations

In order to analyze what separates the mathematical way of looking at a figure from the perceptual one, we must introduce the key notion of FIGURAL UNIT, by distinguishing three things: shapes, basic geometrical figure, and figural unit.

The *shapes* are closed outlines that stand out against a set of plotted or colored elements and that we recognized at first glance as a whole, either a surface or a solid (above, Figure 4).

The *basic geometrical figures* are typical shapes that can be constructed with specific tools (ruler, compasses or software) and which have some notable intrinsic properties such as symmetry, equality or inequality of magnitude, parallelism, orthogonality and so on: circle, square, triangles (equilateral, isosceles, right-angle), parallelogram, rhombus, etc. When introduced at the primary school, their properties are the early contents of the teaching of geometry! Apart from their direct use through a formula for numerical calculation tasks, no problem in geometry can be asked with a single basi geometric shape. *Problems require the spatial combination of at least two basic figure.* In the simple cases two basic figures can be spatially combined in four possible way: one is inscribed in the other, for example a square inscribed in a circle, or the reversa or they overlap each other, or one is the constituent part of the other, like the triangle i comparison to parallelogram. All the problems in geometry are visualized by *configuration* of several basic geometrical figures, which in some cases they may be th same (above, Figure 5). *So we must never confuse the basic figures used in definitior and the configurations given with problems.* There is a well-known cognitive ga between the knowledge of the basic geometrical shapes and their recognition in configuration.

The *figural units* are *all the elements,* which can be visually discriminated in an constructed figure. These elements are not characterized by their shape but *by th number of their dimension.* Thus four kinds of figural units have to be distinguished:

— **1D**/2D, i.e. any line straight or curved,

— **2D**/2D, i.e. any surface wholly or partly closed,

— **3D**/2D, i.e. any configuration of lines that appears in perspective,

— **0D**/2D, i.e. any point recognized as intersect point or a vertex. These notable point are the only 0D figural units. All dots that are added as a mark on a line or outsid belong to the discursive coding of figure. They are not figural units.

For example, in the configuration below (Figure 6), 43 different figural units can b distinguished, even if very few are perceptually recognized apart from the two obviou regular polygons. These 43 figural units are like 43 relevant possibilities of visual focus

In fact, the recognition of figural units 2D or shapes, stops from seeing figural units 1I or the underlying network of straight lines.

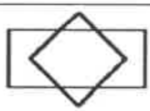
	Figural units **2D**/2D: SHAPES	Figural units **1D**/2D	Figural units **0D**/2D
	**2 regular polygons** (*square, rectangle*) and **5 polygonal shapes** (*2 triangles, 2 pentagons, 1 hexagon*)	**16 segments** or **a network of 8 underlying** *straight lines*	**12 points** *intersection or vertices*

Figure 6. Kinds of figural units whose visual recognitions are mutually exclusive.

Shapes, basic geometrical figures and figural units are almost always confused becaus the «figures» or the « drawings» are considered *in relation to the represented concept* Research and teaching focus only on shapes and basic figures and figural units ar confused with both of them. The various dimensional figural units are always confused under the word «figure».

In the notation nD/nD, the denominator mentions the material support of the figural units. The denominator 2D mentions that all the figural units are semiotic representations on a plane support, either paper or screen, whatever their dimension is. The denominator 3D mentions physical objects. A tight thread and a laser beam are 1D/3D, a thin sheet of paper or the surface of water are 2D/3D, any solid 3D/3D and the point of compasses or a star in the sky are 0D/3D. This notation is very important from an educational point of view. It is with physical objects that we can make concrete operations which allow discovering or «checking» geometric properties. For example, the support of a sheet of paper (2D/**3D**) can be used to introduce the concept of axis of symmetry of any shape or configuration (**2D**/2D). Three concrete operations are mathematically relevant: folding a sheet or return a transparent and superpose it on the sheet for checking the coincidence of the outline of the configuration **2D**/3D with itself. The mathematical difficulty arises when you have to move from the concrete operations with 2D/**3D** physical objects to the semiotic representation **2D**/2D, because the operations for checking and explaining are not at all the same (Duval, 2012). The same cognitive jump was noticed for moving from regular solids 3D/**3D** to their semiotic representations **3D**/2D (Rommevaux, 1998).

2.2 Visualization and Dimensional Deconstruction of Shapes Recognized

Seeing a figure in a mathematical way consists of *discriminating (quickly or at a glance)* all its dimensional figural units possible, anyway the given properties, and not only recognizing shapes or basic figures! The reason is that *a mathematical property cannot be visualized by a single figural unit, but only by the visual relation between two figural units*, from the same number of dimensions, or not. For example, there are several possible definitions of a parallelogram (**2D**/2D), but they involve taking into account properties that each of them refers to a relation between two units **0D** or **1D** /2D, i.e. on *figural units of smaller dimension than what is defined.* I have called dimensional deconstruction of shapes this change of dimension in the focus on any shape recognized or configuration given (Duval, 2005).

The dimensional deconstruction of shapes goes against the spontaneous perceptual way of seeing any visual representation outside mathematics, and especially against the perceptual merging of smaller units 1D or 0D into simple edges or vertices of surfaces 2D, or solids 3D. It is not only conceptual at all, but it is required for understanding and using any geometric concept. It is neither concrete, nor mental. So it is the first cognitive level in the development of knowledge in geometry.

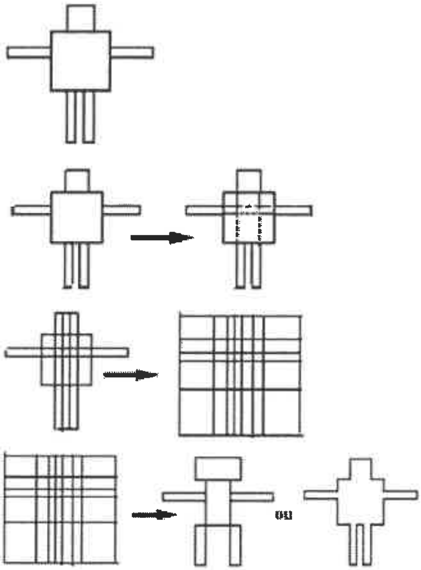
The discrimination of figural units 1D /2D and D0/D2 in shapes 2D/2D requires breaking all the close outlines of the shapes. In fact, this dimensional deconstruction can be described by the three following operations:

— *To extend all their sides for making the underlying straight-line network appear.*

— *Then to make new straight lines appear between the intersection points.*

— *From this wholly developed underlying network we can recognize a lot of quit different basic figures or configurations and see how they can be transformec one into another.*

Here is an example chosen off mathematics to illustrate this process and its cognitiv power.



Non-iconic recognition of shapes (2D) or iconic recognition of a man.

Extension of the hidden sides under the figural units 2D superimposed.

Methodical extension into the **underlying network** of straight lines.

Recognition of other outlines *included in this underlying network*

Figure 7. Illustration of the cognitive process of dimensional deconstruction.

These three visual operations must *become automatic* for the students, like th recognition of letters, words, syntagms and sentences for reading. In other words, the must be carried out at first glance. Visualization lies on the dimensional deconstructio of shapes (and next in the reverse process of integration in higher dimensions). S watching a given figure can immediately and freely come and go between the differen possible units 2D, 1D and 0D and discriminate the relevant figural units in the contex of a problem.

In figure 8 there is a configuration presented as an image on the cover of a schoo notebook in China. Colors enhance the perception of the figural units 2D. You can as any question about the nature of the triangle on the background of the two circles. It wil be a simple problem of dimensional deconstruction of shapes. From an educationa point of view its main interest is: How long does it take to recognize the relevant 1E figural units?

How long does it take for seeing why the triangle ΓAB (*below*) is equilateral: 3 seconds, 10 seconds, 1 minute, 10 minutes, or failure?

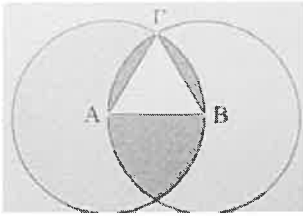


Figure 8. Test of visualization.

The cognitive condition for seeing figures in a mathematical way: is *being able to deconstruct dimensionally* the shapes recognized.

2.3 Visualization and Problems in Elementary Plane Geometry

All the basic figures and configurations have to be seen as particular closed parts of a network of lines. Thus there are two opposite ways but both equally necessary to initiate and develop the cognitive process of visualization: to deconstruct any basic geometrical figure into its underlying network of lines and to recognize the various possible configurations.

Perceptual recognition makes at once some closed outlines obvious and steady. It blocks out any recognition of all the others possible. It is therefore necessary to teach students to focus primarily on the lines and the intersection points in specific tasks of dimensional deconstruction.

Task of inscription of a quadrilateral without any measurement: starting and target configuration.

1.To break the closed outline.

2. To draw all the lines between the given and the new points of intersection.

3. To recognize the closed outlines of the starting configuration in the network amongst several others possible.

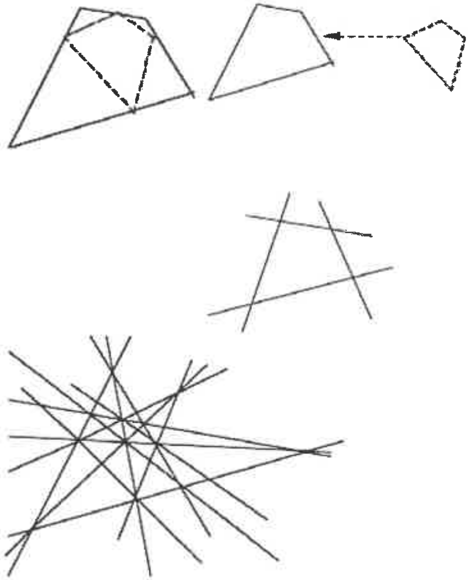
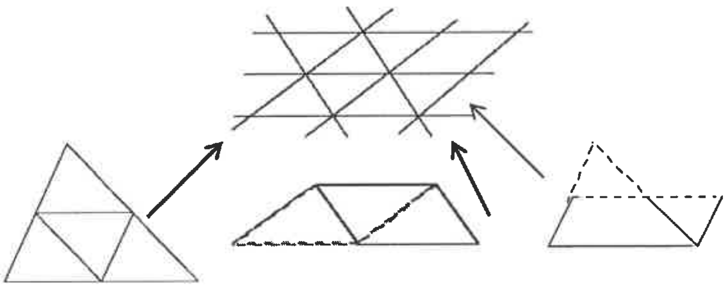


Figure 9. From shapes and configurations to the underlying network of lines.

Recognition tasks are complementary tasks. They do not correspond directly to mathematical activity. However, they are involved in any problem solving since network of lines is a way to generate and solve a lot of problems that can appear on th surface very differently. Take for example the network of straight lines underlying triangle or a parallelogram. In the Figure 10 there are three problems with classica configurations:



According to the given properties (parallelism, parallelogram, the middle of a side) different problems can be asked, involving the same kind of underlying network of lines.

How to cut a triangle in one line to get a parallelogram?

Figure 10. The network of underlying geometrical basic figures as a field of visualization for many problems.

In an organization of teaching which does not take into account the cognitive development of visualization, most of the students cannot recognize the same configuration or subconfiguration if their orientation of the figure that was reversed is right up down or left (Duval 2005).

2.4 Visualization and Qualitative Heuristic Exploration of Configurations

Apart from dimensional deconstruction, there is also a heuristic use of configurations for solving some kinds of problems, mainly those about qualitative comparisons of lengths or areas. The visual process is based on successive transformations of the figural units 2D recognized into other figural units of the same dimension. Unlike the dimensional deconstruction of shapes, this operation can be carried out by using material pieces (2D/3D) as in a jigsaw game. For example, figural units 2D can be moved in the plane or be turned upside down in space. It allows solving problems without explicitly referring to properties and reasoning. There is only a qualitative *calculation on 2D/2D figural units under the assumption of their conservation throughout reconfiguration processes* (Duval, 2005). I called it «operative apprehension of figures ». However, reconfiguration simultaneously uses and goes against the perceptual recognition of shapes as the classical problem mentioned above (Figure 2) shows it.

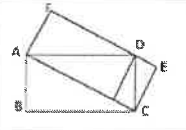
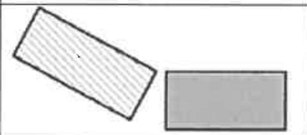
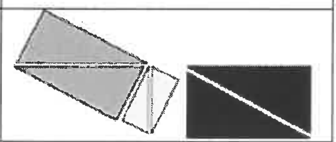
	One figural unit **D2**/D2 *overlaps another*	**6** figural units **D2**/D2 are *juxtaposed*
		

Figure 11. Two visually exclusive ways of recognizing shapes in a configuration.

When there are several possible closed outlines, shapes can be recognized either a *superimposed* shapes or as *juxtaposed* shapes. But both of them are visually exclusive Only one stands out at a glance and it blocks the other. Thus in the problem above th 12-14 years all students have seen by superposition and recognized only the two bi rectangles. So they were reduced in making measurements and they were mistaken.

In this problem, the reconfiguration of these six figural units into quite differen subconfigurations is a qualitative calculation:

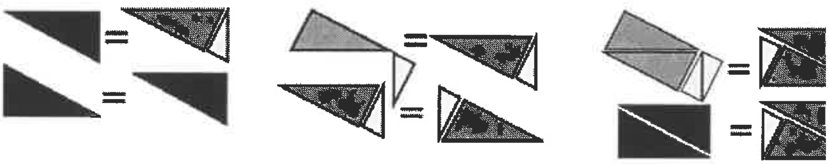


Figure 12. Figural treatment as a qualitative calculation.

We have identified the *cognitive variables that help to see heuristically a solution o prevent from seeing it* (Duval 1995). These cognitive variables can be used as didactica variables for organizing activities in order to develop the heuristic ability.

III. VISUALIZATION, LANGUAGE AND REASONING

Until now, we have highlighted the importance of visualization for understanding an learning in geometry. And we have described the specific way of seeing which shoul be developed as a priority for introducing geometry. This focus on visualization raise the question always said as a refutation: And what about concepts, since we cannot d any geometry without using properties? This recurrent reaction is symptomatic of th prevailing approach in mathematics education. It bypasses the visualization just becaus a "figure" has to be seen from the given properties to discriminate what is relevant fo solving a problem. *Thus the teaching of geometry is organized as if concepts woul guide perception for "reading" figures.* And shapes, objects and words are directl joined together by association as in the perceptual way of seeing, where it run spontaneously for recognizing what pictures, diagrams and maps represent.

The problem is that the use of properties for solving problems can never be carried out without any language production, both explicit and implicit. In fact, the trouble is about the coordination between language with visualization. It is much more complex in geometry than in the other areas of knowledge, because words cannot be associated with figures constructible with tools, as they are with everything that is perceived, either real objects or their images.

Three specific and unusual kinds of language production are required in geometry. They run in opposite ways:

(1) *Definitions* of properties and objects, which can be visualized by basic geometrical figures, and *theorems*, which can be visualized either by a basic figure or by a configuration.

(2) *Descriptions* of an either basic figures or configuration productions. These are mainly asked in the context of teaching at primary or middle school.

(3) *Local valid inferences* from given properties for proving conjectures or for justifying an answer.

The two first kinds of production are naturally closely related to visualization through WORDS. The third lies in deducing new PROPOSITIONS. And deducing *is completely independent of* any figure and visualization. We will consider here only how language and visualization are joined together in geometrical activity.

3.1. Words in Geometry are Related to the Dimensional Deconstruction of Shapes

When one considers the knowledge to be taught throughout the curriculum, even since the primary school, two observations are striking.

Firstly there are *many technical words for very few basic figures*! And these words are *condensations of definitions*. When they are used in a designative or descriptive way, it is not needed to explicit to oneself or to others their definition, i.e. their characteristic conditions of application. But it becomes necessary when words are used to infer or justify.

Secondly there are *two kinds of technical words according the way they refer to figural units:*

— Those that refer directly to a figural unit either 2D, 1D, or 0D. They are associated to each figural unit as a name distinguishing it from others. Visual recognition is sufficient.

— Those that refer to properties or mathematical objects. Here their connection with the figural units is more complex. Properties do not refer to one figural unit, but to the visual relation between TWO figural units 1D or 0D. Objects refer to configurations that require discrimination of figural units 0D, 1D and 2D. Here there is no more possible association. Properties must be either given or inferred from other given. Recognition of a visual relation between two figural units 1D and/or 2D is never sufficient in itself to assert a property.

Thus we get the following classification of the technical words by taking into accou both the dimension of figural units and the way technical words refer to the figur units. Words about magnitudes are not included in this classification since magnitude and quantities are not a matter of visualization.

	OBJECTS	PROPERTIES: relation between TWO figural units 1D or 2D — ***connected to some** figural unit 2D*	PROPERTIES: relation between TWO figural units 1D or 2D — ***independent from** any figural unit 2D*
FIGURAL UNITS 2D **glaringly obvious** ↓	SQUARE, TRIANGLE, PARALLELOGRAM, CIRCLE… ↑	regular, convex, concave, number of sides, Isosceles, equilateral, rectangle,	
FIGURAL UNITS 1D or 0D merged into figural units 2D/2D	straight line, curve, segment, side, diagonal, point, vertex, center, radius, diagonal,		parallel, perpendicular, symmetric

Figure 13. Classification of words referring to visualization.

The arrow indicates the dimensional deconstruction. The dotted arrow indicates that th naming of the figural unit 1D depends on the figural figure 2D in which it is visuall contextualized (Figure 13).

We can observe the separation between the properties that are independent from an figural 2D, and the ones that refer to perceptual shape differences among the figura units 2D. Some differences are about the visual features of the outline of the shape an the others about the equality of magnitude between the constituent units 1D of th shapes. *From a cognitive point of view the properties which are independent of an figural units 2D are the most difficult to be truly understood* for students, because the require some prior development of the ability to deconstruct dimensionally the shape recognized.

3.2 Students' Misunderstanding and the Unnatural Coordination betwee Language and Visualization

Geometrical activity involves at least two registers of semiotic representation. It give rise to quite different tasks according to the register in which some production i required from the students. Here the two registers in question are natural language and non-iconic figures. So we have two cognitive kinds of task according to the direction of the conversion of representation to be carried out:

— *From words to configurations* as, for example, from the given of a problem or from given instructions in a task of construction of «figures».

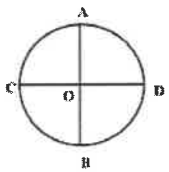
— *From configuration to words*, as in the task of description in order to produce a sequence of instructions that is necessary for constructing some configuration. The sequence depends on the tools to be used.

It is in these tasks of description that the cognitive complexity of the coordination between language and visualization appears. Two points are essential to analyze and develop this coordination. They can be formulated by the following two questions:

— How to name figural units 1D?

— How to string two instructions together for developing a sequence?

Let's give just one very elementary example.



Explain how to construct this figure by using unmarked ruler and compass;

Figure 14. Task of description.

To indicate the figural units we talk about, whatever the number of their dimension, we always use letters. But what matters is how we qualify them or categorize them, i.e. how we name them. So if you are asked to draw AB and then CB, *three kinds of choices are possible to qualify the same figural unit.* We can name each figural unit directly, according to the upper figural unit 2D in which it is seen, or according to its relation to another figural unit of the same dimension.

	OBJECTS		PROPERTIES:
	Directly	as belonging to ***a figural unit of upper dimension :***	according to its relation to ***another figural unit of the same dimension***
FIGURAL UNITS 1D	Segment or straight line ?	Radius diameter Diagonal, Side (of a polygon) ?	Perpendicular Parallel ?
FIGURAL UNITS 0D	point	center, vertex ?	middle, symmetric ?

Figure 15. Three kinds of choices for naming one figural unit 1D.

For each kind of choices there are several possible words. You only have to attend an classroom in primary school to notice that most of the students confuse these words due to the lack of visualization. The deepest difficulty is about the relational wor "perpendicular", because the students associate it with single figural unity 2D and nc the relationship between two units figural 1D (Duval, 2011).

To string two instructions together, you have to use two different technical words fo naming the same figural unit 1D. So if you are asked to draw AB and then the circle you cannot use the same word for the same figural unit 0D:

O is the *middle* of [AC] (segment AC).

Draw a circle with *center* O.

Here we are close to the fundamental law of mathematical thought, not to the logica law of non-contradiction, but the semio-cognitive law of progression in thinking. Frege was the first to have formulated it. *In mathematics, for describing, reasoning o calculating, you have always to use two different semiotic representations of the sam thing.* It is *the basic process for moving forward from any representation or expressio to another one.* Fortunately we do not speak, nor describe nor reason like that outside o mathematics. Therefore, how could students discover by themselves this non-conceptua gap that remains systematically ignored or unspoken in the teaching of geometry?

IV. VISUALIZATION AND SOLVING PROBLEMS FROM REALITY

The opposition between two ways of introducing geometry has established itself with the generalization of the mathematics education for all the students until 15 or 16 years old. On one side, there would be a theoretical approach based on the mathematical construction of objects, and therefore focusing on primitive notions, definitions, axioms and proofs. On the other side, there would be the empirical approach based on the relationship between geometry and physics and focusing on problems solved by applying geometrical knowledge. With younger students, it would be important to improve this approach into a more motivating and pragmatic one, based on the knowledge of the surrounding world and focusing on tasks and problems closed to the concrete reality. Is the specific difficulty of geometrical visualization subdued, or even neutralized, when geometry is introduced from solving practical problems and it is taught as a modeling of reality? Is visualization then still crucial for learning geometry?

There is an unexpressed assumption regarding the empirical and practical approaches of the teaching of geometry. If they are contextualized in concrete situations and used for solving practical problems, basic figures and configurations could be seen in a perceptual way. In others words, when the close relationship between geometry and reality is highlighted, the gap between the mathematical way of seeing and the natural one would be easier to be overcome. However, this close relationship between geometry and reality is always a posteriori. It is most often discovered long afterwards, because the means and methods to develop geometrical knowledge, those to develop physical knowledge of the nature and those to acquire practical knowledge of the surrounding world are not the same. If someone wants the students to become able to use geometrical knowledge in order to improve themselves and solve a wide range of problems of reality, then one question must be asked. It can be done in two reverse ways. How can we move from perception of reality to the knowledge of geometrical properties? How can we recognize which are the relevant geometrical properties to use in any practical situation?

For focusing back and forth between the perception or the image of a concrete situation and any basic figure or configuration, and for recognizing which of them matches, a diagram of the real situation is needed. *The close relationship between geometry and reality appears only when we can compare two different models of a situation:* the one based on the description of the reality and the other based on the choice between different theorems' configurations. From a cognitive and an educational point of view, the development of the ability to schematize the real situations is crucial for enabling the students to use mathematical knowledge.

4.1 Diagrammatic Representation and Geometrical Representation in Solving Real Problem

A classical example in teaching is the use of the intercept or Thales' theorem fc calculating the length of distances that can be physically measured.

To apply this theorem, we must select three items or distances that can be measurec But it depends on both the inaccessible point aimed at and the physical constraints c the situation. Therefore a diagram of the real situation must represent the relationshir between these four types of the relevant required data. Such diagrammati representation is required especially because there are several situations possible.

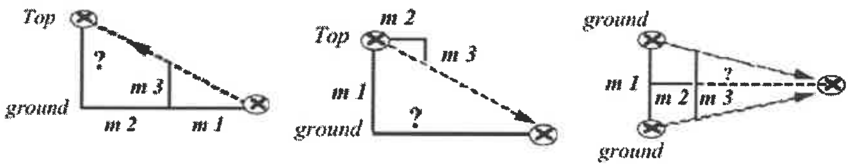
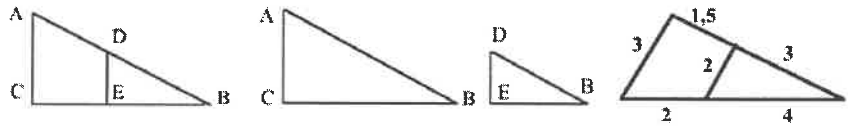


Figure 16. Three diagrammatic models of real situations to collect physical measures.

Then we must take into account the different possible configurations for the theorer and select the one, which matches with the diagram of the situation. And only then w know what equality of ratios can be used, among the six possible, for calculating th distance to the inaccessible point.



3/4.5 = 4/6 ; 4/6 = 2/3

3/4.5 = 2/3 ; 3/4.5 = 4/6

4/6 = 2/3 ; 3/4.5 = 2/3

4/2 = 6/3

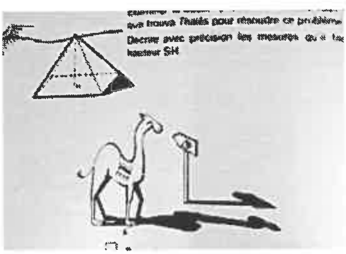
Figure 17. Configurations of the intercept theorem.

There is no direct passage between geometry and reality. For being able to apply geometric knowledge to real situations you have to realize and manage two kinds of variations of representations:

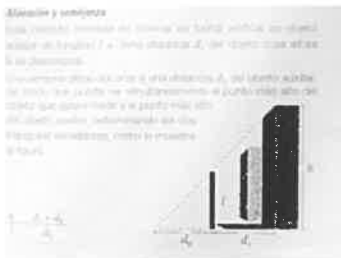
— Diagrams, which schematize real situations according to the relevant data, which must be collected.

— Configurations that correspond to some mathematical knowledge and visualizing all the possible relations between the data.

Diagrams and geometric figures refer to two quite different factors of variation for learning how mathematical knowledge can be applied to solve problems from reality. They should be separated in teaching, but in fact it is quite the opposite that happens. Only mixed representations that superimpose images evoking a concrete situation, diagrams and configurations are given to the students for illustrating the verbal description of a particular situation.



Images mixed up with configuration without diagram



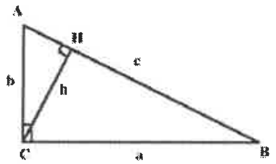
Superimposition of image, diagram and configuration

Figure 18. Typical mixed representations from textbooks.

Furthermore, a methodical exploration of the different situations in which a geometric knowledge can be applied is not organized. Under these conditions, can the students learn to recognize how geometrical knowledge can be matched in problems from reality and what particular knowledge is relevant to a given situation?

4.2 Visualization and Magnitude Calculations

Visualization concerns the dimensional deconstruction of shapes, and all the operations that reconfigure any spatial organization of figural units 2D/2D. It means that visualization concerns first affine properties and not magnitudes. However, visualization can also be essential for distinguishing *all ratios of units 1D magnitudes* that are involved in a configuration of figural units 2D. A classic example is that of similar triangles.



AC =a ; CB = b ; AB = c

AHC and ACB are right angles.

From these given verify the Pythagoras relation
$a^2 + b^2 = c^2$

Figure 19. Calculation task requiring visualization.

The configuration above can be seen either as a triangle divided into two smalle triangles, i.e. two triangles completely overlapping a greater one or as three triangles w can juxtapose (above, Figure 19). And when carrying out rotations in the plane, or i space as if these three triangles were units 2D/3D, we can display them so that th respective sides of the right angles appear in the same direction. So we can see th relevant equalities of ratios between each small triangle and the triangle ABC.

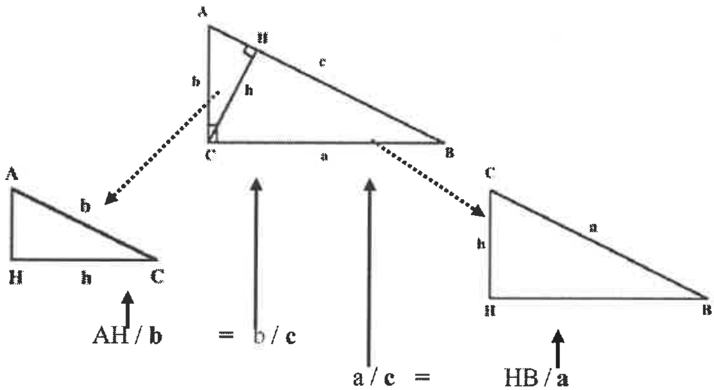


Figure 20. Mapping the four relevant ratios onto the respective similar triangles.

Then it is easy to calculate:

$$b^2 = AH \times c \quad \text{and} \quad a^2 = HB \times c$$

$$a^2 + b^2 = (HB + AH) \times c = c^2$$

In fact, "to read" a configuration means carrying out easily this kind of mapping. Fo experts or trained people it can be carried out directly on the starting configuration. I this case, the configuration can be seen in mind as three triangles juxtaposed. But is i not confusing for young students, who are confined to the perceptual recognition of th figure? Solving geometric problems on relations between magnitudes does not depend primarily on the difficulty of using mathematical properties, but on the cognitive complexity of visualization, which is involved.

V. FOR A NEW APPROACH OF GEOMETRY IN EDUCATION

The mathematical way of seeing any configuration requires the recognition of all figural units 0D/2D, 1D/2D, 2D/2D that perceptually merge into shapes 2D or 3D. This recognition is independent of the knowledge of any geometrical property and does not depend on any knowledge of geometric properties. Instead, it is the prior ability needed to visually and correctly match properties and configuration given in the context of a problem. Visualization lies in this recognition of all the possible figural units and not firstly in the various configurations, which can be produced using specific tools.

Perception goes against this mathematical way of seeing, because it makes some shapes (figural units 2D or 3D) stand out against the recognition of figural units 1D or 0D, and even against the recognition of other possible shapes 2D. That is why visualization in geometry requires a dimensional deconstruction of shapes that goes against the perceptual way to recognize both shapes and the objects they represent. For mathematicians and experts, the dimensional deconstruction of shapes has become automatic and runs almost as fast as perceptual recognition. But this is not the case for young students or anyone else. For them, the dimensional deconstruction of shapes is an unnatural way of looking at figures, which is difficult to imagine, because in all areas of knowledge the recognition of what images, diagrams or pictures represent is simply based on the perceptual recognition of shapes.

At last, it is not possible to move directly between reality and geometrical modeling. The application of mathematical knowledge to real-world problems requires an intermediary representation that must be a diagram of the real situations. And from 2D/3D or 3D/3D concrete objects or images to 1D/2D or 2D/2D figural units, which visualize relations or properties, there is deep cognitive gap, because we cannot carry out the same significant operations with figural units in the plane as with concrete objects in space.

The first crucial challenge for any teaching of geometry is to make students aware of the quite unnatural way of looking at the perceptually recognized shapes and the configurations combining several basic geometrical shapes, anyway they combine or not the same basic geometrical shape. In solving problems, even those that require only applying formulas to numerical data or using equalities of ratios, one must be able to recognize almost at first glance all the figural units which are relevant to the problem. This is the condition for a visual exploration, in which the different possibilities to change these figural units into other ones can be seen. To reach this educational goal, visualization, calculation and justification by inferring properties (in fact: propositions) have to be developed carefully in separate tasks in the choice of problems or activity sequences in classroom. But *why to separate visualization tasks from calculation tasi and visualization tasks from proving tasks by inferring properties?*

Calculation tasks short-circuit visualization, because the focus is on numerical valu about lengths (segment), perimeters (outline of shape), areas (place bounded by a outline) and volumes (space or capacity of a solid). Thus, the spontaneous perceptu way of seeing, which merges shapes recognition and estimation of their magnitudes, reinforced instead of being neutralized. That brings students to make physic measurements on the sides of a figure or empirical verifications directly on th configuration for solving problems. That brings them also to see straight lines a "almost parallel" or obviously parallel.

Tasks *about* inferring properties are completely independent of any visualization. The require a specific level of language, in which what matters is not the words, but th *valid deduction of propositions according to their theoretical status*. It is the secon crucial threshold for understanding in geometry (Duval 1991, 2007). *On the contrar visualization is closely connected to description.* Unlike inferring or reasonin description requires the use of technical words that condense definitions witho expliciting any proposition. And these technical words are related to the various figur units 0D, 1D, 2D, and 3D. Therefore, the description tasks are as important as th justification or proving tasks. Their role in the learning process is *to make studen aware of how technical words match with the figural units.*

The non-separation of visualization tasks from calculation tasks and from inferring task creates difficulties and mental blocks for the students' further learning of geometr throughout the curriculum.

The importance and priority of visualization in learning geometry are misunderstood c even denied, because both the goal of mathematics education and progression i learning are determined solely from the mathematical point of view. Objectives ar some contents of knowledge to be acquired by all the students at primary and middl school. Then these contents are decomposed into prerequisite knowledge content which in turn are broken down and so on, until the simplest concepts. Progression c learning is unidimensionally organized as (re)construction of knowledge that mu follow the reverse order of this decomposition thus obtained. So the organization c mathematics education and the development of understanding in geometry are full subordinated to *a successive acquisition of local concepts* throughout curriculum. An the specific development of visualization that is required for understanding in geometr turns out to be dismissed from teaching, since visualization is independent of an geometric concept.

Two points are indicative of the unidimensional organization of teaching. Properties an technical words about figural units 1D are introduced before the figural units 2D. I other words, it is assumed that the dimensional deconstruction of shapes is obvious t students. The same word «"figure" is used to denote the basic geometrical shapes 2I which are given to illustrate definitions of geometrical objects such as triangle, squar circle, etc. and to refer to configurations which superpose or juxtapose several basi geometrical shapes for visualizing problems. In other words, it is assumed that the students could naturally switch the shapes perceptually recognized into others that this recognition blacks out. But this is not what we observe in the classroom, because most of the students cannot go beyond perceptual recognition of "shapes-objects".

A new approach for the teaching of geometry is necessary to prevent many students from reaching very quickly an impasse. The first steps for introducing geometry at primary school must focus on the development of geometric visualization, that means on dimensional deconstruction tasks, before aiming at the acquisition of knowledge about basic geometrical figures and the heterogeneous technical vocabulary associated with them. For example, tasks of restoration of non-classical configurations that have partly deleted should be preferred to tasks of construction. This kind of task has proved to be exciting for younger students (Duval & Godin, 2005).

The main issue about the teaching of geometry is that of the cognitive access to the way of thinking and seeing in geometry, and not the knowledge content to be taught. The teaching of geometry is organized only according to mathematical standards. And the educational assumptions about the processes of knowledge acquisition are misleading. Figures should be seen from the represented object indicated by the properties coded or stated and the teaching of geometry should be organized as if the knowledge of properties were to change or guide the perceptual recognition of the relevant figural units for solving problems. It does not work for most students at primary and secondary levels, and that leads to a vicious circle didactic. While the perceptual recognition of the figures is the first major obstacle in the learning of geometry, however it is used as the intuitive and obvious way to make young students discover geometric properties from concrete materials and from problems dealing with reality.

This unilateral approach of the teaching of geometry and its educational assumptions must be challenged. The unusual way of seeing in geometry and the complexity of the coordination between visualization and language must be a top priority for research into understanding and learning geometry processes. These are the two decisive factors for developing the imagination and the ability to use geometric knowledge to solve problems in real situations or purely mathematical.

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Unjustified Assumptions in Geometry Made by High School Students

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ABSTRACT: *We investigated Unjustified Assumptions (UAs) made by high school students when proving geometric statements. We investigated how widespread the phenomenon of UAs was. The participants in this research were 246 high-attaining students from 10th grade classes in different high schools in Israel who were enrolled in a full-year geometry course. Data were collected by means of written questionnaires. The main findings are that among all incorrect answers, 87% were based on unjustified assumptions, and that UAs arise due to varied reasons such as the lack of strategic knowledge, the lack of the relevant knowledge needed for proving the statement in question and relying on the particularity of drawings. We also propose a characterization of tasks that invite UAs and a classification of UAs.*

Key words: *Geometry, Proof, Assumptions, Unjustified assumptions*.

INTRODUCTION

Many students have great difficulty with constructing geometric proofs. These difficulties arise due to lack of content knowledge as well as due to lack of general skills and strategies that influence the proving process (see, e.g., Ufer, Heinze & Reiss, 2009; Weber, 2001).

As a result, when proving geometric statements, students tend to make assumptions that are not given in the task, without justifying them while producing the proof. For example, students assume that segments are equal/ parallel/ perpendicular when these properties were not given but were helpful for reaching the statement to be proved.

Such assumptions will be called *unjustified assumptions (UAs)* in the present research. An assumption is considered a UA if it satisfies the following three conditions:

1. It is justified incorrectly or not justified (according to researchers' judgment);
2. It is used to reach the statement to be proved;
3. It is made at the beginning of the proof or in its middle.

Similar phenomena were partly treated in several earlier studies (Küchemann & Hoyles, 2006; Movshovitz-Hadar, Zaslavsky & Inbar, 1987a, b; Pool, 2003; Selden & Selden, 1987). Figure 1 presents tasks from these research studies, in which students made errors that can be considered as UAs.

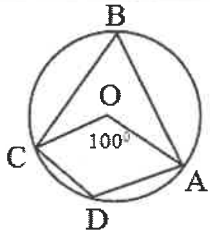
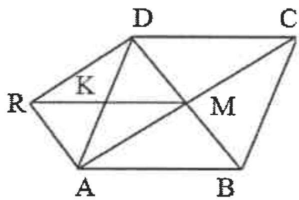
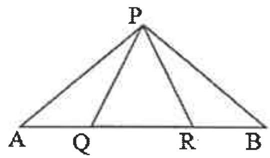
Task	UAs
1) Find: (a) ∠CBA (b) ∠CDA  (Küchemann & Hoyles, 2006)	In part b, students made the following UAs: 1. Assuming that the radius OI bisects ∠COA. 2. Assuming that BD is a diameter.
2) ABCD is a rhombus. Given: ARIIBD, RDIIAC, and K is the intersection point of RM and AD.  Prove: AMDR is a rectangle. (Movshovitz-Hadar et al., 1987a)	Two UAs were found: 1. AMDR is a rectangle because in a rectangle any two opposite angles are equal 90^0. 2. A quadrilateral with paralle opposite sides is a rectangle.
3) ΔPQR is equilateral. A and B are points on the continuation of QR. ∠APB=120^0  Prove: ΔAPQ~ΔBPR (Movshovitz-Hadar et al., 1987b)	Many students assumed that ∠QPA=∠RPB=(120-60)/2=30^0 Then they proved that ΔAPQ≅ΔBPR and got that ΔAPQ~ΔBPR.

Figure 1. UAs from earlier research studies.

The phenomenon of unjustified assumptions was not investigated as such befor However, it was indirectly treated where the studies mentioned above dealt wit situations in which students made different kinds of errors in mathematical reasonin (either initially or later while processing the data). Such errors were classified by both Movshovitz-Hadar et al. (19787a) and Selden and Selden (1987). These classification reveal that most of the errors made by high school students are not accidental but rathe can be shown to have a rational basis and can be derived by a quasi-logical process that makes sense to the student. Consequently, these errors can be considered as UAs since they have a conceptual nature and offer general ideas that students believe in although they do not provide the necessary sequence of arguments.

In this paper we investigate how widespread the phenomenon of making UAs in geometry is. In addition, we attempt to characterize geometric tasks that invite UAs as well as to build a classification for UAs using the classifications of Movshovitz-Hadar et al. (1987a) and of Selden and Selden (1987) as basis.

THEORETICAL BACKGROUND

This section has two subsections: In Subsection 2.1 we discuss the proof skills needed in geometry as well as students' difficulties with the task of proof construction. In Subsection 2.2 we deal with the limitations of visual perception of geometric objects and with how drawings can be obstacles when proving geometric statements.

These difficulties lead students to make different kinds of mathematical reasoning errors when proving, among them UAs.

Difficulties with the Task of Proof Construction in Geometry

In order to be able to construct proofs, certain knowledge and certain skills are required to be mastered by the student. Ufer et al. (2009) addressed particularly three kinds of skills that are essential for proving in geometry:

- *Content Knowledge*: To be able to construct proofs in any mathematical field, the student needs knowledge about the content of the field. This knowledge includes conceptual knowledge connected to central notions of the field and it can be available in addition to procedural knowledge.
- *Deductive reasoning by mental models:* The first step to solve a proof task is to construct an adequate mental model, which is usually linked to a figure in the task formulation. Next, a first conclusion is derived from the model and incorporated into the mental model of the situation as additional information. If the proof is not finished the process starts again.
- *Proving and metacognition*: Apart from content aspects, more general skills and strategies influence the proving process. Among these are strategies like working backwards and forwards, identification of invariants and use of symmetries. Moreover, monitoring and adapting the plan for the problem solving process are needed as well as knowing when and how to use particular strategies.

While it is recognized internationally that it is very important to teach 'proof' to all students, this is not an easy task.

A number of researchers have found that it is difficult for many students to know ho to advance logical arguments (Chazan, 1993; Kunimune & Jones, 2009; Mariotti, 200 Moore, 1994; Senk, 1985; Ufer et al. 2009; Weber, 2001). These researchers report tha

- Students do not have an accurate conception of what constitutes a mathematic proof. Many students believe that verifying that a general claim holds in a specif instance or several instances is a sufficient proof. Others believe that a proof of theorem is valid if and only if it follows a traditional format such as the 'tw column proofs' taught in geometry.
- In many cases students are unable to construct a proof because they do n understand the theorems involved and they do not know how to use them to obtai the structure of proofs. Simply recognizing a theorem does not guarantee one ca apply it properly.
- Students may lack what Weber (2001) calls 'strategic knowledge': the ability t distinguish helpful actions among several alternatives. In constructing a proof, th student is given an initial set of assumptions and is asked to derive a sequence c claims, which conclude with the statement to be proved. And while there are large number of actions that one can perform, only a small subset of these action is useful in completing the task.

Drawings as Obstacles in Geometric Proofs

The potential and limitations of visual media are recognized as part of the mathematic classroom culture. Teachers often use drawings in order to enhance students' thinkin However, sometimes students' attendance to the particularity of these drawings ca narrow their images and lead to prototypical thinking (Yerushalmy, 2005).

Students may create inaccurate, misleading drawings and arrive at incorrect conjecture or they may create correct drawings that are so particular as to inhibit their ability t derive general conclusions and proofs that go beyond the drawings they have create (Schoenfeld, 1986).

Many students are unclear about which aspects of a drawing are general and which ar specific. Some students also believe that a proof is only valid for its accompanyin figure or an accompanying class of figures even if the specific features of the figure ar not used in the proof (Martin & McCrone, 2001).

Yerushalmy and Chazan (1990) grouped visualization obstacles in geometric object according to:

- *Particularity of drawings:* In high school geometry, drawings are meant to b understood as representing a class of objects and contain the essence of th situation. Nevertheless, every drawing has characteristics that are individual an not representative of the class. For example, a specific acute triangle ABC, whic is meant to represent all triangles, is by no means a universally vali representation since it does not depict obtuse angles. This obstacle causes students to be trapped by the one-case-concreteness of an image or a drawing, which may tie to irrelevant details or may even introduce false data (Weber, 2001; Yerushalmy & Chazan, 1990). When learning theorems, students often incorporate information contained in a drawing as part of a theorem, for example, thinking that the exterior angle of a triangle must be obtuse because the drawing given with the theorem pictured an obtuse exterior angle. This information later constrains the application of the theorem. That is, the student is not able to recognize the theorem as relevant for another problem because the drawing contains an acute exterior angle (Clements & Battista, 1992).

- *Prototypical drawings as models:* When examining prototypical examples, one will find in each of them specific attributes, which are dominant and "draw our attention". Hershkowitz (Hershkowitz, 1989b; Hershkowitz & Vinner, 1983) presents evidence of the prototypical right triangle, whose perpendicular sides are in the vertical-horizontal position as well as of the prototypical isosceles triangle, which is "standing on its base" (Figure 2).

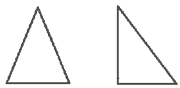


Figure 2. Prototypical isosceles and right triangles.

The orientation effect here is well known and it is an example of a visual perceptual process in which students and even teachers use visual judgment to decide whether a figure is an example of the concept or not.

Hershkowitz (1989b) called this the "prototype phenomenon", where the prototype is a result of our visual perceptual limitations which affect the identification ability of individuals - students as well as teachers.

These prototypes may induce inflexible thinking, thus preventing the recognition of a concept in a non-standard drawing. Students' definitions may include irrelevant characteristics of the standard drawing, causing difficulties in creating or interpreting drawings.

Zodik and Zaslavsky (2008) point to the complexity and subtle considerations involved in choosing an appropriate drawing to accompany a problem. Some choices lead to drawings that are helpful, to the extent that they may actually reduce considerably the cognitive demand of the problem, while others may raise its level of difficulty.

The specific choices a teacher makes may reflect pedagogical goals, such as maintaining the need to rely on the givens of the problem and not on the particularities of a specific visual representation. Some choices are made in advance while others are made spontaneously in response to classroom interactions. Whatever choices teachers make, there is a danger of transmitting some contradicting messages to the students: o the one hand a drawing is provided in order to convey some useful informatio regarding the problem situation; on the other hand, students are taught to ignore som parts and not to rely on everything they see in a drawing. It is not necessarily clear t students where to draw the line.

When categorizing UAs (Subsection 3.2), one might think that assumptions based o drawings should be considered as a separate category of UAs. However, supportin what Zodik and Zaslavsky (2008) claim, we chose not to do this, since the drawing each drawing – influences the proving process of a geometric statement.

RATIONALE AND RESEARCH QUESTIONS

This section describes the motivation for the research questions while providing:

- A general framework for UAs based on our earlier work (Dvora & Dreyfu: 2004);
- An initial characterization of UAs based on two classifications of errors in hig school mathematics as well as our earlier work.

General Framework for UAs

In earlier work (Dvora & Dreyfus, 2004), we investigated UAs in geometry using thre tasks from topics familiar to the students (triangles and quadrilaterals). The mai finding was that among all incorrect answers, 72% included UAs. Despite this higl percentage, it would be inappropriate to conclude that this phenomenon is widesprea since it was investigated among only 93 students using only three tasks.

The current research provides an opportunity to investigate this phenomenon among larger number of students and with a more varied set of tasks in order to answer the firs research question:

Research Question no. 1

How widespread is the phenomenon of UAs among high school students in Israe towards the end of their instruction in Euclidean geometry?

In Israeli high schools, students may take mathematics at three levels: 3 units, 4 unit and 5 units; 3 units is the lowest level and 5 units is the highest level.

Students who are enrolled in 4 and 5 unit classes are called high-attaining students an only they take Euclidean geometry in 10th grade. Both levels (4-5) take the sam geometry course. Since 5 unit students may be expected to have more thorough critica thinking skills than 4 unit students, we made the hypothesis that 5 unit students ma make fewer UAs than 4 unit students. The division of the participants into these tw levels enables investigating the influence of the students' thinking skills on making UAs. Therefore the second research question is:

Research Question no. 2

What is the connection, if any, between the mathematics levels of the students and their making UAs in geometry?

The frequencies with which UAs were made in the three tasks of our earlier research (2004) ranged from 55% to 81%. These differences indicate that there may be tasks that invite more UAs than others. Therefore the third research question is:

Research Question no. 3

- *What are the characteristics of geometric proof tasks that invite UAs?*
- *Why do these tasks invite more UAs than other geometric proof tasks?*

Classification and Characterization of UAs

In this section we present two classifications of errors in mathematical reasoning. These classifications were helpful in classifying as well as building a categorization of UAs.

The Classification of Movshovitz-Hadar and Colleagues

Movshovitz-Hadar, Zaslavsky, and Inbar (1987a) developed an empirical model for errors by Israeli high school mathematics students and demonstrated its reliability. Their basic assumption was that most of the errors high school students make in mathematics are not accidental and are derived by a quasi-logical process that somehow makes sense to the student.

The items covered topics such as linear and quadratic functions, powers and logarithms, plane and solid geometry, elementary statistics, trigonometry etc. In this paper we present and discuss only the categories that may be treated as UAs and show in which sense they may be UAs:

M_1: Misused data

M_2: Invalid logical inference

M_3: Distorted theorem or definition

M_1. Misused data

This category includes those errors that can be related to some discrepancy between the data as given in the item and how the student treated them. Such an error may be made either initially while putting the data together or later while processing the data. Its main characteristics are:

- Adding extraneous data: using as "given" a piece of information that is neith stated nor follows immediately from the given information. For exampl assuming that an arbitrary line through the vertex of an angle is an angle bisector.
- Neglecting some given data necessary for the proof and compensating for the lac of information by adding other data.
- Assigning to a given piece of information a meaning inconsistent with the text, fc example: using the height of a triangle as a median.

M_2. Logically invalid inference

This category includes those errors that deal with fallacious reasoning. Its mai characteristics are:

- Concluding that p implies q when q does not follow from p. This often happen when p follows form q but q does not follow from p
- Making an unjustified jump in a logical inference.

For example, in Task 2 in Figure 1, a student wrote that AMDR is a rectangle becaus "in a rectangle any two opposite angles are congruent and equal 90^0".

M_3. Distorted theorem or definition

This category includes errors that deal with a distortion of theorems or definitions.

Example: Same problem as for M_2. Here the incorrect argument was: "A quadrilatera with parallel opposite sides is a rectangle".

The Classification of Selden and Selden

Similarly to Movshovitz-Hadar et al. (1987a/b), Selden and Selden (1987) described number of types of errors and misconceptions that arose in mathematical reasoni during a junior level course in abstract algebra, taught several times at universities in th United States, Turkey, and Nigeria. The categories are as follows:

S_1: Using the converse of a theorem

This is a classic and extremely persistent reasoning error. The misconception consists i concluding an implication from its converse.

The basis for this misconception seems to be the imprecision of everyday language People often use the "if, then" construction when they mean "if and only if".

S_2: Weakening the theorem

Adding to the hypothesis is a well-known technique of practicing mathematicians: I one cannot prove a conjecture as it stands, one can add to its hypothesis and attempt t prove a weaker result. However, students who add hypotheses rarely realize they ar proving a weaker result.

S_3: Misuse of theorems

In applying a theorem, errors may arise from misunderstanding or partly neglecting the hypothesis or misinterpreting the conclusion. A student's inability to read precisely, combined with a desire to finish the proof quickly, may result in such mistakes.

This error has similarities with S_2. In S_3, the theorem being used is misunderstood, and in S_2, the theorem being proved is misunderstood

S_4: Holes

This type of error consists in claiming that a statement follows immediately from previously established results when in reality a considerable argument is required.

The two classifications discussed here overlap to quite an extent; Categories M_2, S_1 and S_4 deal with logically invalid inference while categories M_3, S_2 and S_3 deal with a distortion of theorems. However, nothing in the Seldens' classification corresponds to M_1.

An Initial Categorization of UAs

The following categories of UAs are intended to describe the origin of the UAs as well as to provide a better understanding of students' considerations while making such assumptions and the reasons for which they make them. We present three categories that are built upon the above classifications. For each category, we provide its theoretical characteristics and then illustrate it by an example:

- *Misused Data (MD).* Unjustified assumptions belonging to this category can be related to some discrepancy between the data as given in the task and how the student treats them. The main characteristics are: (i) assigning to a given piece of information a meaning that is not given, or (ii) neglecting some given data necessary for the proof and compensating for the lack of information by adding other data. For example, in Task 1 in Figure 1, students assumed that radius OD bisects $\angle COA$.
- *Distorted Theorem (DT).* Unjustified assumptions belonging to this category originate in inappropriate content knowledge that leads to a distortion of theorems or definitions. For example, in Task 2 in Figure 1, students assumed that AMDR is a rectangle arguing "a quadrilateral with parallel opposite sides is a rectangle". This is of course a definition of a parallelogram. Thus the correct theorem is "a parallelogram with a right angle is a rectangle".
- Hole (H). Unjustified assumptions belonging to this category consist in claiming that a statement follows immediately from previously established results when in reality a considerable argument is required. Such assumptions therefore indicate an unjustified jump or a hole in the proof. An example for such assumption was in Task 2 in Figure 1 when a student made an unjustified jump by concluding that a quadrilateral with two right angles is a rectangle.

Table 1 presents the above categories and relates them to the classifications discusse above.

Table 1

Initial Categorization of UAs

Movshovitz-Hadar	Seldens	Category
M_1	-----	MD
M_2	S_1	H
	S_4	
M_3	S_2	DT
	S_3	

UAs belonging to each of these categories can be used either in a backward or in forward way:

- A backward way is a process in which the proof chain is constructed from the en to the beginning; a UA may be made because it can be used to lead to th statement to be proved or to an intermediate statement that is known to lead to th statement to be proved.
- A forward way is a process in which the proof chain is constructed from th beginning to the end; while examining the givens, a student may make a UA whe considering how the givens might help in proving the statement.

The proposed categorization of UAs is preliminary. Hence, the goal of the fourt research question is investigating this categorization and improving it.

Research Question no. 4

How can UAs made by high school students be categorized?

METHOD

Participants

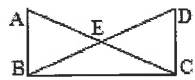
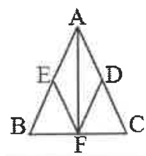
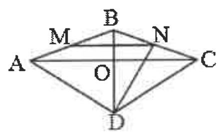
The research reported in the current paper forms part of a larger study, which also had qualitative component.

The participants were high-attaining students from eight 10^{th} grade classes wit different teachers from two different high schools in Israel. These students wer enrolled in a full-year geometry course. The reason for this choice is that according t the Israeli mathematics curriculum, these students finish learning Euclidean geometry at the end of grade 10. Hence, at the time they participated in the research, they already had considerable experience with proving and were familiar with a variety of proving tasks on triangles, quadrilaterals, and circles.

A questionnaire was administered to 204 students. In the 10^{th} grade, students are not divided into separate classes learning at the 4 and 5 unit levels. Therefore the division into levels for the purpose of this research was done according to the teachers' judgments: towards the end of the school year, the teachers knew their students well and had a well-founded opinion for which mathematics level they were suited. Hence, the teachers were asked to classify their students into the levels of 4 or 5 units. The teachers prepared the classifications according to the levels the students would be enrolled in the following year. The results of this classification were that among the 204 students, 103 students were classified as 4 units (C4U) and 101 students were classified as 5 units (C5U).

Proof Tasks

In order to investigate the research questions, data were collected by means of written questionnaires. Six proof tasks were used: Tasks 1-2 were about triangles, Tasks 3-4 were about quadrilaterals and Tasks 5-6 were about circles. The tasks are presented in Figure 3.

Task	Description of the task	
1	Given: BD=AC, AB⊥BC, DC⊥BC. Prove: AE=DE.	
2	Points D, E , F are on the sides of DABC. Given: AE=EF=FD=DA, AF⊥BC. Prove: ED is the midline in DABC.	
3	Point D is outside ΔABC in such a way that: AD=BD=CD. BD and AC intersect at point O. Point N is on BC. Given: AM=MB, DN⊥BC, MN⊥BD. Prove: MNCA is an isosceles trapezoid.	

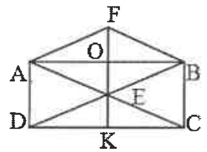
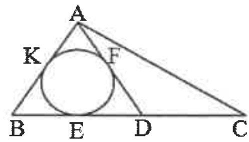
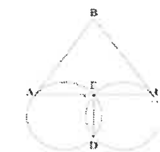
4	The diagonals of rectangle ABCD intersect at point E. Given: BFIICA, AFIIDB. FE and AB intersect at point O. FK passes through E. Prove: DK=KC. 
5	AD is the median to the hypotenuse in a right triangle ABC (∠A=900). A circle is inscribed in triangle ABD. K, F and E are the tangency points. Given: BE=DE. Prove: AB=½BC. 
6	Two circles with the same radius intersect at points D and E. Given: AC passes through point E in such a way that AC⊥ED. The tangents to the circles AB and BC intersect at point B. Prove: AB=CB. 

Figure 3. Tasks for the questionnaire.

Considerations in Choosing the Tasks

The tasks were chosen on the basis of the following criteria:

- The tasks were within the field of experience of the students and of a level the could be expected to prove in class or in an examination. The tasks were chose from textbooks as well as from matriculations exams.
- The tasks were varied in such a way that they invite different categories of UAs.

While the first of these criteria was easy to handle given our knowledge of th curriculum, the second criterion required a thorough a priori analysis of potential task: This a priori analysis had three parts:

- Possible paths for proving the statement, given students' knowledge and pric experience;
- Necessary previous knowledge elements for proving the statement and link students are expected to establish between them in order to construct the proof;

- Students' potential UAs when constructing the proof of the statement.

The a priori analysis showed that Tasks 1, 5 and 6 may be expected to invite UAs from category MD since their drawings might encourage adding lines and thus adding extraneous data. The drawing of task 1 can be seen as an incomplete rectangle with a missing side AD that encourages drawing AD and assigning to it extraneous data such as: AD||BC or AD⊥AB. The drawing of task 5 (6) encourages connecting AE (BE) and assuming that it is perpendicular to BD (AC).

Tasks 2, 3 and 4 may be expected to invite UAs from category DT since they deal with properties of triangles and quadrilaterals that students feel confident they know how and when to use, although they might not. The drawings of these tasks either present figures in their prototypical form or look similar to drawings accompanying familiar theorems. This may lead students to use theorems under wrong conditions.

Moreover, the task choice was validated by means of a questionnaire for teachers. Thirteen experienced teachers of geometry from three high schools were asked to decide for each task whether it was appropriate for 4 or 5 unit students; teachers were also asked what mistakes they expected students to make. The teachers were not familiar with the term "unjustified assumptions" nor with the aims of the research. Nevertheless, teachers pointed out quite a number of statements that they believed the students might assume without explaining why they may be assumed. Here are some important points about the results of this questionnaire:

- All mistakes proposed by the teachers, except for one, can be considered as UAs.

  This demonstrates that teachers are at least implicitly aware that in geometry students assume properties that are not justified and that teachers expect students to make many UAs possibly more so than other errors.

- All tasks were considered by the teachers to be appropriate for the population for which they were intended. Tasks 1, 2, and 4 were seen as more appropriate for the 4 unit level, and Tasks 3, 5, and 6 as more appropriate for the 5 unit level.

Administration of the Questionnaire

Three equivalent versions of the questionnaire, V_1, V_2, and V_3 were prepared and administered. Each version included two of the six tasks in Figure 3 in order to allow the students to answer any of the versions in a 45 minute lesson. V_1 included Tasks 1 and 4; V_2 included Tasks 2 and 5 and V_3 included Tasks 3 and 6.

The questionnaire was administered in the eight classes mentioned above. In each class, the versions were distributed at random, one version to each student. The resulting distribution is described in Table 2.

Table 2

The Versions of the Questionnaire (N=204)

	V_1	V_2	V_3	Total
4 units	37	33	33	103
5 units	35	32	34	101
Total	72	65	67	204

The Analysis of the Questionnaire

The analysis of the questionnaire followed Research Questions 1-4:

- Question 1 was how widespread the phenomenon of UAs was among high schoc students in Israel towards the end of their instruction in Euclidean geometry.
- For answering this question, all answers of the questionnaire were divided int correct answers, incorrect answers and no answer. Among the incorrect answer: we identified those that include UAs according to the conditions in Section 1.
- Question 2 was what, if any, the connection between the mathematics levels of th students and their making UAs in geometry was.
- For answering this question, we used the same categories used for answerin Question 1 in relation to the mathematics levels.
- Question 3 was what were the characteristics of geometric proof tasks that invit UAs. For answering this question, we built a profile for each task that include three aspects: (a) What were the main UAs? (b) How did the students justify th UAs (if at all) and how did they use them in the proof? and (c) Interpretation o what led students to assume these UAs.
- Question 4 was how UAs made by high school students can be categorized Looking at the data, we noticed that not all the UAs were the same; we identifie UAs that originated in inappropriate content knowledge (such as distorte theorems), and UAs that originated in adding properties that were not given. Tha is why, for answering this question, we have used the initial categorization of UA that included the categories: MD, DT and H as basis. However, while carrying ou the analysis, we realized that there were many UAs that could be assigned to botl categories, DT and H, since they both refer to assumptions that originate i inappropriate content knowledge. For example, let's have a look at the followin arguments:

  (1)In a parallelogram the diagonals are perpendicular.

  (2) A quadrilateral with two right angles is a rectangle.

Both claims indicate inappropriate content knowledge. However, the difference between them is that in (1), there is an expansion of the properties of the parallelogram while in (2), there is lack of necessary properties that make a quadrilateral a rectangle.

In order to distinguish between them, we combined the categories DT and H into a single category, DT, with two sub-categories, namely DT_E (for Expansion) and D_{TL} (for Lack). These new categories are defined in Table 3.

Table 3

Categorization of UAs

Category		**Description**
MD		MD refers to assumptions that are related to a discrepancy between the data and how the student treated them. Its main characteristics are: ▪ Adding extraneous data to given lines or to auxiliary lines that neither is stated in the task nor follows from previous statements. ▪ Neglecting some given data necessary for the proof and compensating for the lack of information by adding data. Example: assigning the properties of an angle bisector for an arbitrary line through the vertex of an angle.
DT	**DT_E**	DT_E refers to a case in which students apply a theorem outside its conditions by expanding the theorem to include non- critical properties. Example: In a parallelogram the diagonals are perpendicular.
	DT_L	DT_L refers to a case in which students have only part of the necessary properties to conclude a statement, so they modify the theorem to include only these properties because that is what they have. Such assumptions indicate an unjustified hole in the proof. Example: A quadrilateral with two right angles is a rectangle.

FINDINGS AND DISCUSSION

In this section, we present the findings of the questionnaire and their analysis in four subsections, each one giving the details of the analysis related to one of the four research questions, as explained in the preceding subsection 4.2.3.

The Phenomenon of UAs is Widespread

Table 4 presents the frequencies of the 408 (204 students, each one solving two task answers according to correct answers, incorrect answers and no answer separately fc the two mathematics levels. The numbers in brackets show how many incorrect answei included UAs.

These results show that 38% (11% from 4 units and 27% from 5 units) of the answer were correct. However, we focus on the incorrect answers: out of all answers, 45% (28% from 4 units and 17% from 5 units) were incorrect, and a vast majority of ther included UAs: 87% of the incorrect answers included UAs (159 out of 182).

This provides a positive answer to the first research question: the phenomenon of UA when proving geometric tasks is widespread among high school students in Israel.

Table 4

Frequencies of Students' Answers (N=408)

		T_1	T_2	T_3	T_4	T_5	T_6	Total	
4 units	**Correct**	9 (24%)	4 (12%)	0 (0%)	14 (38%)	7 (22%)	10 (30%)	44 (11%)	20
	Incorrect (UAs)	23[22] (62%)	22[13] (66%)	20[20] (61%)	15[14] (40%)	13[13] (39%)	19[19] (58%)	112[101] (28%)	
	No answer	5 (14%)	7 (22%)	13 (39%)	8 (22%)	13 (39%)	4 (12%)	50 (12%)	
5 units	**Correct**	20 (57%)	20 (62%)	12 (35%)	26 (74%)	17 (53%)	15 (44%)	110 (27%)	20
	Incorrect (UAs)	13[9] (37%)	9[6] (28%)	13[11] (38%)	6[4] (17%)	13[12] (41%)	16[16] (47%)	70[58] (17%)	
	No answer	2 (6%)	3 (10%)	9 (27%)	3 (9%)	2 (6%)	3 (9%)	22 (5%)	
	Total	72	65	67	72	65	67		40

Connection between the Mathematics Levels and UAs

The 204 students were divided about equally into C4U and C5U students. Similarly, the versions of the questionnaire were administered about equally between the two levels (see Table 2).

Table 4 demonstrates that at both levels the vast majority of incorrect answers are UAs (90% at C4U, 82% at C5U). The main difference between the levels is that there are more incorrect answers (and hence more UAs) at the 4 unit level than at the 5 unit level. In fact, the number of UAs made at the 4 unit level was almost twice as large as that at the 5 unit level. Furthermore, the performances at the 5 unit level was much better than at 4 unit level not only in making fewer UAs but in each one of the other categories:

- Success in proving the statements (54% correct answers at C5U; 21% at C4U).
- Not giving an answer at all (10% at C5U; 25% at C4U).

These differences were found in spite of the fact that the students studied in the same classes, with the same teachers, had the same textbooks, were exposed to the same tasks and passed the same examinations.

These findings provide an answer to the second research question: there is a connection between the mathematics level and making UAs.

Characteristics of the Proof Tasks

The third research question aims to characterize tasks that tend to invite UAs. In this subsection, we present in detail the profiles of Tasks 1 and 6 since (i) they deal with different shapes (task 1 is about triangles and quadrilaterals and task 6 is about circles), (ii) both tasks invite several UAs and (iii) the UAs are of different types. Then we summarize the characteristics of tasks that invite UAs, based on all six task profiles.

Task no. 1

a) *What were the main UAs?*

36 out of 72 students who were administered this task produced incorrect answers. Among them, 31 students made UAs. The main UAs were:

1. ABCD is a rectangle (16 students)
2. $AD \parallel BC$ (6)
3. ABCD is a parallelogram (5)

b) *How did the students justify these UAs and use them in the proof?*

UA_1: ABCD is a rectangle

All 16 students who made UA_1 justified their claim that ABCD was a rectangl
Their arguments were:

- A quadrilateral with congruent diagonals and one right angle
- A quadrilateral with congruent diagonals
- A quadrilateral with two right angles/one right angle

UA_2: AD||BC

All six students who made UA_2 started their answer with the assumption th AD||BC. They all justified it by writing "auxiliary line". Their answers illustrate th the UA was made in order to reach the conclusion that ABCD is a rectangle. Havin a rectangle, they used the property that the diagonals are congruent and bisect eac other.

UA_3: ABCD is a parallelogram

All five students who made UA_3 concluded that ABCD is a parallelogram by arguin that "a quadrilateral with congruent diagonals is a parallelogram". Their answe illustrate that this UA was used as a stage to conclude that ABCD is a rectangle.

c) *Interpretation of what led students to assume the main UAs*

$UAs_{1\text{-}3}$ were made with the purpose of reaching that ABCD is a rectangle. Possibl reasons for making these UAs are:

- The students were given that AC=BD and needed to prove that AE=DE. Thi may have led them to think that the diagonals of quadrilateral ABCD had t bisect each other and hence it had to be a rectangle ($UAs_{1\text{-}3}$).
- The students' arguments may have been triggered by the drawing: ABC could be seen as a rectangle (UA_1) since AB||DC, $\angle B = C = 90^0$ and AD look parallel to BC (UA_2). 'Seeing' the rectangle and realizing that it would lead t AE=DE may have made this UA to be tempting and inevitable.
- The students did not remember well what properties were needed to prove th a quadrilateral is a parallelogram or a rectangle; the different arguments impl that they were convinced that ABCD was a rectangle so they used argument according to what they had.

These reasons suggest some characteristics of Task 1 that may invite UAs:

1. *Assigning data to auxiliary lines:* The drawing can be seen as an incomplete figur with a missing side AD that encourages assuming that AD||BC or AD⊥AB.
2. *Prototypical drawing:*

   (a) The quadrilateral ABCD looks like the prototypical rectangle, the one whos sides are parallel to the paper's margins and whose basis is its longer side.

(b) ΔABC and ΔDCB are right triangles in which BE and CE go out from the right angle towards the hypotenuse. Those triangles look exactly like the right triangle that typically accompanies the theorem "a median to the hypotenuse equals half of the hypotenuse" (Figure 4). This might cause students to assume that BE and CE are medians.

Figure 4. A median to the hypotenuse drawing.

Task no. 6

This task had been given in a matriculation exam for 4 and 5 unit students several years ago. One may prove this task by first showing that AD=CD and that ∠BAD=∠BCD=90^0. The former was easier since the circles have the same radius; however, proving the latter was difficult.

In addition to the difficulty of the topic, circles, students sometimes tend to apply theorems that refer to one circle situation in two circle situations (see in the following).

a) *What were the main UAs?*

This is the only task in which all the incorrect answers (35) included UAs. The main UAs were:

1. BE⊥AC (20)
2. AB=CB (9)

b) *How did the students justify the UAs and use them in the proof?*

UA_1: BE⊥AC

The most frequent UA, assuming that BE⊥AC, was made by 20 students. All these students showed correctly that ΔADC is isosceles. However, none of them used the given statement that AB and CB are tangents to the circle. That is why they needed alternative paths: 19 (out of the 20) students concluded that AE=CE from the fact that ΔADC was isosceles and then assumed that BE⊥AC and concluded that AB=CB; the remaining student made the assumption that D, E and B were on the same line and used this assumption in order to prove that ΔABD ≅ΔCBD.

UA_2: AB=CB

Nine students made this UA. Similarly to the case of UA_1, these students did not use all the given statements in the task; they recognized that AB and CB were tangents to the circle, but neglected the given that DE⊥AC. Recognizing that there were two tangents, the students concluded that AB=CB probably since they misapplied the theorem of tw tangents to a circle (see in item c).

c) Interpretation of what led students to assume $UAs_{1\text{-}2}$

Possible reasons for assuming UA_1:

The statement to be proved was that AB=CB; the students concluded correctly th AE=CE. Having E as the midpoint of AC as well as DE⊥AC seemed to lead them t assume that B is on the extension of DE and that BE⊥AC because

- They knew BE was a median but they needed it to be an altitude to prove th ΔABC was an isosceles triangle.
- Points B, E and D look as they form a line, so since ∠AED and ∠CED a congruent to 90^0, it's tempting to assume that ∠BEA and ∠BEC are congrue to 90^0 too.

Possible reasons for assuming UA_2:

The students recognized that AB and CB were tangents to the circle, however the misapplied the theorem "if two tangents are drawn on a circle and they instersect, th lengths of the two tangent segments from the point of intersection to the points c tangency will be the same" (Figure 5). This theorem refers to two tangents that a drawn to the same circle but the students applied it to tangents drawn to two circles.

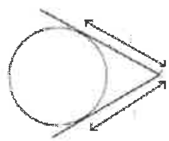


Figure 5. The drawing of two tangents to the circle.

These reasons suggest some characteristics of Task 6 that may invite UAs:

1. *Assigning data to auxiliary lines*: The property of lines DE and AC (DE⊥AC) an the special situation of line BE (which looks perpendicular to AC) which i imposed as if it was given, encourage connecting BE and assuming it i perpendicular to AC.
2. *Prototypical drawing*: ΔABC looks like the prototypical isosceles triangle. Since I is the midpoint of AC, it is tempting to assume that BE⊥AC and conclude th AB=CB.
3. *Symmetry:* The accompanying figure is drawn symmetrically. Hence both th symmetrical claim and the symmetrical drawing cause students to make symmetr assumptions that are 'obvious' but not deductively justified.

4. *Complexity of the task:* This task is considered complex since students need to establish several links in order to reach the statement to be proved.

Summary: Characteristics of Tasks that Invite UAs

We now summarize the task characteristics likely to invite UAs that were identified in the analyses of Tasks 1 and 6 as well as in analogous analyses of the UAs made by students in Tasks 2, 3, 4 and 5. Table 5 describes each characteristic and presents in which tasks it appeared.

Table 5

Description of the Characteristics of the Tasks

	Description
Prototypical drawing	Tasks whose accompanying drawing recalls prototypical examples of known theorems tend to invite UAs. In such cases, students often incorporate information contained in the drawing as part of the theorem, even though the conditions of the theorem are not satisfied (Tasks 1, 2, 3, 4, 6).
Assigning data to auxiliary lines	Auxiliary lines are drawn in a proof with the purpose of modifying the figure to help demonstrate something. However there are some challenges such as: (1) deciding where to draw the lines in order to solve the task in simplest way and (2) not assigning to these lines extraneous information. However, students sometimes do not recognize what the helpful lines are and when adding new lines, they tend to assign to them extraneous properties (Tasks 1, 2, 5, 6).
Symmetry	Tasks whose accompanying drawing implies some symmetry causes students to make assumptions that are 'obvious' but not deductively justified since the claim and the drawing make the conclusion psychologically inevitable (Tasks 2, 3, 4, 6).
Complexity of the task	In complex tasks students need to establish several links in order to reach the statement to be proved. They may get lost with all the information and have trouble organizing their thoughts. As a result, they may not carry out all the necessary steps and tend to make shortcuts using UAs. These shortcuts can bridge stages students need to reach in the proof but don't know how to justify (Tasks 3, 6).

Categorization of UAs

In this subsection, we present another way to investigate the phenomenon of UAs mor deeply by building a categorization of UAs. We present the main UAs in Tasks 1 and and discuss their categories according to Table 3.

Task 1

UA_1 refers to three claims such as: A quadrilateral with congruent diagonals and on right angle is a rectangle, a quadrilateral with congruent diagonals is a rectangle (see th discussion in Subsection 5.3). These arguments belong to category DT_L since th students wanted to prove that ABCD is a rectangle but they used only part of th necessary properties.

These arguments describe the different ways in which students modified the theorem according to the properties available to them, in order to fulfill their goal of proving th ABCD is a rectangle.

UA_2 (ADIIBC) belongs to category MD due to the following characteristics:

- The students added extraneous data to line AD, namely that it is parallel to BC.
- The students did not use the given statement AC=DB in their proofs. Once the made the UA, they reached the conclusion that ABCD was a rectangle and did nc need the given AC=BD any more.

The five students who made UA_3 concluded that ABCD was a parallelogram since had congruent diagonals. This assumption belongs to DT_E since congruent diagonals ar not a critical property of a parallelogram, therefore it indicates an expansion of th property of the parallelogram.

Task 6

UA_1 belongs to category MD since the students added the extraneous property to lin BE that it is perpendicular to AC.

UA_2 belongs to category DT_E since it was justified by the theorem of two tangents to circle.

Analogous categorizations of the main UAs made by students in Tasks 2, 3, 4 and 5 int the categories MD, DT_L and DT_E have been successfully carried out but are omitte here because of space limitations.

CONCLUSION AND FURTHER RESEARCH

As the community of mathematics education works toward giving proof a mor prominent and widespread place in the classroom, we believe it is important to lool upon the failures of proof construction in the area where proof has been most often found – geometry (Mariotti, 2006).

In this research, we investigated the most widespread failure of proof construction in geometry– making unjustified assumptions (UAs). The most significant finding of this research is that most of the errors (87%) that high school students in Israel make when proving geometric statements are UAs (Subsection 5.1). There are tasks, including some use in our research, in which one has to use appropriate auxiliary lines in order to prove the statement at hand. If students did not choose the auxiliary line(s), they either did not prove the statement, or they produced an incorrect proof that relied on UAs. The latter are, of course included in the 87% of errors that include making UAs, while the former are included among the remaining 13% of incorrect answers.

While ours was not a random sample of any larger population, the students who answered the questionnaire were all in high level streams and they answered the questionnaire toward the end of their geometry course. It can safely be assumed that earlier in the course students would have, if anything, made more UAs and that less talented students would have, if anything made more UAs.

Yet, it is important to stress that all students, who made a UA, without any exception, justified their UA. This is a significant observation, if not about the students' thinking, then at least about the socio-mathematical norms in their classes.

In addition, the research produced the following conclusions:

(1) UAs arise when

- Students use theorems under inappropriate conditions (Moore, 1994);
- Students relate to the particularity of drawings (Yerushalmy & Chazan, 1990);
- Students lack strategic knowledge (Weber (2001).

(2) Characteristics of tasks that invite UAs are (Subsection 5.3):

- Tasks that deal with less familiar topics;
- Complex tasks in which proofs consist of several steps;
- Tasks that encourage adding auxiliary lines;
- Tasks whose drawings recall prototypical examples of known theorems (Hershkowitz, 1989b; Yerushalmy & Chazan, 1990).

(3) 4 unit students tend to make more UAs than 5 unit students: In fact, the number of UAs made by the former was almost twice as large as that made by the latter (Subsection 5.2).

(4) Three categories of UAs were found: MD, DT_L and DT_E (Subsection 5.4).

Consequently, this research has some theoretical benefits about content and logic aspects of proof:

- Misconception of what a proof is: When proving geometric statements, making UAs indicates that students do not have an accurate conception of what constitute a mathematical proof (Elliott & Knuth, 1998; Harel & Sowder, 1998; Senl 1985); they do not understand that every statement in the proof must be given derived logically from the previous one, since they do not explain where thes assumptions (UAs) come from.

- Inappropriate content knowledge: When proving geometric statements, some UAs are made since students do not have all the relevant content knowledg needed for proving the statement at hand. According to Moore (1994), even whe students do know a definition or a theorem, this does not guarantee they can appl it properly because they may not be able to produce a proof by working formall with logic and definitions but needed intuitive understanding.

- Particularity of drawings: In this research there were cases in which the drawing that accompanied the tasks inhibited students' ability to derive conclusions that g beyond the drawings. According to previous research (Zodik & Zaslavsky, 200 Dvora & Dreyfus, 2004; Yerushalmy and Chazan, 1990; Hershkowitz, 1989 Hershkowitz & Vinner, 1983), when students use definitions and theorems, the tend to impose characteristics on geometrical objects because of their drawing and consequently, end up making UAs that are based on the drawings.

  Fischbein (1993) provides the underlying theory for this by introducing the ter 'figural concept' to stress the double nature of geometric figures: conceptual an figural. When proving geometric statements, there is tension between these tw systems: The definitions and theorems (conceptual system) are not clear t students and often are forgotten. As a result, the figural component tends to fre itself from the formal control and to act independently.

- Lack of 'strategic knowledge': UAs are made when students do not have wha Weber (2001) calls 'strategic knowledge': the ability to distinguish helpful action among several alternatives. When constructing a proof, students are given a initial set of assumptions and are expected to construct a sequence of statement each of which follows from previous statements, definitions and theorems, so tha the sequence concludes with the claim to be proved. Most of the students fail i choosing the relevant inferences and therefore end up making UAs.

This new knowledge about the most frequent errors (UAs) in geometric proo construction as well as knowledge of other failures may contribute to

- Increasing the chance of success when teaching proof in geometry and in othe areas;
- A better understanding of the notion of proof both by teachers and by students;

- Addressing the knowledge and the skills students have to master in order to acquire the ability to construct proofs.

Yet, here are three suggestions for further study:

- Students participating in this research have been familiar with geometry and proofs for almost three years (grades 8-10). At the time they participated in the research, they were just completing their geometry course. During these years, they have developed their knowledge of geometric concepts and definitions as well as their own perception of what a proof is and how a geometric proof should look. Conducting a parallel research with junior high school students (grade 8) who just begin their experience with geometric proofs will provide an opportunity to investigate how students develop their understanding of geometric concepts and how and when UAs first appear.
- In this research we characterized proof tasks, which invite UAs. One may assume that tasks, which lack these characteristics, do not invite UAs or invite fewer UAs. However, this issue was not investigated and requires further study that investigates the characteristics of proofs tasks that are less likely to invite UAs.
- The phenomenon of UAs in geometry was investigated in this research only from the students' side; there is no reference to teachers and their role in encouraging or preventing students from making UAs. Yackel and Cobb (1996) introduced the term "socio-mathematical norms" to stress how teachers' beliefs and actions in class influence students' understanding mathematics. Thus, additional research should be carried out: (i) on the connection between teachers' attitudes and socio-mathematical norms and students' making UAs in geometry and (ii) on how teachers who are aware of the phenomenon of UAs in geometry and who expose to the findings of this research use it during classroom instruction.

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How Do Flemish Children Solve 'Greek' Word Problems? On Children's Quantitative Analogical Reasoning in Mathematically Neutral Word Problems

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ABSTRACT: *Different from most previous research on students' transition from additive to proportional reasoning, this study departs from the basic idea that both additive and proportional reasoning are types of quantitative analogical (QA) reasoning. We investigated the development and nature of primary school children's QA reasoning by offering two word problems containing three numbers, to 325 third, fourth, fifth and sixth graders. In one problem, ratios between given numbers were integer, in the other they were non-integer. Except for these numbers, these word problems were written in the Greek alphabet, and thus totally incomprehensible to the children. Results revealed that the percentage of QA answers considerably increased with age. Among the QA answers, it was found that younger children focused more frequently on additive relations while the focus of older children was more on proportional relations. Moreover, problems in which the numbers formed integer number ratios evoked more proportional answers, whereas problems with non-integer ratios evoked more additive answers. This effect of numbers was strongest in the fifth grade. The implications of these findings for further research and educational practice are discussed.*

Keywords: *Proportional reasoning, Additive reasoning, Quantitative analogical reasoning, Word problems.*

INTRODUCTION

Learning to solve proportional word problems is an important topic in elementary school mathematics education. From fourth grade on, children are frequently confronted with simple proportion problems, mainly with a missing-value structure (Cramer & Post, 1993), also called rule-of-three problems (Vergnaud, 1983, 1988). In general, the proportional reasoning literature characterizes missing-value word problems a problems in which three magnitudes are given and a fourth one has to be found b identifying the multiplicative relation between two given magnitudes and applying th relation to the third given magnitude (Kaput & West, 1994; Vergnaud, 1997). Howeve as will be explained in detail later on, missing-value problems are not *necessaril* proportional. They can also have another underlying mathematical structure, such as a additive one. In that case, the solution lies in detecting the difference (instead of th ratio) between two given magnitudes and applying this difference to the thir magnitude. However, in both cases, a quantitative relation between two give magnitudes is considered, and is applied to the third given magnitude in order t calculate the unknown one. In this study, we will focus on this broader reasonin ability, which we denominate as quantitative analogical reasoning[1] or, briefly, QA reasoning.

THEORETICAL AND EMPIRICAL BACKGROUND

Before we go into the central idea of QA reasoning in more general terms, we will firs take a closer look at the typical structure of proportional missing-value word problems and the kind of QA reasoning that is used to solve such problems. Second, we wi consider how missing-value word problems that are additive in nature are solved. In third step, we will show the similarities between proportional and additive reasonin vis-à-vis missing-value problems, and we will argue that both additive and proportiona reasoning are utterances of QA reasoning. Fourth, we will explain how we wi investigate children's inclination for this kind of reasoning in missing-value wor problems in the present study.

Solving Proportional Missing-value Word Problems

Vergnaud (1983, 1988) mentioned that the specific structure of proportional missing value problems or rule-of-three problems is characterized by a simple direct proportio between two 'measure spaces'. A measure space can be seen as a physical magnitud (such as speed or price) that can be quantified. Each measure space consists of quantity, expressed with a certain unit of measure (such as 6 seconds or 5 euros). Figur 1 (adapted from Vergnaud, 1983), describes the structure of this problem type. M_1 an M_2 represent two measure spaces. The known magnitudes are *a*, *b*, and *c*, while *d* is th unknown magnitude.

[1] We specifically focused on quantitative analogical reasoning in missing-value word problems. W acknowledge that there are several other relational structures in word problems that are also worth bein studied, such as part-whole relations.

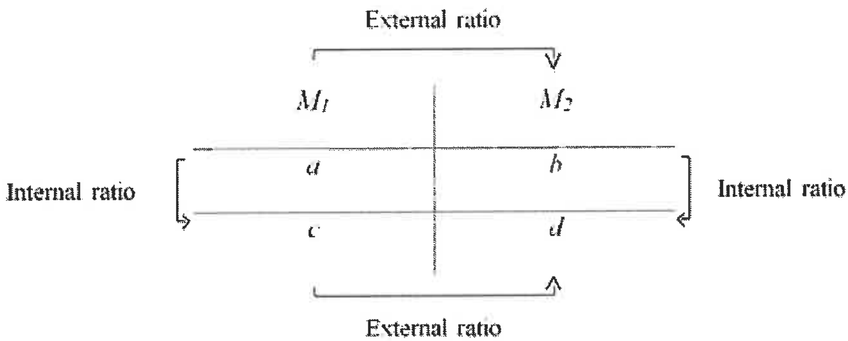
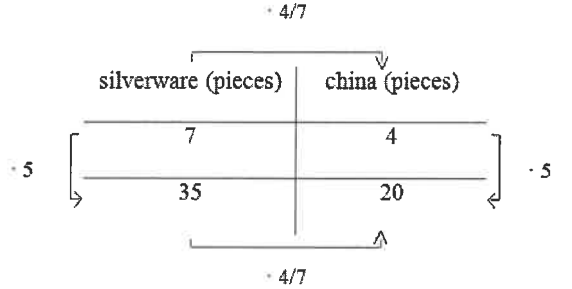
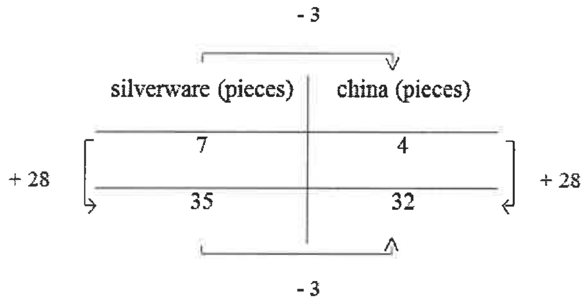


Figure 1. Character of proportional reasoning (adapted from Vergnaud, 1983).

A proportion then can be seen as a multiplicative relationship between two magnitudes in two measure spaces. This relation between magnitudes in two measure spaces, so the relation between *a* and *b*, and between *c* and *d*, refers to the external ratio. A solution method in which the external ratio between *a* and *b* is applied to the relation between *c* and *d*, is termed a 'functional approach' (Vergnaud, 1983, 1988). To illustrate the structure of missing-value word problems, and the functional approach to solve them, we use the 'placemat problem' of Kaput and West (1994): "A restaurant sets tables by putting seven pieces of silverware and four pieces of china on each placemat. If it used thirty-five pieces of silverware in its table settings last night, how many pieces of china did it use?" (p. 245). Proportional reasoners using the external ratio (see Figure 2) assume a proportional relationship between silverware and china pieces (i.e. $7 \cdot 4/7 = 4$), and apply this relationship to the third magnitude (i.e. $35 \cdot 4/7 = 20$). However, solving proportional problems is also possible by working within each of the two measure spaces and by considering the internal ratios. A focus on the relation between *a* and *c* and between *b* and *d* is named a 'scalar approach' (Kaput & West, 1994; Vergnaud, 1983). Proportional reasoners using the internal ratio assume a proportional relationship between the first and second number of silverware pieces (i.e. $7 \cdot 5 = 35$), and apply this relationship to the third magnitude (i.e. $4 \cdot 5 = 20$). As reviewed by Van Dooren, De Bock, Janssens and Verschaffel (2008), the scalar approach is most often used when approaching proportional missing-value word problems, while the functional approach is used less often. Also other approaches are possible, such as the 'unit factor approach' (e.g., 1 piece of silverware for 4/7 pieces of china, so $35 \cdot 4/7$ pieces of china are used) or the 'building up-approach' (e.g., for 7 + 7 + 7 + 7 + 7 pieces of silverware, 4 + 4 + 4 + 4 + 4 pieces of china are used).



a) Proper proportional reasoning



b) Improper additive reasoning

Figure 2. Proper proportional reasoning (a) and improper additive reasoning (b) in proportional missing-value word problems.

In elementary school, children get ample instruction in, and practice with, the solutio of proportional missing-value problems in a diversity of contexts (such as equal sharing constant price, or uniform speed) (Vergnaud, 1983, 1988). However, previous researcl has shown that children experience difficulties in answering these problems (e.g., Hart 1988; Kaput & West, 1994; Karplus, Pulos & Stage, 1983). Younger childrer frequently give additive solutions instead of proportional ones. The aforementione proportional 'placemat problem' of Kaput and West (1994) can be used to illustrat children's incorrect additive reasoning. Those children would assume an additiv relationship between pieces of silverware and pieces of china (i.e. 7 – 3 = 4, see Figur 2), and apply it to the third known magnitude (i.e. 35 – 3 = 32). Alternatively, additiv reasoners could also assume an additive relationship between the two numbers o silverware pieces (i.e. 7 + 28 = 35), and apply this to the third magnitude (i.e. 4 + 28 = 32). Translating these two forms of erroneous additive reasoning in the aforementioned terminology and scheme of Vergnaud (1983, 1988), we could state that children who improperly reason additively, calculate the difference between two given magnitudes (i.e. either within or between measure spaces, see further) and assume that the difference between the two other magnitudes will be the same.

Studies into the improper use of additive reasoning have further pointed out that children's erroneous choice for an additive solution method to solve proportional missing-value word problems is strongly determined by the numbers given in the word problem and by children's age. As far as the impact of the numbers is concerned, children have been found to use additive solution methods in proportional word problems more frequently when the numbers in the problem form non-integer ratios than when they form integer ratios (Kaput & West, 1994; Karplus et al., 1983; Vergnaud, 1983, 1988). With respect to children's age, it has been found that especially younger children overuse these additive solution methods in proportional missing-value word problems.

Most research that focused on improper additive reasoning used these mistakes to demonstrate that the children who committed those mistakes have not reached the stage of proportional reasoning yet (or at least not yet completely). Their answers still depend on the nature of the ratios between the numbers given in a word problem, instead of on the underlying mathematical relation in the word problem, which is in fact the only relevant task characteristic on which their answers should be based. While we agree with this conclusion, a basic tenet of the present article is that children who reason additively in those proportional word problems have already taken a valuable step in their development towards proportional reasoning, in the sense that they focus on the quantitative relation between two magnitudes that are given in the word problem, and apply this relation to a third given magnitude in order to calculate the missing one (rather than, for instance, merely performing an arithmetic calculation on all given numbers in the problem). In other words, they are doing what we call quantitative analogical (QA) reasoning. The only problem with their reasoning is that they look for an additive rather than a proportional relation between the four magnitudes in the problem.

Solving Additive Missing-value Word Problems

Of course, not every missing-value problem should be solved by means of proportional reasoning. In some word problems, another type of reasoning (e.g. quadratic, exponential, ...) is required. In this paper, missing-value problems where additive reasoning is required are of specific interest. An example is the word problem that Cramer, Post and Currier (1993) gave to pre-service elementary education teachers: "Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?" (p. 159). Here, the additive relation (i.e. a relation of difference) is appropriate and can – just as described above for proportional missing-value problems – t calculated either between or within measure spaces. Children considering the additi relation between magnitudes in two measure spaces (i.e. Julie's number of laps ar Sue's number of laps), look at Sue's and Julie's initially given number of laps (i.e. 3 + = 9), and apply this relation to Julie's second mentioned number of laps, to find Sue final number of laps (i.e. 15 + 6 = 21). As described in Figure 3 in a more general wa the relation of 'external difference' between *a* and *b*, which are magnitudes in tw different measure spaces, is applied to the relation between *c* and *d*. Another way t solve the problem is by calculating the difference between Julie's first and secon number of laps (i.e. 3 + 12 = 15), and apply this difference to Sue's initial number c laps (i.e. 9 + 12 = 21). The latter method relies on the 'internal differences' withi measure spaces, so on the difference between *a* and *c*, and *b* and *d*.

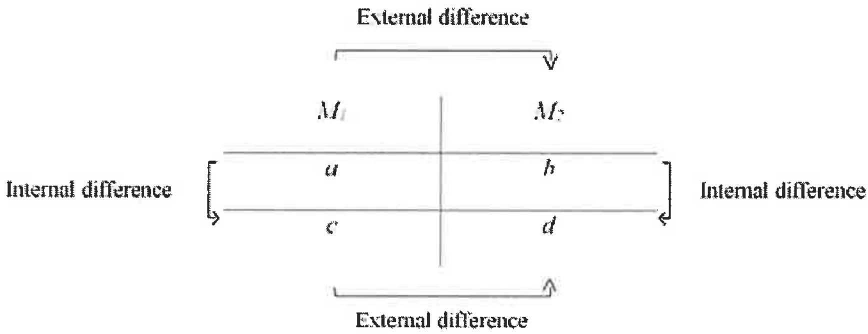


Figure 3. Character of additive reasoning.

Analogously to our overview of the incorrect use of additive reasoning to proportion missing-value problems as described in the previous paragraph, it has been empiricall shown that many children erroneously use proportional solution methods to additiv missing-value word problems, such as the aforementioned runner problem of Cramer e al. (1993). Previous research also pointed out that their improper use of proportion reasoning is strongly determined by task and subject characteristics, similar to what wa found in the literature on the improper use of additive strategies in proportional wor problems (Fernández, Llinares, Van Dooren, De Bock, et al., 2010; Van Dooren, D Bock, Hessels, Janssens, & Verschaffel, 2005; Van Dooren, De Bock & Verschaffe 2010; for a review, see Van Dooren et al., 2008).

First, children's erroneous choice for a proportional solution method to solve an additiv word problem is significantly affected by the numbers given in the word problem. Th application of proportional methods occurs more frequently when the numbers in th word problem form integer ratios, whereas children are less inclined to erroneously us proportional reasoning methods in word problems where the numbers form non-intege ratios (Van Dooren, De Bock, Evers & Verschaffel, 2008; Van Dooren et al., 2010

Quite remarkably, Van Dooren et al. (2010) found that the impact of the numbers in the word problem is even stronger than the effect of the proportional or additive nature of the problem itself. Second, while people from various ages fall prey to the improper use of proportional methods (Nunes & Bryant, 2010), the overuse of proportional methods to additive problems tends to increase with age during elementary school and the first years of secondary school (Fernández et al., 2012). Moreover, age and the previously explained number characteristics interact. Van Dooren et al. (2010) showed that between the stage where children overuse additive methods on proportional problems (as described in the previous paragraph) and the stage where they overuse proportional methods on additive problems, there is a stage of simultaneous overuse of additive and proportional methods. Children in this intermediate stage give additive answers to word problems with non-integer ratios and proportional answers to problems with integer ratios, independent of their actual mathematical structure. In Flanders (Belgium), this intermediate stage typically occurs in 5th grade of primary school, which is not that surprising given the intensive practice of proportional missing-value problems from the beginning of fifth grade on (see conclusion and discussion).

Quantitative Analogical Reasoning as an Important Problem Solving Skill

As outlined above, previous research into additive and proportional missing-value word problems has often focused on children's choices for additive and proportional reasoning on, respectively, word problems requiring a proportional and additive solution, and on the task and subject characteristics that determine these erroneous choices. In this previous research, and in the theorizing of additive and proportional reasoning more generally, the focus was on the *differences* between both types of reasoning, and children's inability to correctly distinguish where to apply each of them. However, when conceptualizing additive and proportional reasoning as in Figure 1 and Figure 3, it stands out that additive missing-value reasoning shows important *similarities* with proportional missing-value reasoning. Kaput and West (1994) already emphasized that children who improperly use the additive approach for proportional reasoning problems of the missing-value type, still "distinguish the quantities, construct units, and correctly identify the unknown quantity" (p. 251). In other words, improper additive reasoners demonstrate insight into the different known and unknown magnitudes and their mutual relations. The only difference with appropriate proportional reasoning is that the former focuses on a different kind of mathematical relation between a and b (i.e. a difference instead of a ratio, as argued by Nunes and Bryant, 2010). The same claim could be made for the improper use of proportional reasoning to additive missing-value word problems. So, regardless of the correctness for a given problem, additive and proportional missing-value reasoning have in common that children focus on the analogical relations between the four magnitudes in the word problem.

To sum up, the abovementioned types of additive and proportional reasoning ai quantitative analogical, in the sense defined above. Thus, in order to get a complet picture of the development of children's QA reasoning, one needs to simultaneousl consider both its additive and proportional form. Still, it is of course also interesting t investigate on which relation children focus (i.e. additive or proportional) if they reaso analogically.

FOCUS OF THE CURRENT STUDY

In this study, we applied a novel approach to investigate the development of Q reasoning and the factors that may affect it. Previous research has shown that children' choice for an additive or proportional solution method is often based on a salient bu essentially irrelevant criterion, namely the ratios between numbers in the word problen rather than on the mathematical model underlying the problem situation. However, i all previous studies, this underlying mathematical model could be determined clearl and unquestionably by carefully reading and processing the word problem. In th current study, besides the development of children's quantitative analogical reasonin per se, we also wanted to investigate children's choice for an additive or proportiona approach in situations where they were not directed whatsoever by the mathematica structure of the word problem. This allowed us to get a view on children's general an spontaneous inclination towards QA reasoning, and, in case such reasoning occurrec which type of reasoning then would be used. For this reason, we used an atypical kin of items, namely mathematically neutral word problems. One way to design suc neutral problems is to phrase them in a language that makes absolutely no sense fc these children. In the present study we posed the problems in Greek literal symbol which were completely inaccessible to the children, except for the numbers which wer presented in their usual Arabic form. Still, children were asked to try to solve thes 'incomprehensible' word problems. Our intention was thus to analyze to what exter they would look for a quantitative analogical relation between the given numbers, and i so, if they would opt for an additive or a proportional one.

Research Questions and Hypotheses

Our first research question was: To what degree do children apply quantitativ analogical reasoning in neutral word problems, and how is this affected by age Because of elementary school children's increasing classroom experiences with solvin missing-value word problems, we expected that even our neutral word problems woul elicit a substantial amount of QA reasoning (hypothesis 1), and that this amount woul increase with age (hypothesis 2).

Our second research question was: What is the nature of children's QA reasoning, an how is it affected by age and by number characteristics of the neutral word problem Given that both additive and proportional types of answers to missing-value problem were observed in previous research, we hypothesized that we would observe both type of QA answers to our neutral word problems (hypothesis 3). Furthermore, based on the aforementioned previous research results into clearly additive and proportional word problems, we anticipated that among the QA answers, there would be a development with age, from a dominance of additive answers towards a dominance of proportional answers for neutral word problems too (hypothesis 4). We also expected a reliance on the characteristics of the numbers in the word problem. More specifically, we predicted that problems containing non-integer ratios would lead to a higher number of additive answers than problems with integer ratios, and that the latter problems would lead to a higher number of proportional answers than problems with non-integer ratios (hypothesis 5). Finally, we anticipated that this number effect would be different for the different age groups. More specifically, the sensitivity to the numbers in the problem was expected to be the strongest between the initial and final stages wherein we expected, respectively, mainly additive and proportional answers (hypothesis 6).

METHOD

Participants

Participants were 325 children from third to sixth grade from two primary schools in Flanders (88 third graders, 78 fourth graders, 81 fifth graders and 78 sixth graders). The number of boys and girls was approximately equal in the sample. Both schools attracted children from different socioeconomic backgrounds and were average in size. One school was situated in a small village, whereas the other was located in a middle-sized city.

In Flanders, the standards for elementary school mathematics are the same for all schools (Ministerie van de Vlaamse Gemeenschap, 1997). So, although different textbooks are used by different schools, the contents and instructional approaches are still quite similar. This also holds for the curricular topic of proportional reasoning. In second and third grade, attention is paid to solving simple proportional word problems (e.g., of the type '1 kg of apples costs *x* EUR. How much would 5 kg of apples cost?'). The essential curricular contents related to proportionality are already introduced at the end of the fourth grade, but are thoroughly rehearsed, further developed, and intensively practiced in fifth and sixth grade. The solution of additive missing-value problems receives little or no attention in the elementary school mathematics curriculum.

Materials and Procedure

All children solved two paper-and-pencil tests. Each of the tests contained one neutral word problem, along with 15 buffer items (related to various parts of the children's curriculum). Both neutral word problems were stated in Greek literal symbols, except for their numbers, which were given in Arabic form, as shown in Figure 4. These Greek problems were considered as mathematically neutral, because they did not contain any intrinsic indication for an additive or proportional solution method. Given that th Flemish children could absolutely not read nor understand the text of these problem: neither the proportional nor the additive solution method – nor any other solutio method – could be considered as correct or incorrect[1]. The two word problems onl differed with respect to the numbers used in the problem. For one problem, the give numbers formed integer (internal and external) ratios (e.g., 4, 16 and 8 as give magnitudes). For the other problem, both ratios were non-integer (e.g., 4, 14 and 6 a given magnitudes). To minimize the influence of the specific numbers being used i both problems, several sets of numbers forming integer and non-integer ratios wer used. In all sets of numbers, the outcome of the QA calculations (whether don additively or proportionally) was always integer. To prevent difficulties with th classification of children's responses, the sum of the three numbers being chosen wa never equal to the outcome of the proportional or additive calculation (see results).

The two tests were administered on two separate moments in time, with one week i between. The instructor told children that the test was aimed at assessing their genera mathematics achievement. For the neutral word problems the test merely mentioned tha the problems were in Greek but that children should nevertheless try to fill them ir Children were asked to record their calculations on the answer sheets, and were told tha a pocket calculator could be used (as in their usual word problem solving lessons an tests).

This word problem is a Greek one. Try to fill in a number on the dotted line.

Αδα καλκα πορελαντορα λικτουν κοττορ.

Νοϖεργανιχα τινεσταρι 4 ποσστορ ιο χηιον ανπερα τον ϖορχον 16 στατον εστανο τυ ϖ μαγχανετο.

Προβαλεντι μογρονατεσ 8 ογροντ ο γνοστον καλκονο τοτ λινδεναν, ναγ κιφ νισϖορκ ς χκρινον λοπεναδο μαορν εωεινστ?

Answer:

Γελομαλ λοπανδορα ριτ .... νιφφ τοτο.

Figure 4. 'Greek' word problem.

RESULTS

Quantitative Analogical Reasoning

Our first research question comprised the degree to which children gave QA answers t the neutral word problems, and the extent to which this was affected by age. In a firs step of the analysis, the responses to the two neutral word problems were therefor classified in the following two broad categories:

- QA answer: when either proportional or additive operations were executed on given numbers (i.e. calculating x in $b / a = x / c$ or in $b - a = x - c$)
- Other answer: when the given numbers were combined in another way than specified above, or when the problem was only partly, or not at all answered.

While coding the responses, a third category was added:

- 'Sum-of-three' answer: when the three given numbers were added (i.e. calculating x as $x = a + b + c$)

Although this solution method is not of specific interest for the present study (as it is not a QA answer in the sense explained above), this 'sum-of-three' category was included because it turned out that a large number of children had just added the three given numbers in the word problem. Therefore, we decided to separate these sum-of-three answers from the category of other answers. When a child committed a technical calculation error in an answer that otherwise could unambiguously be considered as a QA answer (e.g. writing that $8 \cdot 2 = 14$), we still classified it as a QA answer.

Table 1 gives an overview of the percentage of all QA, other and sum-of-three answers in different grades. This table reveals that 20.5% of all answers were QA answers. Another 42.6% was of the sum-of-three type, and the remaining 36.9% were other answers. So, in line with hypothesis 1, we found a substantial number of QA answers, especially given that the two neutral word problems were completely incomprehensible to these children. However, even more interesting is the effect of age on the percentage of QA answers.

A generalized estimating equations analysis with number characteristics as within-subjects variable and grade as between-subjects variable, on the percentage of QA, sum-of-three or other answers, revealed that children's age affected their answers. The percentage of QA answers significantly increased from 9.1% in third grade to 41.1% in sixth grade ($\chi^2(3)$= 43.858, $p < .001$), which was in line with our second hypothesis. As shown in Table 1, the initially low percentage of QA answers was due to the remarkably large percentage of sum-of-three answers. Almost half of the answers (48.9%) was characterized as such in third grade, and still almost a quarter in sixth grade ($\chi^2(3)$= 24.579, $p < .001$). The percentage of other answers also decreased with age, from 42.0% in third grade to 35.9% sixth grade, but this decrease was much smaller and non-significant. In sum, these results showed that, with age, children developed a stronger inclination to solve the two neutral problems by QA reasoning, while their tendency to add the three given numbers gradually decreased and their tendency towards other kinds of solutions remained more or less the same.

Table 1

Percentages of Quantitative Analogical (QA), Other and Sum-of-three Answers Given by Children in Different Grades

		QA	Other	Sum-of-three
All problems				
	3	9.1	42.0	48.9
	4	7.7	35.9	56.4
	5	25.3	33.4	41.4
	6	41.1	35.9	23.1
Total		20.5	36.9	42.6

Proportional or Additive Quantitative Analogical Reasoning

In a second step, we focused on the subset of answers being coded as QA answer (20.5% of all answers, i.e., 133 out of 650), to answer our second research questio about the precise nature of QA reasoning. Therefore, all QA answers were furthe categorized as follows:

- Proportional answer: when multiplicative operations were executed on give numbers (i.e. calculating x in the expression $b / a = x / c$)
- Additive answer: when additive operations were executed on given numbers (i.e finding x in $b - a = x - c$)

Table 2 gives an overview of the percentage of additive and proportional answers o neutral word problems, of all children in different grades. As expected in our thir hypothesis, the neutral word problems elicited both proportional and additive answers Of all QA answers, half were additive (49.6%), whereas the other half were proportiona (50.4%). Moreover, and related to our remaining hypotheses, the percentage of additiv and proportional answers differed depending on children's grade and on the nature o the numbers given in the word problem (see Table 2). This was further analyzed by generalized estimating equations analysis with the number characteristics (integer o non-integer) as within-subjects variable, grade (third until sixth grade) as between subjects variable, and with the percentages of proportional answers as dependen variable (the percentage of additive answers being the complement of this variable).

Table 2

Percentages of Additive and Proportional Answers Given by Children in Different Grades

		n	Additive	Proportional
Integer				
	3	12	60.0	40.0
	4	11	80.0	20.0
	5	20	26.1	73.9
	6	23	17.6	82.4
Total		66	30.6	69.4
Non-integer				
	3	4	100.0	0.0
	4	1	100.0	0.0
	5	21	77.8	22.2
	6	41	56.7	43.3
Total		67	72.1	27.9
All problems				
	3	16	75.0	25.0
	4	12	91.7	8.3
	5	41	48.8	51.2
	6	64	35.9	64.1
Total		133	49.6	50.4

First, the results of the analysis indicated that the percentage of proportional answers significantly increased with age, from 25.0% in third grade, to 64.1% in sixth grade ($\chi^2(3) = 884.927$, $p < .001$, see Table 2). Accordingly, the percentage of additive answers significantly increased with age, from 75.0% in third grade to 35.9% in sixth grade. These findings were consistent with our fourth hypothesis. Second, the nature of the numbers affected the nature of the QA answers, as expected in hypothesis 5. The integer problem evoked significantly more proportional answers than the non-integer problem (69.4% vs. 27.9%, $\chi^2(1) = 1349.979$, $p < .001$). Third, the number effect interacted significantly with the effect of grade ($\chi^2(2)$= 452.825, $p < .001$). For the interpretation (this interaction, one can compare the difference in the percentage of proportion: answers to the integer and non-integer variant in different grades. The difference in thi grade (40.0%) and fourth grade (20.0%) is not reliable due to the very low absolu number of QA answers. However, in line with hypothesis 6, the number effect was tl largest in fifth grade (leading to a difference of 51.7%) and decreased towards sixt grade (39.1%).

CONCLUSION AND DISCUSSION

This study focused on children's quantitative analogical (QA) reasoning in wor problems that could be considered neutral in terms of their underlying mathematic: model, given the completely unknown language and unfamiliar literal symbols in whic they were posed, except for their numerals. We investigated the extent to which childre gave answers based on QA reasoning, and the development of their answers with ag We also studied the additive or proportional nature of children's QA answers, an whether children's age and the numbers characteristics in the problem influenced tl choice for an additive or proportional answer. Hereafter we first summarize the mai research results related to these two research questions, and we end with som theoretical, methodological and practical implications.

Quantitative Analogical Reasoning

Based on the analogy between additive and proportional reasoning in typical missing value problems, in a first step, we analyzed children's tendency to give answers base on QA reasoning. This kind of analysis is rather unique, because previous research int this topic has mainly focused on either additive reasoning or proportional reasoning without explicitly recognizing the common nature of additive and proportiona reasoning, namely that they are both forms of analogical reasoning in terms c quantitative relations. Also, previous studies offered word problems with a clea mathematical underlying model, and thus were unable to investigate children' spontaneous reasoning about quantitative relations. The results from our study indicate that the neutral word problems did elicit answers based on QA reasoning, i approximately one out of five cases. After all, this was a substantial number of Q/ answers, especially given that the word problems were posed in an inaccessibl language and thus completely nonsensical to the children. The percentage of Q/ answers moreover considerably increased with age. Consciously or not, older childre more frequently looked for a relation between two given numbers in the word probler and applied this to the third number, in order to calculate a fourth one.

The finding that children became more focused on quantitative relations relates to th notion of SFOR introduced by McMullen, Hannula-Sormunen and Lehtinen (2013) They studied what they call children's 'spontaneous focus on relations' (SFOR) b using specific non-explicitly mathematical tasks, whereas we have conceptualized Q/

reasoning in the context of missing-value word problems, which are clearly explicitly mathematical tasks. Future research should study the relation between their SFOR-construct and our notion of QA reasoning. Moreover, and as also remarked by McMullen et al. (2013), future research on children's spontaneous focusing on quantitative (analogical) relations should include a measure of children's actual quantitative skills, to confirm that the increase in their (analogical) relational answers is mainly due to an increase in whether and how they perceive a situation in quantitative (analogical) relational terms, rather than an increase in their skill to operate with quantities and quantitative relations. In the current study, we assumed that younger children's insufficient quantitative skills may not have been a prominent explanation for the observed lack of QA answers, because several previous studies (as reviewed by Nunes and Bryant, 2010) have already shown that children at the ages involved in our study typically succeed in solving both additive and proportional reasoning problems.

Proportional versus Additive Reasoning

Up to now, we discussed our results into children's inclination towards quantitative analogical reasoning in general. In a second step, we discuss on which quantitative relation those quantitative analogical reasoners relied, and what task and subject characteristics accounted for children's choice for a certain quantitative analogical relation.

Our results showed that approximately the same overall percentage of quantitative analogical answers was additive or proportional, but that the percentage of additive and proportional answers further depended on subject and task characteristics. The percentage of additive answers decreased with age, while the percentage of proportional answers increased accordingly. With respect to the number characteristics in the problem, problems with integer ratios evoked more proportional than additive answers, whereas there reverse was true for problems with non-integer ratios. We also observed an interaction between children's grade and the characteristics of the numbers in the problem, as the number effect was most prominent in fifth grade. These results resembled our previous results (e.g., Van Dooren et al., 2005, 2010; Van Dooren, De Bock, Evers et al., 2008), but we want to stress that our present results are obtained by offering children mathematically neutral items, whereas the problems used in those previous studies were in a clearly additive or proportional missing-value format.

Although the exact explanation for our findings is still open for discussion, it may at least partly be found in the current elementary mathematics education curriculum (as suggested by Van Dooren, De Bock, Evers et al., 2008; Van Dooren, De Bock, Janssens et al., 2008). Children currently encounter in the elementary mathematics classes a restricted and stereotyped diet of word problems, and are taught to solve these problems by recognizing the precise problem type and activating the arithmetic solution method that is associated with it (e.g., Verschaffel, De Corte & Lasure, 1994; Verschaffel, Greer & De Corte, 2000, 2007). As argued by Van Dooren et al. (2010), the vast majority of word problems with a missing-value structure that children encounter in elementary school can be solved by focusing on the proportional relations. Moreover, whe proportional reasoning is introduced, problems typically first involve numbers formin integer ratios (Van Dooren et al., 2010). This way, it is not surprising that older childre (who have been subjected more frequently to this restricted set of word problem: increasingly reason proportionally, and that children tend to connect certain superfici: cues in the word problem (i.e. number characteristics) with concrete solution methods.

Regardless of the fact that additive analogical reasoning often inappropriately occurs i proportional missing-value problems, it is still a valuable step in children's developme towards proportional reasoning. Additive reasoning is after all already a way of Q. reasoning. Therefore, we suggest that both additive and proportional missing-valu problems should be offered in the elementary school curriculum, and that childre repeatedly should be stimulated and helped to distinguish between additive an proportional problems. We are convinced that offering both types of missing-valu problems would help children to gain an understanding of what quantitative analogic: reasoning means, as well as what it thus implies to determine the precise nature of th: relation in a word problem.

NOTE

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Endnote

[1] This is not the first study wherein children solving word problems have bee confronted with incomprehensibe text. In 1997 D'Amore confronted Italian elementar school children with regular word problems in which the word of the objects bein bought and sold by a shop keeper, namely "pencils") was replaced, in a first small-scal study, by a pseudo-word ("orettoles") and, in a later study, by a non-wor ("przxetqzyw"). But, besides this particular word, the remaining text remained intact i D'Amore's research. Moreover, the goal of this research was quite different from our: i.e. to confute the prevailing opinion among researchers that a child, when facing a wor problem, has to build a complete and correct model of the situation described in th problem text in order to solve the problem correctly.

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University Students' Conceptualizations of the Infinite Decimal Expansion of Rational Numbers in Teaching Settings

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ABSTRACT: *The study explores mathematics department students' understanding of the double decimal representation of a rational number, utilizing the theoretical framework accompanying the notion of 'procept'. The analysis of the participants' responses to the questions of a teaching classroom scenario show that very few attributed both a number and a process meaning to the notation provided, revealing their own fragmented and limited understanding of the relevant mathematical idea. Given that the latter constitutes an important base for the forms of teaching practice that they will go on to develop, important questions emerge as to the professional experiences that would challenge and strengthen prospective teachers' disciplinary knowledge.*

Keywords: *Prospective teachers, Rational numbers, Mathematics teacher knowledge, Procept.*

INTRODUCTION

Teachers' disciplinary mathematics knowledge makes a difference in their professional practice and their students' achievement but its relationship to teaching of the subject matter is neither simple nor straightforward (e.g., Rowland, Martyn, Barber, & Heal, 2000; Hodgen, 2011). In trying to explore this relationship, the relevant research has focused either on the mathematics necessary for teaching or on the mathematics pedagogy or on both.

With regard to the mathematics knowledge for teaching (MKT), the main research interest has been directed towards identifying the degree to which future or practicing teachers are in a position of an advanced understanding of school mathematics, i.e., whether they could go beyond school mathematics and relate it to the underlying advanced academic mathematics. The research reported here constitutes an attempt to look into this issue, concentrating especially on the mathematical idea of the infinite decimal expansion of a rational number as viewed by prospective teachers, adapting the procept framework (Gray & Tall, 2001). Previous work on this particular mathematica idea has focused only on students' difficultly with it.

In the following, a theoretical framework for the study is first sketched, consisted of presentation of some central features related to the research on MKT, a theoretica perspective utilized in the study and a literature review on students' difficulties with th mathematical focus of the study (the infinite decimal expansion of a rational number The methodology adopted and the results of the analysis of the data follow, leading t the discussion and conclusion section.

ON TEACHERS' MATHEMATICAL KNOWLEDGE

The main body of the relevant on teachers' knowledge, elaborating upon the classica notions of subject matter knowledge (SMK) and pedagogical content knowledge (PCK introduced by Shulman (1986), attempted to establish systems of classification for th mathematics knowledge for teaching (MKT) evoked through paradigmatic examples For example, Ball, Thames and Phelps (2008) subdivided these two types of teachers disciplinary knowledge into further categories (e.g., SMK is seen to includ mathematics knowledge exploited in any setting, used specifically in classrooms an related to how mathematical topics are related over the span of the mathematic curriculum). Aiming to provide a heuristic to guide attention to the mathematica knowledge-in-use within teaching, Rowland, Huckstep and Thwaites (2005) identifie four dimensions of this knowledge: foundation (concerns mathematics pedagogy transformation (related to teachers' and students' representations), connection (refers t links made between components of the mathematics teaching) and contingenc (concerns teachers' readiness to respond to students' needs).

The line of thinking adopted by the above studies has been criticised in two ways. Th first is related to their emphasis upon identifying types of MKT, which can veil th essential mathematical activity in which different kinds of knowledge relate and inforn each other (Watson, 2008). The second type of criticism argues that these studie approach teachers' knowledge in individualistic terms rather than as a product of th educational system in which it is located (Petrou & Goulding, 2011).

Independently of the approach adopted, the research on MKT verifies that it bear distinctive features, shaped by the particular context in which it is activated and it i interactively constructed, based on the enactment of the mathematical activit forwarded by teachers in the classroom (Ruthven, 2011). According to Mason (1998) these distinctive features of MKT can be attributed to particular 'levels of awareness' awareness-in-action (acting in the moment, without being necessarily able to justif action/ ability to 'do'), awareness-in-discipline (awareness of awareness in action ability to instruct others) and awareness-in-council (awareness of awareness i discipline/ ability to instruct others to teach).

The present paper can be seen as an attempt to contribute to the studies which explor MKT as knowledge in practice at the awareness-in-discipline level. In particular, i focuses on perspective secondary mathematics teachers' understanding/knowledge of the infinite representation of a rational number, a topic of notable significance for mastering advanced mathematical thinking, which can be seen as part of SMK (in the approach suggested by Ball and her colleagues) and as belonging to the 'transformation' dimension (in the model proposed by Rowland and his colleagues).

For the purposes of the study, the framework suggested by the University of Warwick team was exploited. In particular, Gray, Pitta, Pinto and Tall (1999) suggest a model for the spectrum of performance of different individuals in different contexts when using mathematics procedures (a specific sequence of steps carried out a step at a time – focus on the steps of a procedure), processes (procedures having essentially the same effect – focus on the effect of a procedure) and procepts (Tall et al., 2001). "The portmanteau word 'procept' … (refers) to this amalgam of concept and process represented by the same symbol […]. An elementary procept is the amalgam of three components: a process which produces a mathematical object, and a symbol which is used to represent either process or object" (Gray & Tall 1994, p. 120). According to Gray and Tall (1994), process and concept are combined in a single notion by who are successful in mathematics.

Gray and Tall (2001) elaborated further the spectrum procedure-process-procept, introducing four different levels of sophistication: pre-procedure, procedure, process and procept (Figure 1).

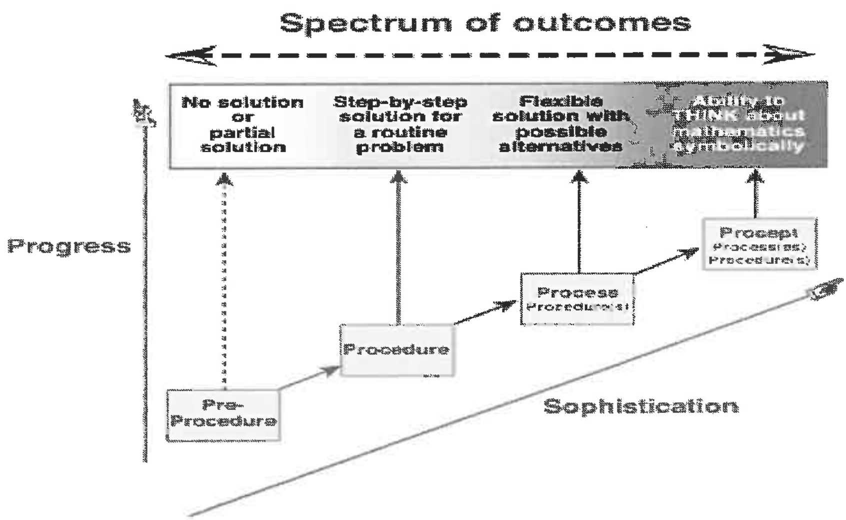

Figure 1. Spectrum of increasing symbolism compression.

DECIMAL EXPANSION OF RATIONAL NUMBERS: STUDENTS' AND PROSPECTIVE TEACHERS' DIFFICULTIES

The decimal expansion of a rational number is finite (non zero) or infinite and periodi However, every rational number with finite decimal expansion has also an alternati infinite decimal periodic expansion with period 9. Previous studies (e.g., Tall Schwarzenberger, 1978; Mamona, 1987; Edwards & Ward 2004; Artique, 200 Dubinsky, Weller, Mc Donald & Brown, 2005a,b; Giannakoulias, Sougoul Zachariades, 2005) have indicated that this type of decimal expansion gives rise students' difficulties and misconceptions and suggested possible sources for these.

In particular, Tall and Schwarzenberger (1978) attribute the difficulty of first-ye university students in comparing '0.999…' with 1 to a) a lack of understanding of th concept of limit, b) misconceptions of the representation '0.999…' seen as long b finite sequence of 9, c) students thinking in terms of infinitesimal and d) a ration number having two different decimal representations. Artique (2000) found th university students who recognize the density of real numbers sometimes reconcile th property with the existence of numbers just before or just after a specific real number, view that explains their difficulty in understanding the nature of '0.999…'. Working i the same area, Giannakoulias, Sougioul and Zachariades (2007), investigating th difficulties in understanding real numbers encountered by students at the beginning their first university semester (all with mathematics major in A-level studies), found th about half of them contradicted themselves considering that'2.999…' is less than 3 an there is no number between these two numbers.

In their own study Li and Tall (1993) found first year trainee mathematics teache (with nominal minimum A-level grade C in mathematics) who agreed that '1/9 0.1+0.01+…' but not that '0.1+0.01+… = 1/9'. Their explanation was that the quotie of 1 by 9 is initially 0.1, then 0.11 and so on, while the terms 0.1, 0.01, … could not b added together to give 1/9 because the procedure is potentially infinite. Edwards an Ward (2004) argued that students accept more easily the equality '1/3 = 0.333...' tha the equality '1 = 0.999...' because '0.333...' is the quotient of the division 1:3, wherea '0.999...' is not quotient of any division. Much along the same lines Mamona (198 found that decimal numbers with infinite digits are sometimes seen by students bein prepared for their university entrance examination (studying A-level mathematics) n as specific numbers, the equality '0.999... = 1' is viewed as incorrect because "the lim cannot be reached in practice", while every decimal number with infinite digits i considered to be an irrational number. Finally, Dubinsky, Weller, Mc Donald an Brown (2005a, b) relate students' difficulty in conceiving the nature of '0.999...' wit historical arguments related to the potential and actual infinity.

A common students' misconception identified in the above studies is that the infinit periodic decimal expansion of a rational number is viewed not as another expansion the rational number but as representation of a number just before it. These studies tak for granted that students consider this representation to stand for a number. The wor presented here is part of a study examining prospective mathematics teachers' makin sense of the mathematical nature of the representation at hand.

METHODOLOGY

The sample of this study consisted of 114 third year students of a mathematics department, who attended an optional module on the Teaching of Calculus as part of their (four years) BSc course. In the context of the module, students were invited to explore issues related to teachers' mathematical knowledge and its relation to teaching quality via a series of tasks, like, for example, analyzing real and hypothetical lesson extracts, teacher interviews, lesson plans, and so on. Students' discussions and individual as well as collective reflections (spoken or written) articulated in a variety of occasions related to tasks evolving during the course were regularly transcribed and studied, sometimes feeding into follow up interactions with the students.

The task which provided the data for the present work was completed in the context of an examination paper and described a hypothetical teaching scenario with the following structure: reflecting upon the learning objectives within a mathematical problem (and solving it); interpreting flawed (fictional) student solution(s); and, describing, in writing, feedback to the student(s). For a more elaborate description of the theoretical origins of this type of tasks see Biza, Nardi and Zachariades (2007). The actual teaching scenario had as follows:

The Task: A final year secondary school teacher gave the following question to his students: 'Which is the meaning of the representation 0.3999... (infinite number of 9)'? Four students gave the following answers:

(i) *Student A:* The representation '0.3999...' means a process that tends to 0.4,

(ii) *Student B:* '0.3999..'. is a number that tends to 0.4,

(iii) *Student C:* '0.3999...' is the number just before the 0.4, and

(iv) *Student D:* The representation '0.3999...' is the sum 0.3+0.09+0.009+... but, as it continuously increases, it cannot be equal to a number.

(a) What could be the teacher's goal in asking this question?

(b) Comment each student's answer according to his/her thought process, which are the positive and the negative points in his/her view and which are his possible misconceptions (if there are any), and

(c) If you were a teacher in this class, how would you help these students to overcome the misconceptions you identified?

The analysis of the participants written responses to the questions of the task aimed to explore their: a) views with respect to the representation '0.3999…' through their reflections on a range of related views of hypothetical students built on the findings of the relevant research and b) didactical and pedagogical management to help hypothetical pupils understand this subtlety and overcome their possibl misconceptions. The present study focuses solely on (a).

Based on the dominant emergent view regarding the nature of the symbol '0.3999…', a documented by extracts identified in various parts of each individual's answers, a attempt was made to classify the teachers of the sample along the four levels o sophistication of the model suggested by Gray and Tall (2001). To this purpose, th following criteria were utilized:

Pre-procedural: participants who give incoherent or irrelevant responses.

Procedural: participants whose response indicates the adoption of only one procedura view among the following: (a) '0.3999... = 0.3+0.09+0.009+…' but not 0.3999...= 0.4 (b) '0.3999...' signifies the sequence 0.3, 0.39, 0.399 … (c) '0.3999...' tends to 0.4

Process: participants whose answer incorporates more than one of (a), (b) and (c above, all having though the same effect.

Procept: participants whose answers indicate that they understand 0.3999…. as the limi of the sequence 0.3, 0.39, 0.399,… also as the infinite sum 0.3+0.09+0.009+… whicl is equal to 0.4.

RESULTS

Based on the criteria described in the methodology section, the distribution of th prospective teachers of the sample along the spectrum suggested by Gray and Tal (2001) is a presented in Table 1 below.

Table 1

Perspective Teachers' Classification

Pre-procedure	**Procedure**	**Process**	**Procept**
22 (19%)	53 (46%)	27 (24%)	12 (11%)

From Table 1, a notable number of the participants appear to be at the pre-procedura stage (poor or irrelevant responses). A characteristic example of this category is th answer provided by student S14, who wrote: "Teacher's aim was to see if the pupils ha understood the notion of integration and convergence […]. Student's A problem is tha he does not check the derivative and the continuity […]. Student B had to checl monotonicity, continuity and of course the limits … Student D seems to use the notio of integration, that is."

About half of the participants (46%) are at the procedural stage, often exhibiting intuitive or empirical approaches. Some of the responses were quite straightforward, as for example, the one provided by S66 who, commenting on the four students' answers and arguing about how he would help them wrote: "Student A seems to have understood what he has been taught and thus he does not misconceives concepts, neither confuses them with pre-existent [...]. For students B and D, I would try [...] to help them understand that a representation of the type '0.3999...' is not simply a number but an increasing process continuously approaching (infinitely) 0.4. For student C, I should support him understand [...] that a process '0.3999...' would never be a number precisely before 0.4". Another participant in the same group, S72, also commenting on the students' answers, wrote: "Student A sees it as a process, because he feels that it lasts, since 9s are infinite. Student B views it simply as a number very close to 0.4. Student C sees it as the number just before 0.4, disregarding that, even if it was the number just before 0.4, there would be between them another number. Student D is the most right of the four. What he describes is exactly what happens".

The responses of part of the participants in this group indicated a view of '0.3999...' as a 'variable number'. For example, students S13 and S23 wrote respectively: "Number '0.3999...' can go infinitely close to 0.4, but it will never reach it [...]. This number grows continuously, but this is not the reason why it doesn't become 0.4". Another participant, S77, tried to clarify what is this variable number using the real number line, arguing "The number 0.3999... is something different from the expression '0.3999'. It is a 'no fixed mathematical entity', such as the number π=3.14..., which is difficult to represent. If we said that number '0.3999...' is specific, we would be able to see between 0.39 and 0.41 on the real number line. But this is a 'trap' which is easy to fall in, if we do not understand that the decimal expression '0.3999...' is not exactly a number but a (mathematical) amount continuously changing (growing). That is, there is a formula inside this representation according to which it varies, such as the sum '1+2+3+...' or, more specifically, the sum '0.3+0.09+0.009+...'.

About 1 to 4 (24%) of the participants was found to provide an answer denoting a process approach to the issue at hand. That is, these prospective teachers understood the representation '0.3999...' as indicating more than one procedure having though the same effect. A typical example of these participants is S51 who expressed this view commenting the answers provided by students B, C and D: "A serious misconception of students B and C is that they think of '0.3999...' as a number. They do not think that '0.3999...' expresses various numbers depending on the number of times that the digit 9 appears [...]. I would indicate to these students that the representation '0.3999...' can take various values, according to the number of times that the digit 9 appears. I would draw some of the numbers of the above procedure in order to help them notice how close the procedure comes to number 0.4, without, though, ever becoming equal to it [...]. Student D very rightly states that the representation '0.3999...' cannot be equal to a number, but escapes him that this representation is infinitesimally close to number 0.4". Another example of this group is S8, who, interpreting students' A and B views wrote: "Student A knows that the representation '0.3999...' is a number which, if rounded, will give us number 0.4, but it will not be equal to it, they will not be the sam [...]. Student B rightly accepts the representation as a number which is very close t 0.4, tends to it [...]. He understands that '0.3999...' is a number very close to 0.4 comes close to it, it is almost the same with it, but they are different and not equal t one another".

It should be noted at this point that some interesting responses classified under th 'process' category led to a thorough reading of the arguments raised within them by a three researchers, finally resulting to the recognition of the need to allow for a group c participants within this particular group to be identified as formulating a distinct sub group. The participants making up this sub-group provided responses incorporating th view expressed by those in the "process|" group, but also suggesting that the responder viewed '0.3999...' as a number, although rather vaguely and not in a completely correc manner. We argue that these prospective teachers are in a kind c transitional/intermediate stage from a process to a procept facet of the spectrum.

The analysis showed that about 1 to 10 (9%) of the participants can be seen a belonging to this sub-group. Participant S79 is a typical example of this sub-group. H wrote about student's A answer: "It is possible that student A thought of a procedure c determining '0.3999...' with progressively greater accuracy, that is, like the sequence c numbers '0.39, 0.399, 0.3999, 0.39999, ...'. He noticed, then, that this numbe approaches number 0.4 and arrived at his conclusion. It is positive that the studen 'grasped' the concept of asymptotic approach [...]. That is, he understood that in thi infinite procedure (sequence of infinite terms), each new number, each next step, bring him closer to 0.4, tends to become 0.4 and he knows that it will never become equal doesn't matter how many steps are perfomed. I see no misconception from the student' part." However, he noted in his response: "Number '0.3999...' is a real number an holds a specific position on the real numbers line. I cannot move it and bring it close and closer to 0.4. It is an irrational but real number. It is not a set o numbers." Anothe participant, S32, commenting on the positive points of students' B and C answer argued: "It is positive for student B that he faces '0.3999...' as a number and also that h apprehends its 'boundary'/ limit value [...]. A positive point for student C is that h understands that 'no matter how close I come to 0.4, '0.3999...' is a number which cannot say exactly which one it is' [...]. Since ('0.3999...') is between 0.3 and 0.4 which are numbers, then it is going to be itslef a number, like all the others".

Only about 1 to 10 of the participants was classified as being at the 'procept' stage Their answers indicate flexible understanding and use of symbols. Also, includ suggestions of interesting explanations for the four students' misconceptions an indicate that the responders were aware of the different approaches and the equivalence between them. A typical example is participant S5, who, articulating his view abou '0.3999...' wrote: "0.3999... expresses a procedure which 'tends' to 0.4 and is also 0.4 [...]. The concept 'tends to' does not exist for numbers. Furthermore, the set of rea numbers is continuous. They do not have previous and subsequent numbers.". Th response of another participant, S2, who was classified in this group, disclose a dee knowledge of the structure of rational numbers. In particular, discussing student's C response, he argues: "Student C has in his mind the order of the set of natural numbers, where every number has a number just before [...]. However, this property cannot be transferred in the set of rational numbers. For example, there is rational number (3/8+4/8)/2 between 3/8 and 4/8 and we could find another rational number between 3/8 and (3/8+4/8)/2.

...Then, in rational numbers we cannot find exactly a next number". S2 goes on to suggest an interesting explanation for why student C considers '0.3999...' less than 0.4. In particular, he wrote: "Student C applies the rules of comparison learned (he first compares the integer parts of numbers, next the digits of tenths, hundredths, ...), concluding that 0.4 is bigger than '0.3999...'". In the last part of his answer, S2 proves the equality 0.3999... = 0.4 and explains to student D that "every geometric progression $\sum_{n=0}^{\infty} a^n$ with $|a| < 1$ converges".

DISCUSSION AND CONCLUSIONS

The results show that, overall, the student teachers of the sample are distributed to all four groups of the spectrum suggested by Gray and Tall (2001) regarding the way they conceive the notation '0.3999...'.

In particular, almost 1 to 5 students argue without making sense as to the meaning of this representation. About 1 to 2 students exhibit a uniquely procedural view of it, that is, they tend to see this representation as a specific procedure, failing to understand it as equivalent to other procedures of the same effect. Some of these students claim this procedure to be the sequence 0.3, 0.39, 0.399,... and others the infinite sum 0.3+0.39+0.399+..., without being able though to see either as equal to 0.4. There are also students of this group who understand this representation not as a specific but as a variable number.

Almost 1 to 4 of the perspetive teachers conceive the representation '0.3999...' as a process, i.e., they view it as designating more than one procedures of the same effect. A noticeable subset of this group appear to be in a transition stage between the "process" and the "procept" conceptualizations. They are capable of identifying not only more than one equivalent procedures but also the concept of number embodied in this particular representation, but in a vague if not incorrect manner.

Only few students, just a little more than 1 to 10, understand the given representation as a procept, a noteworthy finding bearing in mind that the subjects of this study had a strong mathematical background (at least theoritically). These were university mathematics students in their third year of studies, who had been taught an advanced course of calculus, where they were involved with tasks dealing with the decimal expansion of real numbers as a series as well as with the alternate expansion of rational numbers of finite digits as infinite periodic number with period 9. Despite this, notably few appear to hold an accurate understanding of this particular infinite decimal representation of a rational number, whereas a significant number view it only as a procedure or articulate an incoherent or irrelevant view about it. The three dots '.. appear to mainly direct to a process and not to a number. We believe that the teachin scenario exploited for the collection of the data methodologically facilitate the abov conceptions spectrum to emerge mainly by discouraging students to view '0.3999... only as a number.

The findings of this study draw a picture that does not differ substantially from the on emerging for high school students according to the relevant literature and could b attributed to both the difficulty of the idea itself and the poor way is taught over th span of the school mathematics curriculum.

With respect to the difficulties inherent to the meaning of the notation, in the earl history of the development of the mathematical ideas and at an elementary stage th number was seen as the result of counting. Later the number was understood as th result of measurement. And even later constituted a specifically defined element in a se of numbers. In a similar course of development, the representation '0.3999...' i initially conceived by pupils, via a construction procedure (written or mental), as th sequence '0.3, 0.39, 0.399, ...'. However, no measurement exists that could result t '0.3999...' and the rational numbers are defined as the result of the division of intege numbers. But there is no division resulting in '0.3999...'. Thus, it becomes difficult fc students to distinguish between the result of infinite procedures (i.e., the number) an the procedures themselves. It seems that '0.3999...' is a theoretical mathematica construction.

As to the way in which the knowledge related to the notation '0.3999...' is approache in school and university mathematics, the fragmentation, the limited emphasis o relationships between mathematical topics or between advanced and elementar mathematics and other curricular, social and cultural factors encapsulate this knowledg to low or distorted growth. As a consequence, future or practising teachers' relevar knowledge has limited opportunities to develope sufficently, thus framing negativel their own pupils' relevant learning.

It is evident that teachers should have adaptable mathematical knowledge, comprisin school mathematics, but going beyond it and relating it to the underlying advance academic mathematics. The results of the present study show that school and universit experiences of many prospective and practising teachers in mathematics are of limite type, creating reflexes that are difficult to change and failing them in acquiring a deepe mathematical knowledge to dismantle school-related misconceptions and solv mathematical problems competently (Ruthven, 2011). Providing teachers with adequat learning opportunities for the mathematics taught to be acquired and understood i breadth and depth is a challenge waiting to be taken over by teacher education an professional development programmes.

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Stories as a Springboard for Multiple-Solution Mathematical-Tasks in a Technological Environment: The Case of Equality

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ABSTRACT: *Multiple solution tasks have the potential to promote mathematical creativity among students. Three problem situations embedded in stories were purposefully developed on the basis of previous studies, and presented to one dyad in a technological environment. The activities shared the underlying mathematical concept of equality, a key idea for developing algebraic reasoning among young children. The study examined the interaction between the two first-grade students while they were engaged in the process of solving the problems, and analyzed their construction of knowledge and their mathematical creativity. The unique environment provided by Microsoft® PowerPoint™ and Virtual Manipulative software enabled students to think about different solutions as well as different solution methods. Our findings show how students make the transition from pictorial representation to numeric-symbolic representation as well as how they develop abstract concepts and generalizations. The tasks helped to expand the children's mathematical knowledge. The unique role of the technological environment in promoting creative mathematical thinking is discussed.*

Key words: *Creative mathematical thinking, Knowledge constructing, Technological environment, Stories.*

INTRODUCTION

Young children have the potential to learn complex mathematical ideas (Sarama & Clements, 2009). Engaging in mathematical tasks during early childhood helps children learn mathematics and deepens their understanding (NAEYC, 2002, 2006; National Mathematics Advisory Panel, 2008). "Students' understanding of mathematical ideas can be built throughout their school years if they actively engage in tasks and experiences designed to deepen and connect their knowledge" (NCTM, 2000, p. 21). Visual computerized manipulatives may promote knowledge construction regarding core mathematical ideas (Clements & Sarama, 2007). Children's mathematical knowledge before entering school is a predictor of academic success in elementary and high school (Duncan, Dowsett, Claessens, Magnuson, Huston, Klebanov, et al., 2007).

Introducing arithmetical ideas through stories and pictures may provide children background and experience with mathematical ideas (Ginsburg & Seo, 1999) and meaningful context for learning mathematics (Murphy, 2000; Welchman-Tischle 1992). Stories and pictures have the potential to create enjoyable experiences along wit emotional and cognitive involvement (Scribner & Cole, 1973) that can motivat learning mathematics (Hong, 1996; Usnick & McCarthy, 1998).

Working with multiple solution problems plays an important role in developing creativ mathematical thinking (Leikin, 2009). Solving problems using different methods wa found to be a tool for assessing creative mathematical thinking (Karp & Leikin, 2011 Leikin & Berman, 2010; Tabach & Friedlander, 2013) and for promoting creativ mathematical thinking in kindergarten children (Tsamir, Tirosh, Tabach, & Levenson 2010).

The current study refers to the concept of equality, which is one of the core ideas i algebraic thinking. "The notion of equality also should be developed throughout th curriculum… they should come to view the equals sign as a symbol of equivalence an balance" (NCTM 2000, p. 39).

The current case study aims at developing children's creative mathematical thinkin regarding the notion of equality by using stories as springboards to multiple solutio tasks embedded in a technological environment. The study closely traces one pair c first graders as they collaborate in a computerized environment on three activitie involving the creation of equality. The activities have multiple solutions that stem fror stories in rhymes and pictures. The analysis traced how the children constructe knowledge while working together and identified the levels of creative mathematica thinking observed. The study adds to the literature by exploiting the use of technolog in multiple solution tasks to facilitate learning and allow for more creative solutions b children.

LITERATURE REVIEW

The current study refers to four research domains: mathematical creativity; th development of mathematical abilities; stories as a medium to promote mathematica learning; and technological environments for early mathematics education amon children.

Mathematical Creativity

Creative mathematical thinking can be thought of as the ability to solve problems o develop structural thinking while referring to the logical-deductive nature of a domai that has connections to mathematics (Ervynck, 1991). Ervynck identifies three levels o creativity while working on a novel problem or concept: low, high and very high.

Two thinking processes that play a major role in problem solving have been identifie as a basis for creative thinking (Guilford, 1973): convergent thinking and divergen thinking. Convergent thinking takes place while the solver strives logically to find solution to a problem. In other words, the solver seeks to understand the logica connections among knowledge elements in the problem and to apply the relevant standard algorithm. Divergent thinking is relevant in cases marked by several strategies. Guilford (1973) claimed that divergent thinking is related to creative mathematical thinking. A student's ability to solve one problem in several ways and to find different solutions is considered to be an aspect of creative mathematical thinking.

Creative mathematical thinking can be analyzed along four dimensions (Guilford, 1973; Torrance, 1969). The current study examines these dimensions with respect to children's ability to present strategies and solutions to arithmetical problem.

Fluency: The ability to provide a number of solutions for a given problem. Fluency is related to an individual's active and available knowledge with respect to a problem.

Flexibility: The ability to modify one's way of thinking regarding a given task. The solution strategies may derive from different mathematical domains or different mathematical principles.

Elaboration: The ability to expand a given idea, by refinement or by combining other ideas.

Originality: The ability to relate to a given problem in a novel way, producing solutions that are unexpected, unusual or uncommon to the problem at hand.

Mathematical activities with the potential to promote creative mathematical thinking can be used to evoke, condense and reorganize prior knowledge (Mumford, Baughman, Maher, Costanza, & Supinski, 1977). Specifically, open problems such as construction problems may elicit learning because: (a) creative production requires learners to reactivate their prior knowledge and use it to derive the necessary parts relevant to the task; (b) as learners make and verify conjectures, they actively construct their knowledge; and (c) while evaluating their own conjectures and reorganizing their knowledge, learners also develop a general problem-solving strategy. Ervynck (1991) claims that creative mathematical thinking allows individuals to build a schema from which they can create an unlimited number of connections that can lead from one point in the schema to any other desirable target.

Silver (1997) refers to creative mathematical thinking in relation to problem solving. He claims that creative mathematical thinking can be developed by working on relevant mathematical activities. The use of open-ended problems with different solution methods or solution outcomes may provide a learner valuable experience in analyzing and creating new solutions on the basis of known solutions. While engaging students in such problems, teachers may influence their students' thinking processes toward becoming flexible, original, novel, and fluent. Silver et al. (2005) considers creative mathematical thinking to be relevant for all students, not just for those who are gifted.

Developing Mathematical Abilities and the Notion of Equality

Several mathematical abilities outlined in the elementary school curriculum are relevant to the current study (Israel National Mathematics Elementary School Curriculum, 2006):

Using mathematical tools to solve word problems: Solving word problems calls fc careful reading and understanding of a text. Word problems are phrased in everyda language and hence are as rich as the language itself. Students can build a mathematic model using pictorial, representational or other manipulatives (p. 12).

Concrete representations of mathematical situations: Most elementary school student exhibit concrete thinking. Most are able to generalize in order to derive one thing fror another. Students who reflect on their own actions are able to achieve an abstract leve of mathematical concepts (p.13).

Understanding and connecting concepts: This ability refers to the elements c mathematical thinking, such as drawing conclusions while reflecting on what wa learned (p. 14). For example, students can observe connections between even and od properties of addends and their sum.

The notion of equality is central to mathematics in general, and to the current study i particular. For several decades scholars have emphasized the concept of equality as core idea for learning algebra (Behr, Erlwanger, & Nichols, 1975; Erlwanger & Berlanger, 1983; Falkner, Levi & Carpenter, 1999; Ginsburg, 1997; Kieran, 1992 Saenz-Ludlow & Walgamuth, 1998). Elementary school students see the equal sign a an invitation to calculate, e.g. *2+3=* means "*add up the numbers and write the sum o the right hand side of the equal sign.*" That is, the equal sign is perceived as an operato rather than as a relation marker (McNeeil & Alibali, 2005). Hence, students ma misunderstand number sentences representing mathematical relations and arithmeti equivalence (Falkner et al., 1999). Moreover, a lack of understanding of the relationa aspect of equality may result in future difficulties in solving equations (Kieran, 1992 McNeeil & Alibali, 2005).

Stories as a Medium for Promoting Mathematical Learning

Reading picture books is considered important for children's development, especiall for language and literacy (Anderson, Anderson, & Shapiro, 2005). In the last tw decades, tying children's literacy to learning mathematics has become increasingl popular (Haury, 2001). Children's literacy may help motivate children's learnin (Usnick & McCarthy, 1998), connect mathematics with emotional aspects (Griffiths & Clyne, 1991), and raise children's interest (Murphy, 2000). Children's literacy ha proven to provide a meaningful context for learning mathematics (Braddon, Hall, & Taylor, 1993; Lovitt & Clark, 1992; Murphy, 2000; Schiro, 1997; Welchman-Tischler 1992).

Stories and pictures contribute to developing children's attitude toward mathematics Children may demonstrate a mathematical way of thinking, inventiveness and creativ ability (Griffiths & Clyne, 1991). Hong (1996) provided evidence that children' mathematical thinking and performance take place on a higher level in the presence of supportive story that they can relate to and make sense of.

Technological Environments for Early Mathematics Education among Children

Providing children with access to and experience with technology to foster their mathematical thinking is highly recommended (NAEYC, 2002, 2006; NCTM, 2000).

Today one can find many web-based applets for learning mathematics. A sub-set of these can be considered a virtual manipulative (VM): "*A virtual manipulative is best defined as an interactive, Web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge*" (Moyer, Bolyard, & Spikell, 2002, p. 373). Researchers point to several benefits of VMs: (a) are free and easily available via the Internet (Clements & McMillen, 1996; Leathrum, 2001; Moyer & Bolyard, 2002); (b) focuses on specific concepts (Leathrum, 2001); (c) allow children to do things that are not possible or hard to do with physical manipulatives or with paper and pencil (Clements & McMillen, 1996; Reimer & Moyer, 2005); (d) from the pedagogical perspective, VMs may provide immediate feedback and guidance towards corrections (Clements & McMillen, 1996; Durmus & Karakirik, 2006; Jacobs, 2005); (e) VMs may simultaneously provide more than one representation for the same concept (Clements & McMillen, 1996; Moyer & Bolyard, 2002; Suh & Moyer, 2005); (f) VMs may promote generalization (Clements & McMillen, 1996; Durmus & Karakirik, 2006; Jacobs 2005; Moyer & Bolyard, 2002; Suh & Moyer, 2005); (g) VMs are flexible and efficient, and increase learners' motivation and attention (Clements & McMillen, 1996; Leathrum, 2001; Reimer & Moyer, 2005); (h) VMs are appropriate for learners at different levels (Miller, Brown, & Robinson 2002; Riley, Beard, & Strain, 2004).

THE STUDY

Research Questions

The current case study traces one pair of first graders as they collaborate in a computerized environment around three activities involving the creation of equality. The activities have multiple solutions and were presented via a story in rhymes and pictures. The two related research questions are as follows:

1. What processes of knowledge construction took place?
2. What level of creative mathematical thinking can be identified?

Research Population

Two first-grade students, a boy named Eli and a girl named May, participated in the study. Both children had high verbal and mathematical abilities. They did not attend the same school, but they knew each other through family connections. The children and their parents gave their consent to participate in the study. The mathematical content learned at school during the data collection phase of the study included addition and subtraction of numbers from 1 to 20 without breakdown into tenths and acquaintance with natural numbers up to 100.

Research Environment

Microsoft® PowerPoint is a promising software for developing and supportin classroom activities in the domain of literacy (Hourcade, Parette Jr., Boeckmann, & Blum, 2010). This study found that young learners were able to achieve a high level c reflection on their own work, express their reflective thoughts about the experience, an express themselves explicitly in a way that was not observed with othe communicational means.

The flexibility of the Microsoft® PowerPoint software allows educators to embed visua pictures according to the needs of their students and of the curriculum. By doing sc educators can provide important support in literacy studies for young learners. Student become engaged and active while learning in the Microsoft® PowerPoint environmen as they interact with each other, with the teacher, and with the software (Parette, Blur & Watts, 2009).

The current study used Microsoft® PowerPoint as a medium for using stories in rhyme and pictures to introduce three mathematical activities designed for the current study The three activities are described in detail in the next section. The PowerPoir presentation was in the design mode, thus allowing the children to move pictoria objects (See Figure 1 and Figure 2b as examples).

In addition, the second and third activities involved showing the children a VM in th shape of an interactive dynamic balance scale[1]. Using this VM the children are able t enter sentences representing numerical expressions sentence on each pan of the balance The feedback includes the resulting number after the calculation on each pan i performed, as well as a visual display showing the larger result number, represented b a lower (heavier) pan. In addition, if the number-sentences on each pan are equal, thes balanced equations appear automatically on the right-hand side of the screen with a equal sign between them. This allows the user to keep track of number sentences tha turned out to be equal in their previous trials, enabling them to reflect on their action and perhaps to generalize.

Research Tools

Each of the three activities begins with a story taken from a children's literacy book The children listen to the story up to the point at which it digresses into the world o mathematics. The underlying mathematical structure in each of the three activitie involves two given sets with unequal numbers of objects, and the children are asked t create two equal sets of objects.

In the first activity (following Tsamir et al., 2010), two sets are given, with an od

[1] Pan Balance – Numbers, http://illuminations.nctm.org/activitydetail.aspx?id=26

number of candies each. The children are asked to share the number of candies equally between two koalas. No extra candies are available (no stockpile is provided). The sets of candies are presented visually, and the learners can move the candies from one set to the other. In the second activity, two cookies are given, one with an odd number of raisins and the other with an even number of raisins, along with an unlimited stockpile of additional raisins. The sets of raisins and the stockpile are presented visually, and the learners can move them to create an equal number of raisins on each of the cookies. In addition, the children may create a numeric-symbolic representation of the situation by using the Pan Balance VM. The third activity begins with two sets, each with an even numbers of dragons, and an unlimited stockpile. Note that the pictorial representation is suitable mainly for work with relatively small numbers. The Pan Balance VM, on the other hand, has no such restrictions. Table 1 summarizes the conditions of the three activities.

Table 1

Mathematical Structure and Representation of the Three Activities

Activity	First set	Second set	Stockpile*	Pictorial representation	Symbolic representation
1	3 candies	5 candies	-------	Given	---------
2	5 raisins	6 raisins	Given	Given [leading representation]	Given
3	4 dragons	6 dragons	Given	Given	Given [leading representation]

* A stockpile is a "bank" containing an unlimited number of extra objects, in our case raisins or dragons

Data Analysis

For each activity, the expected knowledge construction while the students engaged in each activity was analyzed a priori in view of the expected construction. Table 2 presents the expected knowledge construction for each activity.

Likewise, the evidence for creative mathematical thinking in each activity underwent a priori analysis. The students' work was analyzed in light of these creativity criteria. Tables 3, 4, and 5 show the scores indicating creative mathematical thinking with respect to fluency, flexibility and elaboration, respectively. The decision about the levels in each category was inspired by Leikin and Levav-Waynberg (2008). The authors reached agreement regarding the levels and also consulted with another mathematics educator who is an expert in creative mathematical thinking.

Table 2

Expected Knowledge Constructions for Each Activity

Activity	Main knowledge elements
1	There is more than one solution and strategy to solve the problem.
	The parity of the total number of candies and the possibilities for making pai are connected and related to the solutions.
	The pictorial representation of the situation is transferred to a numeric symbolic representation.
2	Solution represented via numerical sentence.
	Solution represented via verbal description.
	Students understand that the range of solutions varies between zero an infinity.
	Problem connected with previous problem.
3	Verbal problem represented via numerical sentence.
	Students understand that the range of solutions varies between zero an infinity.
	Problem connected with two previous problems.

Table 3

Fluency – Number of Correct Solutions

Activity	Low level	High level	Very high level
1	0 to 1	2 to 3	4 or more
2	0 to 5	6 to 9	10 or more
3	0 to 5	6 to 9	10 or more

Table 4

Flexibility – Number of Correct Solution-Methods

Activity	Low level	High level	Very high level
1	0 to 1	2 to 3	4 or more
2	0 to 2	2 to 3	4 or more
3	0 to 2	3 to 4	5 or more

Table 5

Elaboration – Level of Solution Characteristics

		Low level	High level	Very high level
	Activity 1: Create solution...	using both sets	[using both sets and one set] or [no set]	using both sets, one set and no set
Activities 2, 3	(a) numbers	Up to 10	up to 20 with no converting	with converting or more then 20
	(b) operations	addition or subtraction	addition and subtraction	multiplication or division
	(c) # of operations	Up to one operation on each side of the equation	one side of the equation with two operations	more than two operation

Originality was analyzed based on the number of original solutions found. If no original solution was found, originality was considered low, one original solution was high, and more than one original solution was very high. The solution (0,0) was considered original for all three activities, as the notion of an empty set is known to be counterintuitive even for mathematics teachers (Linchevsky & Vinner, 1998). A solution with fractions was considered original for the first activity due to the young age of the participants. For the second and third activities, mentioning the unlimited option was considered original for the same reason.

Data Collection

Each of the three activities took place on a different day, in a home environment. The children worked together on one computer. The first author acted as facilitator by helping the children read the story and by providing technical support. The activity was videotaped and transcribed verbatim. For all three activities, the PPT file that the children worked on was saved. During the second and third activities, screen shots of the VM were saved as well.

FINDINGS

This section is organized by activity. First, each activity is described, along with the children's main actions. Next, the creative mathematical thinking observed during the activity is analyzed, followed by an analysis of the knowledge constructed by the children.

First Activity

Activity Description

The activity began with the facilitator reading the story. The children were familiar wit the story, and they laughed and completed the last word in some sentences based on th rhymes and their knowledge of the story. When reaching the point where the stoı digressed into mathematics, the facilitator made sure the children understood the task providing the same number of candies for each koala.

Eli immediately suggested *"four and four"* [30, with a questioning tone] and Ma repeated after him in a confirming tone *"four and four"* [32]. The facilitator showed th children the PowerPoint slide with the two bags of candies and the two koalas an explained how to move a candy (Figure 1). Eli moved four candies from one bag to on koala, and handed the mouse over to May to move the other candies.

The facilitator asked if they could find another solution, and Eli said "*I don't think sc* [54]. May did not say anything, but on the slide she moved three candies to one koal and then moved three other candies to the second koala. Only after she finished movin the candies did she and Eli both say, "*three and three*" [62]. The facilitator showed th children how to move between the slides and asked if they could find another solutioı Eli took the mouse and moved two candies to each koala, saying "*two, two candies* [98].

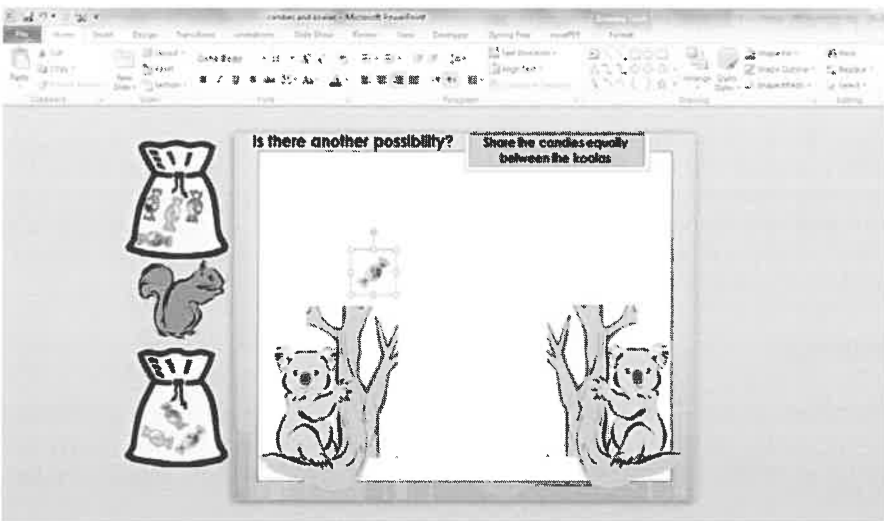

Figure 1. The first activity.

The facilitator repeated the question: "*Can you find another solution?*" May suggested "*four and four*" [104]. Eli replied "I think we already did that one" [106]. The facilitator encouraged May to look at the previous slides to see what answers they had already suggested. May did so, and then suggested a new solution – "*one and one*" [114], and demonstrated it on one slide. The facilitator again asked for another solution and Eli immediately replied "*Wow, each can have zero candies*" [122]. The following discussion took place:

Excerpt 1: First activity, zero as a solution

Turn	Speaker	Utterance
124	May	Is zero also a number?
125	Facilitator	Zero, zero candies, if we give each one zero candies, how many will each one have?
126	Eli	The same.
127	Facilitator	The same, right.
128	Eli	He got nothing, no candies.
129	Facilitator	He got no candies, right!
130	May	Yes, they got the same number.

May seemed not to question the legitimacy of zero as a possible solution; rather, she questioned whether zero is a number. If zero is not considered a number, it cannot serve as a legitimate solution to the problem. (A similar attitude toward zero is a well reported phenomenon, e.g., Levenson, 2013). May eventually accepted the idea of each receiving zero candies as a solution to the problem. It is not clear, however, whether she also accepted zero as a number. It is quite clear that for Eli, who suggested the idea of (0,0), the idea is elaborated through talking with the facilitator.

After the children claimed to have found all the solutions, May explained that "*there isn't another candy*" [171]. To the facilitator's question "*and what would you do if you had another candy?*" [172], May suggested "*to divide it in half*" [173]. This is an unexpected solution given the young age of the participants and the representations of the candies as discrete objects.

The facilitator now directed the activity in a new direction – trying to look at the numbers and what is special about them. The children saw nothing special about the numbers. In the next couple of slides the children tried to organize different numbers of candies in pairs. They soon associated the candies that can be paired with even numbers and decided that if one candy is left alone, the number is odd.

Next the facilitator encouraged the children to write a numerical sentence in the form of 3+5=8 and also to generalize that an odd number plus an odd number yields an even number. She then asked them to consider other sums, and they suggested that an eve number plus an even number yields an even number. It is not clear whether thos generalizations were based on the given number sentences or on an understanding c why two odds make an even.

Later in the discussion the facilitator raised the following question: "*How can we tell a number is even or odd?*" May answered that if we group the number of candies i pairs and see we are left with a candy, the number is odd.

The activity ended after 55 minutes, with the children signing their names on the la slide and saving the file. The children claimed they had fun and wanted to know whe they would meet again and do another activity.

Analysis

According to our criteria for creative mathematical thinking (Tables 3, 4 and 5), th children exhibited a very high level of fluency, finding six different solutions to th problem. Yet it is much less clear whether they distinguished between the differer solution methods. They seemed to relate to both bags of candies as one united pool c candies, dragging candies as they needed, always starting from the upper bag tha contained five candies. This may be a result of the design of the PowerPoint slides, i which the two bags appeared one above the other, with the upper bag containing mor candies. Hence, the level of flexibility is considered to be low. The elaboration level i high and the originality level very high. The children not only found the (0,0) solutior but also came up with the idea of dividing a candy in half.

Concerning the knowledge targeted by the activity (Table 2), it seems that the thre knowledge elements were achieved by the children, at least to some extent. The childre found more than one solution to the problem, though it is less clear whether the employed more than one solution method. They seemed to make a connection betwee parity of the numbers and whether a number is odd or even. Further, there is som evidence that May understood that in a specific case of an odd number of candies, th solution must include half a candy. It is not clear, however, if Eli understood this idea c if this idea was generalized. The children did move successfully from a pictoria representation of the story to representation via a symbolic-numeric sentence.

Second Activity

Activity Description

The second activity began with the children reading the story. When they reached th point where the story digressed into mathematics, the facilitator made sure the childrer understood the task: how to place an equal number of raisins on each cookie. Th facilitator encouraged the children to clearly state the problem, and they wrote it dowr on one slide: "*Each has a different number of raisins. They want to have an equa number, the same*".

Next the children suggested a solution – dividing the extra raisin in half and putting each half on one of the cookies. Note that the same line of thought was evident in the first activity when the children tried to justify their claim about finding all the solutions and considered what would happen if they could use one more candy (see Section 4.1.1, 171-173). The facilitator acknowledged the children's suggestion as a solution, but then called their attention to a different issue by asking whether they could recall having encountered a similar problem before.

The children claimed not to recall a similar problem. After the facilitator repeated this question for the third time, May suggested the following story: "*One girl had two schnitzels, and her friend had one, so the first girl can cut one schnitzel and then they will both have the same.*" Note that May suggested a story based on an everyday situation. She kept the difference between the numbers to one, so in that sense her story and her provided solution were indeed similar to the problem of the number of raisins on the cookies.

The facilitator acknowledged this similarity and opened the PowerPoint presentation the children had saved from the first activity. As she showed the slides, she asked the children to recall the story and the question from the first activity. Although the children were able to recall the problem from the first activity, they still could not see the connection to the present one, as evident by Eli's statement: "*But why did you bring it up?*"(227). The facilitator kept repeating the conclusions drawn during the previous activity about the parity of the numbers and their sum. Next, the facilitator drew the children's attention to the numbers in the current situation, five and six. The facilitator tried to point out the difference between the current case of a total odd number of raisins and the previous case of an even number of candies. Yet this was not clear to the children.

Then the children were shown how to use the Pan Balance VM. The facilitator instructed the children to write the number 5 on one pan, representing the five raisins on the first cookie, and to write the number 6 on the second pan to represent the second cookie. Eli suggested adding 3 raisins, and the facilitator asked how many they would have to add to make the other side equal. The children suggested adding four raisins to the first pan, in response to which the VM displayed a corresponding numerical sentence "5+4=6+3" (Figure 2a). The children also added the raisins to the pictorial representation on the slide and wrote the numerical sentence below the illustration (Figure 2b).

The facilitator asked for more solutions. The children suggested removing one raisin from the cookie with six and doing nothing with the other cookie. This was represented on the VM as "6-1=5+0". The children did not depict this solution pictorially.

To the facilitator's question of whether they had any more solutions, Eli answered "*no we don't*" [629], but as he spoke he changed his mind and called out, "oh, I do have, I do have!" The children then began working on the VM, creating the following solutions:

5-1=6-2; 6-3=5-2; 6-6=5-5; and 6-5=5-4

At this point Eli again claimed that "*there are no other solutions*" (703). At th facilitator's prompt, "*wait, think before you speak*" (704), Eli suggested 6+0=5+1. Th children then revised what they had already suggested and came up with 5+5=6+4. Thi opened up a discussion of how many more solutions they could find. Eli reported that i class they learned up to 100, and that this is as much as can be added. May tried 20(and Eli continued with 2000. This carried the discussion forward to the point wher May said 20000, and Eli stated there is no end to the numbers.

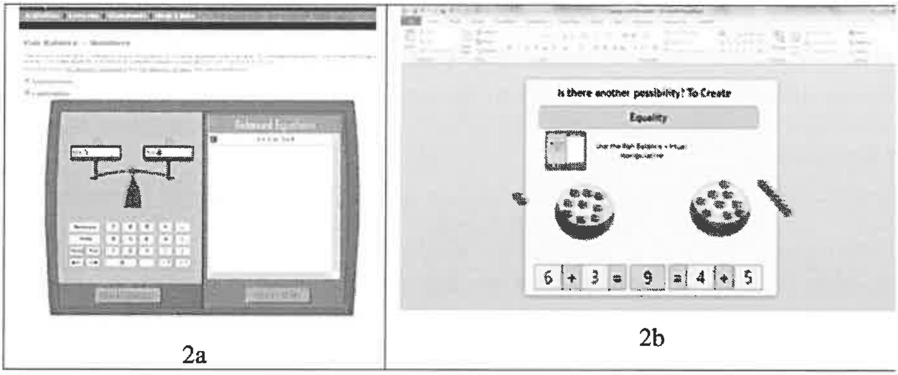

Figure 2. The second activity in two environments – (2a) VM and (2b) PowerPoint.

The children looked back at what they had already written and added three mor solutions:
5-3=6-4; 5-4=6-5; and 6+14=5+15. They also responded to the facilitator's prompt b stating that the smallest solution was zero and the largest was infinity. The childre were also able to say that the solution methods they used were either addition o subtraction.

Analysis

According to our criteria for creative mathematical thinking (Tables 3, 4 and 5), th children exhibited a very high level of fluency, finding twelve different solutions to th problem. Their level of flexibility was high, as they used two methods for solving th problem – addition and subtraction. Their elaboration level (a) on numbers was ver high, as the students worked with numbers higher than 20, and their elaboration leve (b) on operations was high, as the children used both addition and subtraction. Thei performance on the third elaboration criterion (c), number of operations, was low, as th children used up to one operation on each side of the equation. Their originality wa very high, as the children found (0,0) as a solution and also came up with the idea o dividing a raisin in half.

With respect to the knowledge targeted by the activity (Table 2), it seems that the children achieved all four intended knowledge elements. The children moved from solving with the help of both the picture and the VM to using the VM only, yielding many solutions. That is, they did not need the support of the pictorial representation. The children acknowledged the infinite number of solutions to the problem at hand, even if they actually suggested solutions only within the range of numbers they had learned in school. The extent to which the children connected between this problem and the previous problem is not so obvious.

Third Activity

Activity Description

The children and facilitator read the story together, and again the facilitator made sure the children understood the situation – two sets of four and six dragons – and the problem – creating an equal numbers of dragons. Two solutions were suggested first: adding two dragons (waiting for two baby dragons to be born) or removing two dragons from the larger set. At first the facilitator tried to make the children write a plan of action: adding or removing dragons with the aim of creating equality.

The children opened the VM and wrote the second solution, 4=6-2. Then May suggested the idea of moving a dragon from one set to the other set, first formulating it verbally and then symbolically as 4+1=6-1. To the facilitator's question, "*do you have a different solution?*" Eli replied that he had a complex idea, 4+1+1=6. However, when Eli wrote 4+1+1+1 on one pan of the VM and wrote 6 on the other pan, the pans were not balanced, He then suggested a correction, 6+1, obtaining another solution. The last solution raised the idea of 4+3=6+1.

The next idea was to "get rid" of all dragons. The children tried to express this using the VM, and wrote 4-4=6-6. Next, May wrote 6-3=4-1. Eli followed by writing 4+6=6+4. May then wrote 4+5=6+3. These ping pong solutions raised the following discussion:

Excerpt 2: Third activity: How many solutions there are?

Turn	Speaker	Utterance
515	Eli	How many possibilities there are? How many?
516	May	Ten
517	Eli	Not ten
518	May	Infinitely many
519	Eli	Infinitely many
520	Facilitator	Very good.

From the adult perspective, the idea of an infinite number of solutions had already been mentioned and discussed during the second activity. And yet for the learners, this was not necessarily the case. Also, once it became clear that there are at least sever solutions, the question of how many pointed to a mathematical analysis of the situatic and as such was welcome. Further, it seems that here the two children shared the vie of an unlimited number of solutions.

Next the facilitator asked for more solutions, and this time Eli tried a new directio using multiplication as an arithmetic operation. May was hesitant to use multiplicatio stating "*but I did not learn it in school yet*" [530]. The facilitator encouraged tl children to try, and Eli began with "*four times two*" [533]. The facilitator repeated "*fo times two, eight. And what goes on the other side?*" The two children answered togeth "*six plus two.*"

Next May proposed the solution "4-3=6-5," and Eli tried "4+11"on one side an "6*9"on the other and seemed to be surprised by the calculations. He then correcte himself to "4+11=6+9". Next May wrote "4+41", looked at the resulting number on to of the pan, and said, "*Oh, I was going to write 14, I was confused*" [598]. The facilitat stopped her and said, "*Wait, wait, so what should be on the other side? Would you lik to try with me? It is ok if we get it wrong*" [599]. May began, "*six and um ...51*". Sh wrote this and saw the scale was not balanced. The facilitator asked her whether sh needed a larger or a smaller number. May then found the correct number: 39. Next E tried a larger number, 100. He wrote 4+100 on one side and 6+99 on the other side. Th scale was not balanced, and Eli looked at the numbers and corrected his answer t 4+100 and 6+98.

The last two solutions led the facilitator to draw the children's attention to the relatio between the two addends in the case of a numerical sentence of the type "*four plu something equals six plus something else.*" After looking at and discussing th constructed examples, the children noted that the second number was "*two less.*" In th next example, "*4+60*" was on one side, and they suggested "*66 minus 2*" for the othe side. The facilitator asked how much that would be. The children tried 58 as th addends on the second pan and were happy to see a balance.

Next the facilitator asked about the smallest and largest solutions, and the childre immediately responded with zero and an infinite number.

Analysis

According to our criteria for creative mathematical thinking (Tables 3, 4 and 5), th children exhibited a very high level of fluency, finding many different solutions to th problem. Their level of flexibility on this activity was high, as they used three method for solving the problem – addition, subtraction and multiplication. All three elaboratio levels, (a), (b) and (c), were very high. Originality was very high, as the children foun (0,0) as a solution, and also came up with the idea of an unlimited number of solutions.

With respect to the knowledge targeted by the activity (Table 2), it seems that th children achieved the first two intended knowledge elements. They were able to expres their solutions verbally and via numeric-symbolic sentences, and they demonstrated clear understanding of the range of solutions. The extent to which they saw connections among the three activities was less clear. Nevertheless, they developed unexpected knowledge regarding the connections between the second addends that will result in equal sets sizes. Also, they attempted to use multiplication, a topic not yet presented at school. It is not clear what triggered Eli to think about multiplication. It could be that the notion of infinitely many possibilities for achieving an equal number of dragons was related to adding infinitely many numbers, and as they had finished discussing this, he tried to think of another path towards solving the problem.

DISCUSSION

At the outset of this paper we formulated the following two questions: (1) What processes of knowledge construction took place? (2) What level of creative mathematical thinking can be identified? This section begins by outlining the answers to the research questions and then continues by discussing related issues.

The sequence of the three activities the children carried out was purposefully designed to evoke construction of knowledge elements. Specifically, the activities shared the need to create equal sets from a starting point of unequal sets. The details varied from one activity to the other in terms of the starting numbers and the presence of a stockpile, allowing more complex considerations and a larger range of solutions.

Another element that was gradually introduced was the Pan Balance - Numbers VM. During the first activity the children worked only on the pictorial representation of the sets of candies. The VM was introduced during the second activity. At first the children worked on the pictorial representation of the raisins and on the numerical-symbolic representation of the VM, but soon the children began working solely on the symbolic VM representation. In the third activity, the children worked directly on the numerical-symbolic representation of the VM. That is, although each activity began with a verbal story and pictorial representation of the situation, during the solution process the children shifted from a more concrete to a more abstract representation.

All three problems had more than one solution, and more than one method to achieve the solutions. Yet the solutions of the first activity were more limited, as no stockpile was provided in the problem. The children were able to find all five solutions with whole numbers, namely (0,0), (1,1), (2,2), (3,3) and (4,4), and one solution with fractions $(\frac{1}{2}, \frac{1}{2})$. For the next two problems, the range of solutions was not limited, as a stockpile was available, and the children were able to take this into consideration.

We also noted development in the range of methods used by the children to find new solutions. Although the children acknowledged that in school they had only worked on addition and subtraction up to 20, they were able to go beyond this and try multiplication and numbers larger than 20. This was accomplished with the facilitator's encouragement and with the support of the VM, which provided them the results of the calculations.

In terms of creative mathematical thinking, a complex picture emerged (Table 6). Th levels of fluency and originality were very high throughout all three activities. The lev of flexibility was low at first and increased to a high level during the second and thir activities. A similar shift was noted between the second and third activity for two of th elaboration categories, reflecting the use of more complex methods to obtain solution This may be attributed to some extent to the children's growing familiarity with the VM tool, allowing them to consider numbers and operations beyond their school knowledge

Table 6

Levels of Creative Mathematical Thinking Found for Each Activity

	Activity 1	Activity 2	Activity 3
Fluency	Very high	Very high	Very high
Flexibility	Low	High	High
Elaboration	High	--	--
(a) numbers	--	Very high	Very high
(b) operations	--	High	Very high
(c) # of operations	--	Low	Very high
Originality	Very high	Very high	Very high

What, then, can be generalized from the current case study to inform future research The current study adds to the growing acknowledgment of the power of stories t motivate mathematical activity and engage students in problem solving (Zazkis & Liljedahl, 2009). Furthermore, this study tested a new aspect: the use of PowerPoin presentations as an environment to present stories and carry out activities. This aspec has important implications for mathematics teachers as designers and for children' mathematical activities.

Designing a PowerPoint presentation does not require programing knowledge. Man teachers know how to design PowerPoint presentations, and they are well aware of thei students' needs. Our findings suggest that the design mode of a PowerPoint presentatio can be considered a technological platform that fosters mathematical thinking. Hence creating a sequence of PowerPoint presentations to promote specific mathematica knowledge and tailoring them to the explicit needs of a class or group of students can b considered an important tool in the hands of mathematics teachers, specifically wit young learners.

From the point of view of students as mathematics learners, three points should b considered. First, the children were actively engaged in moving the objects on th screen to achieve mathematical goals. In other words, the children treated the pictures a physical objects. Hence, the objects in this environment played the role of physica mediation towards learning. Second, the children were able to record their work and refer to it during the same activity and also across activities. Note that the activity recording resulted from the actions themselves and did not required extra effort. Finally, the environment was very simple to use and required minimal guidance.

The combination of the Pan Balance – Number VM also contributed to learning. On the one hand, the VM was very simple for learners to use. It provided the resulting number on each pan, as well as the entire numerical sentence once the pans were balanced. The record it kept of the numerical sentences found thus far allowed the students to reflect on what they had done. On the other hand, the use of this VM allowed the students to go beyond their current mathematical knowledge, as acknowledged by both children. That is, it served as a tool that had the power to support a higher level of mathematical thinking.

The National Governors Association Center for Best Practices (2010) in the common core mathematical standards stated the following:

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary (p. 6).

This case study provides confirmation of the potential of a sequence of activities in eliciting knowledge construction and creative mathematical thinking. We consider it a starting point for a larger study concerning the same issues with a larger and more varied population.

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Investigating University Mathematics Teaching

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ABSTRACT: *This paper is a part of an ongoing study that investigates mathematics teaching at the university level in a lecture form. The focus of this paper is on a lecturer's teaching practice during a first year calculus course in a department of mathematics. The characteristics of his teaching and the factors that affect this practice were investigated. Through a teaching episode, aspects of his teaching are discussed and through group discussions and interviews the rationale behind these was traced. From the data analysis it emerged that lecturer's prior mathematical experiences as school and university student as well as his experiences as a research mathematician framed his teaching.*

Key words: *Calculus, Teaching practice, Identity.*

INTRODUCTION

Mathematics Education Researchers' predominant concerns at the University level are students' cognitive processes and their difficulties with advanced mathematical concepts (Artigue, Batanero & Kent, 2007). However, a small number of studies focus on university mathematics teaching and most of these studies deal with specific teaching interventions such as project work (Niss, 2001; Vithal, Christiansen & Skovsmose, 1995) or with a small number of students mainly at the tutorial level (Jaworski, 2003; Nardi, 2008; Nardi, Jaworski & Hegedus, 2004). However, teaching large groups of students in a lecture format, the usual instructional activity at the university level, has very little been investigated (Speer, Smith & Horvath, 2010). Speer et al. (2010), through a systematic literature review, reported that most studies relate to college instruction focused on the impact of instructional activities on students' learning while very few on the actual teaching practice. Through an analysis of five research examples related to teaching, they clarified what they considered as teaching practice and its particular meaning at college level. In particular, they made a distinction between instructional activities and teaching practice. According to this distinction the lecture, the context of our study, is an instructional activity while teaching practice concerns teachers' thinking, judgments and decision-making in planning, teaching and reflecting on the lesson. Weber (2004) also investigated teaching practice at the university level through a case study of a lecturer teaching a mathematical analysi course. He distinguished three teaching styles related to the teaching of proof: th logico- structural, procedural and semantic. The styles that the lecturer adopted in hi teaching seemed to be influenced by his beliefs about mathematics as a researc mathematician and about students and teaching as an experienced mathematics lecture Tsay, Judd, Hauk and Davis (2011) studied patterns of classroom discourse in th teaching of a university professor. Their findings suggested that that the professor di intend to have his students take more responsibility but he had not yet determined ho to support the discourse space to do it. Bergsten (2012) investigated through questionnaire how the students themselves viewed a successful lecture. He found th "success" from a student's point of view could be seen as a working balance betwee mathematical exposition (mathematical content, process and institutionalisation teacher immediacy (inspiration, humour, everyday language, diagrams, gestures) an general quality criteria for teaching (coherence, clarity, pace of the lecture). The studie of Weber (2004) and of Tsay et al. (2011) address the nature of lecturing while Bergste (2012) studies students' views about lecturing. In order to investigate further th outcomes of a lecture, attention is needed to the actual teaching practice and i particular to what affects this practice. In a study of university mathematics teaching o linear Algebra in a large cohort, Jaworski, Treffert-Thomas and Bartsch (2009) poin out that issues emerged by studying teaching enable us to address the rationale abou how they are tackled and to open up a dialogue within the mathematical communit Some current research in Europe is also looking at university teaching trying t characterise teaching practices (Ioannou & Nardi, 2009; Petropoulou, Potari & Zachariades, 2011). We attempt to investigate this issue further in the context of firs year Calculus teaching and we are aware that micro and macro dimensions fram mathematical activity and both need to be addressed in the study of mathematic teaching (Valero, 2009; Jaworski & Potari, 2010).

This paper is based on a wider study of university mathematics teaching and it development in a collaborative frame among lecturers and researchers. Here, the focu will be on one lecturer's teaching practice in a Calculus course and on how this practic is related to different identities that he has formed through teaching and researc activities.

THEORETICAL BACKGROUND

Calculus Teaching and Learning

Gueudet (2008), synthesizing various research contributions on the subject o secondary-tertiary transition concluded that students face difficulties with the meanin of mathematical concepts, with semiotic representations, and with mathematical tasks In general, first year students experience difficulties in aligning with the advance mathematical practices they enter. One might expect that, in their first steps in th academic environment, students would be drawn into mathematicians' practices

However, Burton (1999) observed that most of the university mathematics teachers who were respondents in her research, were "not giving learners a sense of the fun, excitement, challenge which holds them in the discipline", although she warned that "it is impossible to speak about ... mathematical practices as if they are uniform" (p. 141). University teaching has particular characteristics: big cohorts of students, expository mode of lecturing, variable attendance, tensions (Jaworski et al., 2009). We aim to explore further these characteristics and we believe that, taking into account the current reality of mathematical practices from both micro and macro point of view may contribute to a discussion about what kind of learning experiences we want for our students.

The Theoretical Construct of Identity

In their work, Lave and Wegner (Lave, 1992; Lave & Wegner, 1991; Wegner, 1998) see identity as a function of participation in different communities arguing that it is only through social processes and shared experiences that people gain a sense of self and meaning. They also argue that people do not have one identity, but different identities that are more or less salient in different situations. In a recent study of mathematics teaching Jaworski (2012), draws on several research projects and highlights the theoretical construct of "identity". She adopts Gee's (2001) definition that identity is "what it means to be "a certain kind of person" with "...more social characteristics"" (p. 614). Jaworski (2012) recognises that teachers are humans with personal characteristics and she considers aspects of teachers' identity in teaching. She writes that "...we can see a teacher as an interactive participant in worlds of mathematics, school, classroom, students and educational expectations" (p. 616). We see these worlds with their social characteristics to form lecturer's identities and affect teaching. With other words the lecturer engages in a multitude of simultaneous practices in the sense described by Skott (2010) related to the teaching and learning of mathematics when he designs, acts and reflects on his teaching. As Jaworski (2012) advocates, in seeking to characterize university mathematics teaching we need to consider the thinking and intentions of the teacher in choosing particular approaches to mathematics for students and all this thinking and intentions are closely related with teacher's identity. Furthermore, this teaching identity is related in the literature with experiences the teacher brings into classroom. For example, Walshaw (2004) explores aspects of pre-serving teaching in schools focusing on the constitution of teaching identity. She concludes that the concept of teacher identity is best thought of, as complex and multiple. She also argues for centring teacher's differing experiences as the organizers of his perceptions and claims that this "scripts identity as synonymous with teacher's role and function" (p.65). We see identity as a unifying concept that can bring together multiple and interrelated elements that a teacher can bring in the learning environment including beliefs, attitudes and personal experiences. With this lens, we intend to trace the key elements of one university teacher's teaching actions, decisions and reflections and how his teaching practice is related to his different identities.

The Mathematical Concept of Infinite Series

Infinite series is a complex, often counter-intuitive but significant concept wi applications in mathematics and science that are wide ranging and crucial (Nardi, Bi & González-Martin, 2009). Nardi et al. (2009), investigating the learning and teachin of this concept and summarizing previous research results conclude that "studen appear to have little understanding of what the concept actually means, have no visu imagery associated with it and see little or no relevance to it in mathematical and oth situations" (p. 194). In the same paper it is discussed how pedagogical practice ca assist students' overcoming of persistent perceptions such as "the sum of infinitel many quantities is always infinitely great" or "all I need is for the terms to get small and then I have convergence". In order to meet students' cognitive needs with regard t the learning of series the authors propose the use of key examples in teaching, includin examples of divergent series, the use of high evocative representations and a epistemological-historical perspective of the concept with more applications and intr mathematical references to the concept's significance.

In this paper, we focus on a lecturer's teaching about infinite series. In particular, w seek to identify characteristics of his teaching practice and to investigate the reasons f which he considers these crucial for mathematics teaching.

METHODOLOGY

This paper is a part of an on-going study with aim to investigate the first year calculu teaching in mathematics departments by observing the actual teaching and analysin what the lecturers say about their teaching. In this study, the teaching in tw departments in Greek Universities (D1 and D2) is investigated. Calculus is compulsory course in both departments and it is taught in a lecture format in classes c approximately one hundred students. The main difference between these courses is o the emphasis given to the content. In D1 the course has a theoretical focus, emphasizin the concepts and the proofs of the theorems, while in D2 the emphasis is o computations and applications. Seven lecturers participated in the study (five from D and two from D2). Data were collected through class observations, after clas interviews with the lecturers conducted by the first author and group discussions amon lecturers and researchers. All lectures, interviews and discussions were audio-recorde and transcribed.

In data analysis, each lesson is divided in episodes according to the accomplishment c a specific teaching goal that is made explicit by the lecturer (in the lesson or in th interviews) or identified by the researchers in the process of analysis. These episode are analysed to identify characteristics of university calculus teaching. The rationale c teaching is investigated through the analysis of the interviews and discussions.

In this paper, we refer to the teaching of one lecturer (L) from D2. He is an experience mathematics researcher and university teacher (having more than 20 years research an teaching experience). During last year a two-hour lesson of this lecturer was observed and videotaped and a one-hour discussion after the lesson was audio-recorded and transcribed. In the same semester the lecturer participated in three two-hour discussions with another lecturer from D1 and the three of the authors of the paper. This year, an additional one-hour discussion with the lecturer clarifying issues from last year's data was also conducted. In this paper, we analyse one episode that appear to be typical of the way that the lecturer interacts with the students, as this emerged from the data. The episode is from a lecture about the convergence of series of real numbers. The extracts of the episode and lecturer's comments translated from Greek to English by the first author.

DATA ANALYSIS: EMERGING ISSUES

We present below the episode and its analysis. In this episode the lecturer attempted to facilitate students to connect infinite series with rational numbers with periodic decimal expansion.

A Teaching Episode and its Analysis

At the beginning of the lesson the lecturer, after reminding the definition of a convergent series (a series converges when the sequence of partial sums converges), wrote the geometrical series (G from now on, i.e. G is the series $\sum_{k=0}^{+\infty} x^k = \frac{1}{1-x}$, when $|x|<1$) as "a foundational example" (lecturer's words) of a convergent series. The dialogue below followed right after:

1. L: Let's see an application of all of these in a classical example you already
2. know from school: the number 0.333...
3. You accepted that this number is finite and you have calculated it.
4. But why is it finite?
5. S: (no response)
6. L: What kind of number is it? What does it represent?
7. S1: It's a periodic number.
8. L: How does this number come of ?
9. Let's recognise what this number represents. What does it represent?
10. If we have the number 300. What does the number 300 represent?
11. S: (no response)
12. L: i.e. if we write 300 in the decimal numeral system what do we take?

13. S2: $300 = 3 \cdot 100$.

14. L: So, $300 = 3 \cdot 10^2 + 0 \cdot 10 + 0$.

15. Consequently, let's write the decimal expansion of the number 0.333…

16. [He writes $0.333... = 3 \cdot 10^{-1} + 3 \cdot 10^{-2} + 3 \cdot 10^{-3} + ...$]

17. What we write here is nothing else but the sum of a series with the

18. general type $\sum_{k=1}^{+\infty}(3 \cdot 10^{-k})$ despite that you did not call it a series. Ok?

19. Now the question is: Is this sum finite?

20. Does it remind you something from what we did?

21. S1: The ... previous ... where $x = 1/10$.

22. L: Indeed. If we choose $x = 1/10$, actually $|x| < 1$.

23. [He is going on proving that $\sum_{k=1}^{+\infty}(3 \cdot 10^{-k}) = 1/3$ using the above type]

24. So, when you were saying at school that 0.333... is 1/3 you were right!

25. The essential thing here is that although the number 0.333... has infinite

26. decimal digits, it is exactly equal with 1/3

27. so, we have a different representation of the same number.

By analysing the above episode we see the following:

- In the introduction of the episode the lecturer reminds students the necessary topic from the last lesson.
- In lines 1-5 the lecturer, in order to connect the concept of series with anothe concept, provides a familiar example to students and poses a question (line 4 challenging them to look deeper in what they already know. The students do nc respond.
- In lines 6-11 the lecturer asks more focused questions. A student (S1) provides relevant response and the lecturer provides a simpler example to make clearer t students what is asked. The students do not respond again.
- In lines 12-16 the lecturer used another simpler example trying to direct th discussion towards his goal and brings back the students to consider the mor complex case. Then he connects the decimal representation of a specific numbe with a series.
- In lines 19-27 the lecturer returns to his initial question and prompts students t reflect on what was discussed in the lesson before. A student responds and th

lecturer evaluates and gives a mathematical proof of the result. Finally, he connects the two representations of the same number (lines 24-27).

In this episode, the lecturer attempts to encourage students to connect concepts (lines 17, 18) and different representations (line 27) by using and linking familiar examples accessible to them (lines 1, 2).

Nardi et al. (2009), argue that the representation of a rational number with infinite decimal periodical expansion like the number used in the above teaching episode, show that the concept of series has important applications and therefore is central to mathematical education of a wide range of scientists and professionals. Through this example the lecturer encourages students to make connections between something they already know and a new concept. His goal is to make students gain a better understanding of both. The exemplification of the decimal expansion through an easier example that the lecturer is ready to adjust (line 10) in order to clarify his questions and promote mathematical meaning is also apparent. He also tries to interact with the students by asking questions and waiting for an answer (lines 4, 10, 19, 20). Having a poor feedback the lecturer asks rhetorical questions (line 8, 9) and uses the simpler example (line 10) as scaffolding towards his aim. It seems that he takes into account his students' difficulties, he "acts in the moment" (Mason, 1999) and he adjusts his teaching to these.

Lecturer's Reflections

Both, connections and exemplifications are key elements of the lecturer's practice as it comes out from the whole lesson's data and the data from the interview and discussions as well. In the discussions, he is clear about the central role of connections in mathematics:

"Mathematics is nothing else but connections. What I do very consciously is that I try to pass to the students that making connections is what mathematics is about. I do it through simple examples. Making connections is what we are trying to do."

Using appropriate examples for exemplifying mathematical ideas is one of the lecturer's aims, as he says in the interview after the lesson:

"Students may not understand that the teacher tries to give examples that reveal some things. An example is maybe another exercise for them but not for me. I give an example with a specific goal. ... I deliberately construct something."

Exemplifying through simple specific examples is what the lecturer also does in his research:

"Exemplifications with quite specific examples help to catch the essence of a problem. I want to be clear for everyone why we use a specific strategy. I try to encourage students to identify this proper strategy by asking questions. ... Then I start to build on their findings... The examples I use are deliberately chosen for helping students pass to what

I evaluate as a higher level of understanding. … This is common with my research. Th is the way I produce work."

The insights he gains from his students as he interacts with them are important for th lecturer as he claims in the group discussions. Taking into account students' difficultie and adjusting the teaching to address students' cognitive needs is important for th lecturer, as it is emerged from a group discussion:

"What I consider very important is to have a sense of your audience anytime; it is whor you address to. We try to educate the students. So, you design your teaching based o them. You must not make decisions without to take into account the students. We ma have a concept to introduce in our mind but the audience is very specific. So, if yo have the A audience the introduction of the concept may be totally different from th introduction of the same concept to an audience B or an audience C. For me, th audience is of crucial importance."

Reflecting on the lesson, the lecturer talks about students' difficulties to understan complex mathematical concepts as the concept of a series, and he is concerned abou how to deal with these difficulties:

"I saw that there is a problem with the concept of the series but I don't know at wha point students loose the contact."

The lecturer also refers to tensions he experienced during the lesson:

"Of course you, as a teacher, try to help students make connections by themselves bu how exactly can this be achieved…? You make several attempts but you need to know how to achieve this. It's not obvious what helps students make connections."

In a group discussion, the lecturer reflects on his teaching actions after he has seen hi video-taped lesson:

"In this lesson, students asked me questions and it is true that I answere straightforward. I don't know if it was good. I have to say I was very concerned. I wa interested in one thing. I hadn't realized it that way."

During the group discussions the lecturer connected elements of his teaching practic with his own experiences as a student of mathematics and as a research mathematician:

"I give problems to my students through which they can experience the joy o discovery. If they work with the problems they will enter into this process. They won' discover unknown things but they will discover something. I was doing the same as a student of mathematics. I was wasting a lot of time about four or five hours to solve tw or three problems. But after that I was happy."

He also connected his current teaching with the way he was teaching at the beginning o his teaching carrier:

"If you are a novice teacher, you teach different. For example, teaching my first course I believed that by facing a problem from several perspectives would help my students. was wrong. After some years, I realized that my students were losing the game at th emotional level. "Do you know what does it mean to have in front of you a person who can give one, two, three different proofs and you cannot even think at a basic level? It breaks you down!" This is what one of my students told me, and for that reason, I have never done the same."

Teaching is situated in a given sociocultural and institutional frame. From a macro stance there are also external factors that have an impact on teaching. The lecturer was aware of these factors and of the effect they had to his actual practice:

"The basic problem in our department is the low achievement of our students. Given this, we decided to reform the curriculum aiming to link what is taught at high school with that at the university... We wanted to communicate with our students and help them to learn... But the curriculum did not facilitate the transition.

"If a lesson is obligatory then it is somehow easier because all the students have to take this course. I myself shall teach differently in an obligatory course than in an elective course. An obligatory lesson is addressed to everyone and the students have no chance of giving up the course."

CONCLUDING DISCUSSION

From the above teaching episode and the other data we have collected and analyzed, we have identified teacher's actions that seem to be rather typical of his teaching. For example, he tries to make connections and to exemplify the main ideas through examples accessible to students. Nardi et al. (2009), argue for the need of showing applications of the concept through examples and connections that are of crucial importance for mathematical learning at post-secondary level. Connections are also key components of the practice of research mathematicians (Burton, 1999). The lecturer also encourages interaction with the students by asking questions and giving them time to respond. His ultimate goal is students' conceptual understanding and he tries to facilitate it in different ways. For example, he tries to build on students' prior knowledge and experiences. Generally, as indicated by the data, he believes that teaching and learning are interrelated processes. However, he often faced tensions caused by the large number of students that were related to the management of students' ideas on the one hand and covering the content on the other.

Trying to interpret the underlying reasoning of the lecturer's teaching actions and decisions we see that these are framed from his perspective of mathematics, his perspective of teaching and external factors from the wider institutional frame in which the teaching is situated. His thinking and his intentions behind the approaches he uses to bring mathematics closer to his students are related to his identities formed in the context of two simultaneous practices in which he has been participating: mathematical identity (MI) and teaching identity (TI). In particular, MI is expressed through what he considers important in mathematics, as the key ideas of a proof, the connections between different mathematical concepts, the role of exemplification in mathematics, and the importance of informal approaches. TI contains all of his perspectives for teaching and learning, such as that teaching has to build on students' prior experience; learning is achieved through students' active participation; teaching and learning ar interrelated. It seems also that his MI affects his TI, and both affect the lecturer' teaching practice. For example, he is a research mathematician believing "that makin connections is what [mathematicians] try to do" and a teacher who tries "to engage hi students to this activity". Thus his teaching identity (e.g. students need to connec different content areas or different concepts) is affected by his mathematical identit (e.g. connections are important in mathematics) and influences what he does in th classroom. External factors affecting the lecturer's teaching practice include the socia and institutional frame in which teaching takes place including students' lov mathematical achievement, cultural differences between school and university or th general educational system.

The following diagram may be helpful to conceptualize the above issues:

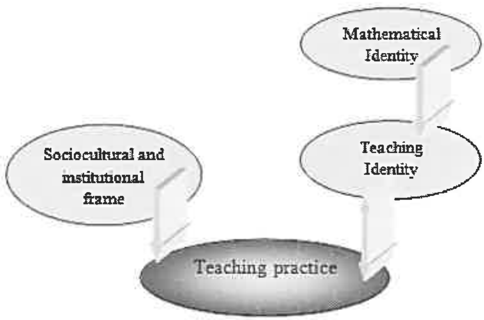

Figure 1. How different identities influence teaching practice.

MI and TI are connected with lecturer's experiences. In particular, we see tha experiences as a research mathematician (e.g. 'exemplifications are what I use t produce work') and as a student of mathematics (e.g. 'I did the same' (solving problem to experience the excitement of discovery)) affect his mathematical identity. Similarly his teaching experience and his experiences as a student affect his teaching identity. W seek to develop the above network by analysing data from more cases and to investigat certain relations.

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Primary School Students' Structure of Ability in Transformational Geometry Concepts

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ABSTRACT: *Early studies in the field of transformational geometry suggest that performing transformations is a multi-faceted mental operation. Moreover, there were suggestions that the structure of transformational geometry concepts ability in children can be viewed from a mathematical perspective, in contrast to cognitive. However, the components of this ability as well as its structure appear not to have been defined in the literature. The purpose of this paper is to investigate: (a) the components that synthesize primary school students' ability in transformational geometry concepts, and (b) the structure of this ability. The sample was 166 students from the fourth, fifth and sixth grade. The validity of a theoretical model which considers abilities in the mathematical concepts of translation, reflection, and rotation as the subcomponents that synthesize ability in transformational geometry concepts was investigated. For this purpose, different types of tasks were used for measuring the sub-abilities of each of the three geometric transformations' abilities, namely: (1) recognition of image, (2) recognition of transformation, (3) identification of parameters of transformation, and (4) construction of image. The validity of the components and of the theoretical model structure was tested with subsequent CFA tests. The results suggest that the study's theoretical model is statistically significant and can describe primary school students' ability in transformational geometry.*

Key words: *Transformational geometry, Translation, Reflection, Rotation, Primary school.*

INTRODUCTION

The growing emphasis on geometry teaching during the last few decades has modified its traditionally Euclidian-based content, by introducing new types of geometry, such as transformational geometry (Jones, 2002). Transformational geometry refers to mental or physical transformation of shapes. The most common types of geometric transformations in literature and in primary school textbooks are translation, axial reflection, and point rotation. According to NCTM's *Principals and Standards for School Mathematics* (2000), "Instructional programs from kindergarten through grade 12 should enable all students to apply transformations and use symmetry to analyze mathematical situations" (p.41). Transformational geometry is considered important i supporting children's development of geometric and spatial thinking (Hollebrand 2003). Moreover, it is related to a variety of activities in academic and every-day lif such as geometrical constructions, art, architecture, carpentry, electronics, mechanic geography, and navigation (Boulter & Kirby, 1994).

In the early studies of transformational geometry, there have been some disagreemen among researchers regarding the structure of transformational geometry abilit Specifically, the arguments evolved around whether investigating ability i transformational geometry concepts from a mathematical structure aspect is always i accord with the child's cognitive structure (Lesh, 1976; Moyer, 1978). Nevertheless, was suggested that performing geometric transformations is a multi-faceted ment operation (Kidder, 1976). Over the last few years, research has focused in studying th development of knowledge and understanding of transformations (Molina, 1990; Soo 1989; Thaqi, Giménez, & Rosich, 2011; Yanik & Flores, 2009), and various theoretic frameworks have been used (Hollebrands, 2003; Molina, 1990; Portnoy et al., 200 Soon, 1989). For this purpose, different types of tasks have been identified and used i these studies. However, the components that synthesize this ability and its structur from either a mathematical or cognitive perspective, appear not to have been clearl defined or empirically tested. This seems to be critical in order to study student development and understanding of transformations.

This paper is based on the pilot results of a large scale project investigating student ability in rigid transformations (translation, reflection, rotation). Its purpose is t investigate: (a) the components that synthesize primary school students' ability i geometric transformations, and (b) the structure of this ability, from a mathematic concepts perspective.

To this end, Section 2 provides an overview of the research done on the development c transformational geometry ability by considering the different types of tasks which ma constitute different mathematical abilities in transformational geometry. The sectio ends with specifying the study's objectives. Section 3, gives information regarding th procedure and setting of the study, the tests used, and the analysis employed. Th theoretical model and the results are presented and discussed in Section 4, while i Section 5 we draw the conclusions and some implications for teaching, as well as som possible directions for further research.

LITERATURE REVIEW

One of the first debates in the transformational geometry field of research was th structure of geometric transformations learning. According to Piaget (Piaget & Inhelde 1971), children learn transformations in the following order: translations, reflection and rotations, and that the anticipatory level precedes the representational level c understanding. Kidder (1976), Lesh (1976), Moyer (1978), and Schultz (1978), in more general context, have argued against the assumption that the mathematic structure is always in accord with the child's cognitive structure. Therefore, one cannot be absolute that children conceive the transformations as translations, reflections, and rotations in the mathematical sense. It seems possible that they may be using some entirely different system of relations to describe geometric transformations (Lesh, 1976). This may explain why there appears to be a disagreement regarding the level of difficulty in geometric transformations learning. While Moyer (1978) suggests that translation is at least as easy as reflection, even though reflection can be considered mathematically primitive, Schultz and Austin (1983) suggest that translations seem to be the easiest transformations for students and that certain configurations can influence the relative difficulty of rotations and reflections. The effects that each of these configurations may have on geometric transformation tasks can be different in respect to the type of geometric transformation and student age (Schultz & Austin, 1983).

Lesh (1976) suggests that the appropriate way to analyze the transformational geometry tasks is not to focus on translations, reflections, and rotations. Moyer (1978) suggests that one important property may be related to "up-down" and "left-right" changes and Kidder (1976) suggests orientation in the sense of vertical, horizontal, and diagonal. Towards this direction, Schultz (1978) tried to include properties of the transformation and of the figure to further investigate the development of students' understanding of transformations. The properties of the transformation were the direction (horizontal, vertical, and diagonal) and size of the displacement, and the properties of the figure were the familiarity of the shape and size. Her results, however, cannot lead to a conclusion regarding the differences in the way that children conceive these tasks, and are still very much adherent to the translations-reflections-rotations structure. One of the main disadvantages of these studies is that, since they were influenced in a large extent by the methodology of Piaget's experiments, they only investigated students' ability in only one type of task, that of finding the image of a shape or the position of a certain dot on a shape's image.

The first attempts that used a variety of tasks to study the development of transformational geometry concepts as a sequence of levels were based on the van Hiele model of geometrical understanding (Molina, 1990; Nasser, 1989; Soon, 1989). These studies aimed to investigate the development of transformational geometry concepts based on the van Hiele model. These investigations did not only include tasks of executing and identifying geometric transformations (Edwards, 1990; Hart, 1981) or performing and inversing geometric transformations (Kidder, 1976). They referred to a variety of tasks, matched to each level, including visual recognition of the three geometric transformations, understanding the properties of the transformations, relating the properties of the transformations, and tasks of formal definition and deductive proof. However, the focus of these studies was to confirm the applicability of the van Hiele theory in transformational geometry, rather than to confirm the components that synthesize this ability.

The most recent attempt for describing the development of knowledge and understanding of geometric transformations based on different types of tasks and different abilities in transformational geometry concepts was performed by Yanik and

Flores (2009). This study focused mainly on the learning of translations, and based i findings on a case study. Specifically, it investigated the hierarchical development (translation by focusing on different sub-abilities. Again, a variety of tasks were use which were organized into the categories of recognizing, describing, performing, an representing a translation. The findings of this study indicated that the development of prospective teacher's thinking about translations started by (i) referring t transformations as undefined motions of a single object, followed by (ii) usin transformations as defined motions of a single object, (iii) understanding abo transformations as defined motions of all points on the plane, and (iv) mapping of th plane onto itself. The researchers guided research into revising and improving th theoretical model for other subjects and for other concepts, as well as for the oth geometric transformations. This raises some questions regarding possible similariti and differences among the structure of the three geometric transformations.

Even though many of the most recent studies were carried out in secondary and highe education, they seem to have a tendency into using different types of tasks in order t measure students' abilities in the three geometric transformations. Moreover, the saliently suggest the existence of different types of abilities within transformationa geometry, at different levels. However, no research has so far confirmed that suc abilities, and moreover which of these abilities actually consist sub-factors of the sam general ability, which is considered to be transformational geometry ability. Th clarification of the sub-components of transformational geometry concepts ability i primary school education constitutes the first part of what this study aims to investigat The second part aspires to investigate the structure of transformational geometry abilit from a mathematical perspective.

METHODOLOGY

Participants

The participants of the study were 166 primary school students (78 boys and 88 girls Specifically, fifty-two were fourth-graders, fifty-three were fifth-graders and sixty-on were sixth graders. The students came from both urban and rural schools.

Instrument and Procedure

The instrument used in the study was a transformational geometry test, develope especially for the purpose of the project. The test had three sections: the first was abou translation, the second about reflection, and the third about rotation. Each section had similar structure and included four different types of tasks (see Table 1 for examples (1) recognising the image of a translation/ reflection/ rotation among other choices; (2 recognising a translation/ reflection/ rotation among other choices; (3) identifying th parameters of a given translation/ reflection/ rotation; and (4) constructing the image fo a given translation/reflection/rotation. Following the suggestions by Schultz and Austi

(1983) regarding the configurations that influence the level of difficulty of isomorphic similar tasks, at least three tasks with different configurations were given for each of the four types. In respect to the configuration of direction, three tasks were given: one in horizontal, one in vertical, and one in diagonal direction. In types 3 and 4, there was an additional task with overlapping image to respect the configuration of distance, and in type 4 an additional task with an unfamiliar shape in horizontal direction to respect the configuration of familiarity with the shape. The tasks were split and administered to all the students in two equally difficult parts. Students were given 40 minutes to complete each part of the test. To avoid practice effects, half of the students received one part of the test first, while the rest of the students received the other part.

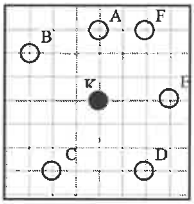
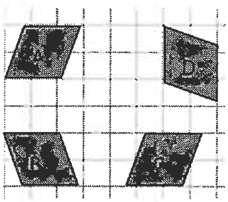
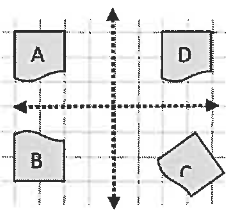
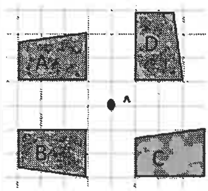
After completing the test, students' responses were graded. Types of tasks (1) recognising the image of a transformation, and (2) recognising a transformation were of multiple choices, with one correct response out of four alternatives. In these tasks, 1 mark was given to each correct response and 0 marks were given to each incorrect response. In type 3, identifying the parameters for a given translation, and type 4 tasks, constructing the image for a transformation, 0 marks were given for incorrect response and 1 for correct. Partial credit was given to responses with some correct elements. Items with no response received 0 marks.

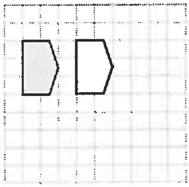
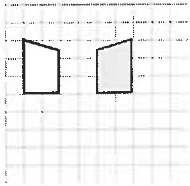
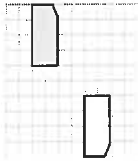
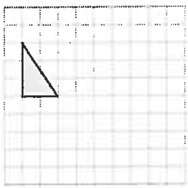
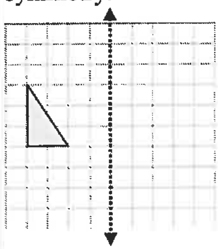
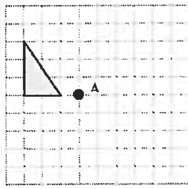
Statistical Procedures

For testing the fit of the theoretical model regarding the structure of transformational geometry ability, MPLUS was used with ML estimator. More than one fit indices were used to evaluate the extent to which the data fit the theoretical model under investigation. Specifically, the fit indices and their optimal values were: (a) the ratio of chi-square to its degrees of freedom, which should be less than 1.96, since a significant chi-square indicates lack of satisfactory model fit; (b) the Comparative Fit Index (CFI), the values of which should be equal to or larger than 0.90; and (c) the Root Mean Square Error of Approximation (RMSEA), with acceptable values less than or equal to 0.06 (Muthén & Muthén, 2004).

Table 1

Examples of the Tasks in the Transformational Geometry Ability Test

Type of task	Translation example	Reflection example	Rotation example
1. Recognition of a transformation image	Which of the following images is the translation of the pre-image K, when it translates 3 units up? A C D E 	Which of the following shapes is the reflection of shape Z over a vertical line of symmetry? 	Which of th following shapes i the rotation of th grey figure at ¼ of turn? 
2. Recognition of a transformation	Which of the following pairs of shapes show a translation? a) A and D b) B and C c) C and D d) A and C 	Which of the following pairs of shapes show a reflection? a) A and D b) B and C c) B and A d) C and D 	Which of th following pairs o shapes show rotation? a) A and D b) B and C c) C and D d) A and C 

3. Identifying a transformation's parameters	Give the instructions for the translation of the shaded figure to the position of the white figure. 	Draw the line of symmetry for every case. 	Find the point of rotation and the fraction that shows how much the shape turned to the right.  1/4 2/4 3/4
4. Construction of an image under transformation	Translate 4 units to the right. 	Draw the reflection of each shape over the given line of symmetry. 	Rotate the shape ¼ of a turn to the right. 

RESULTS

This section presents the theoretical model and the results of the statistical analyses. The aim of the study was: (a) to investigate the components that synthesize primary school students' ability in transformational geometry concepts, and (b) to investigate the structure of this ability based on the mathematical concepts of translation, reflection, and rotation.

Regarding the first objective about the components that synthesize primary school students' ability in transformational geometry concepts, confirmatory factor analyses were performed in order to confirm that the operationally isomorphic tasks with different configurations form separate factors, and that these factors load separately on the three factors of ability which constitute the mathematical concepts of transformational geometry: Translation Ability, Reflection Ability, and Rotation Ability. Figure 1 presents the structure for each of these factors. In this diagram, the rectangular shapes represent the tasks and the elliptical shapes represent the factors c the theoretical model. The arrows show the factor for which ability in each iten contributes. The numbers over the arrows are the coefficients, which serve as a indicator of the importance of each item for each factor. The first number indicate factor loading and the number in parenthesis indicates the corresponding interprete dispersion (r^2). On each arrow, the numbers in the first row represent the coefficients fc the translation items, the numbers in the second row represent the coefficients for th reflection items, and the numbers in the third row represent the coefficients for th rotation items.

According to Figure 1, each of the items loads only on the expected correspondin factor, for each of the three geometric transformations. These are the factors "Recognis image", "Recognise transformation", "Identify parameters", and "Construct Image". A item factor loadings are statistically significant at level α=.05. However, in the case c rotation, two of the items (Item 4 and Item 11) did not significantly load to th corresponding factors and were withdrawn from the model. Two of the expected factors "Recognise image" and "Recognise transformation" constitute a second order facto which contributes significantly to each of the three geometric transformations abilit factors at primary school level. This implies that the two types of tasks share som common abilities and syllogisms. This factor was named "Recognise properties", sinc the common characteristic shared by these tasks is the recognition of each geometri transformation's properties regarding the orientation, position, and size of the figure "Recognise properties", "Identify parameters", and "Construct image" seem to load t their corresponding factor of geometric transformation concept, namely "Translatio ability", "Reflection ability", and "Rotation ability". As described earlier, th coefficients serve as indicators of the importance of each factor for ability in translation reflection, or rotation. The fact that all the factor loadings to the Translation/ Reflection Rotation Ability factors are high (loadings are greater than .70, at α=.05) suggests tha all factors have a considerable contribution towards ability in transformational geometr concepts.

Table 2 presents the values of the fit indices for each one of the models in Figure 1 Moreover, it provides the values of the fit indices for the corresponding alternativ models, which assume that there are no sub-components and that all translation item load on a single factor, all reflection items load on another single factor, and all rotatio items load on another single factor.

This means that each of these abilities is uni-dimensional. The comparison of the fi indices suggests that the proposed theoretical model of structure for each geometri transformation fits the data much better than the one-factor model for each of the thre geometric transformations ability. This confirms that the items of each geometri transformation do not form the same factor, and that this ability is comprised by distinc sub-factors. Hence, there are twelve first-order factors for transformational geometr ability, four for each of the three geometric transformations of translation, reflection and rotation: (i) "Recognise image", (ii) "Recognise transformation", (iii) "Identif parameters", and (iv) "Construct Image".

Table 2

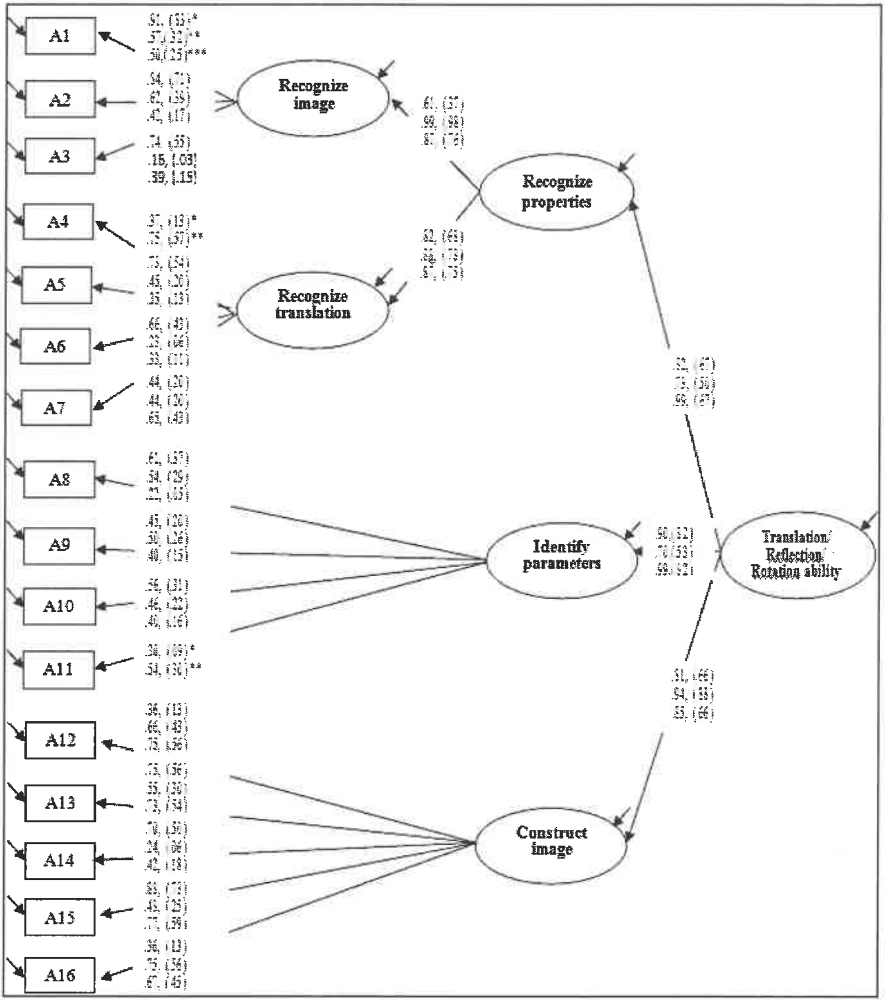
Comparison of the Theoretical Models to the Single-Factor Models

		x^2	df	x^2/df	x^2 diff.	df diff.	CFI	RMSEA
Translation	One factor model	379.13	104	3.65			.63	.13
	Theoretical model	136.30	98	1.39	242.82	6	.95	.05
Reflection	One factor model	200.15	104	1.93			.77	.08
	Theoretical model	126.60	98	1.29	73.55	6	.93	.04
Rotation	One factor model	170.75	77	2.22			.80	.09
	Theoretical model	90.62	70	1.30	80.13	7	.96	.04

Figure 2 presents the theoretical model for ability in transformational geometry concepts, based on the mathematical perspective of abilities in translation, reflection, and rotation. Based on the findings of the models in Figure 1, students mean performance in the twelve first-order factors was calculated to form the basic components of "Transformational Geometry Ability". After subsequent model tests, the model presented in Figure 2 proved to have a very good fit to the data (CFI = .96, x^2=72.78, df=52, x^2/df=1.39, $RMSEA$ = .05).

From a mathematical structure point of view, the components load on higher order factors that form three separate mathematical abilities: "Translation Ability", "Reflection Ability" and "Rotation Ability", and these three abilities constitute "Transformational Geometry Ability".

Figure 2 shows that the statistical analyses confirmed that the three factors of abilities in the transformational geometry concepts "Translation Ability", "Reflection Ability", and "Rotation Ability", load significantly on the higher order factor which is considered to be "Transformational Geometry Ability". Moreover, all the loadings to this factor are very high (loadings are greater than .90, at α=.05), which indicates their considerable contribution towards transformational geometry ability.



* Factor loading values for the translation ability model (*CFI* = .95, x^2=136.30, *df* =98, x^2/df =1.39, *RMSEA* = .05)

** Factor loading values for the reflection ability model (*CFI*=.93, x^2= 126.60, *df*=98, x^2/df=1.29, *RMSEA*=.04)

*** Factor loading values for the rotation ability model (*CFI*=.96, x^2 = 90.62, *df*=70, x^2/df=1.30, *RMSEA*=.04)

Figure 1. The proposed model of ability for the three geometric transformations (translation, reflection, and rotation).

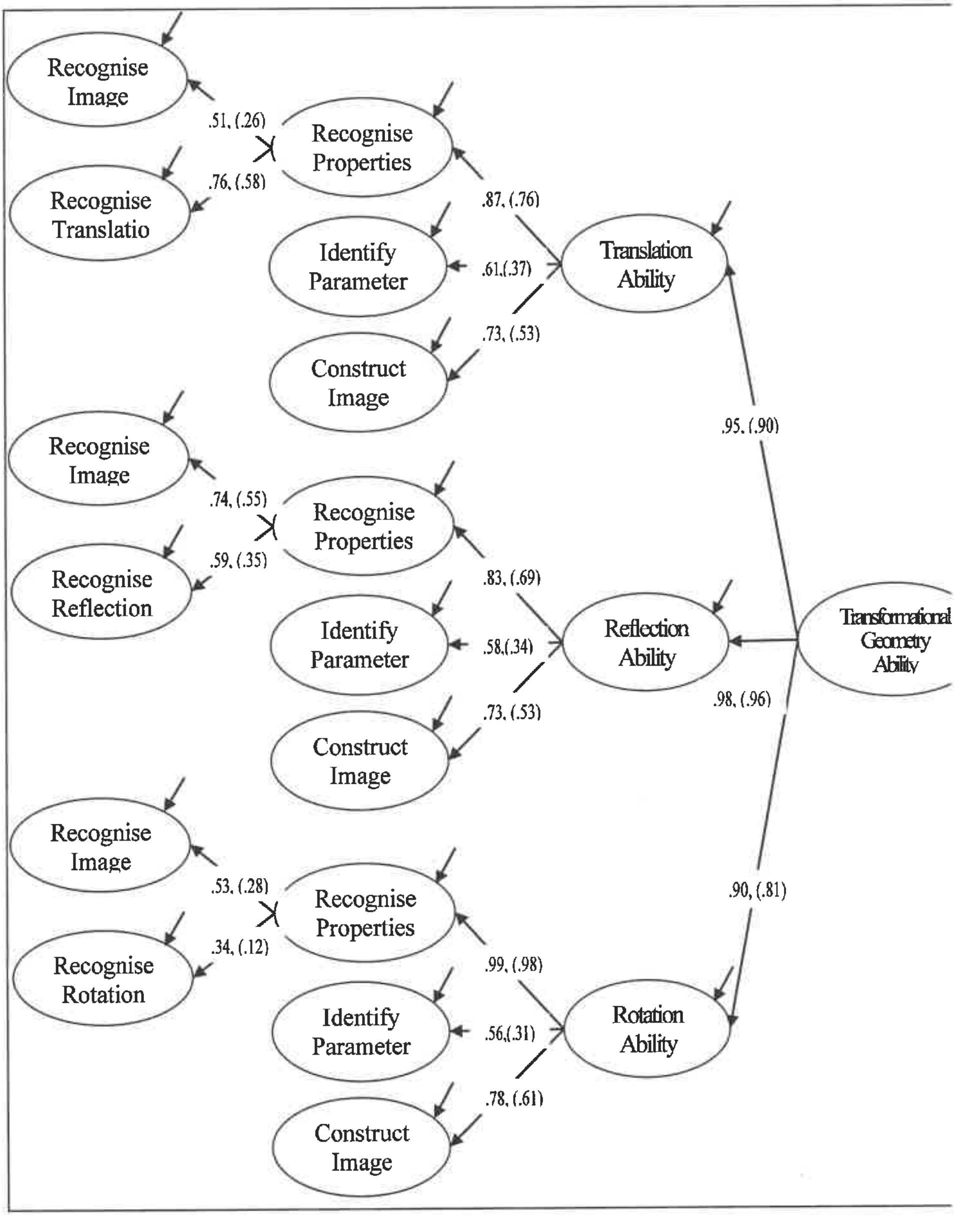


Figure 2. The proposed model of ability in transformational geometry concepts.

DISCUSSION AND CONCLUSIONS

This paper aimed to investigate: (a) the components that synthesize primary schoo students' ability in geometric transformations, and (b) the structure of this ability, fro a mathematical perspective. This section discusses the findings of the study an summarizes the main conclusions.

First of all, the findings of this study confirm the position that performing geometri transformations is a multi-faceted mental operation. Even though this has bee theoretically proposed by Kidder (1976) in some of the earliest studies in geometri transformations, it has never been empirically verified before. The findings of this stud not only verify the multi-component nature of transformational geometry ability, the also describe the components of this ability and the structure that seems to conne them. Specifically, the findings of the study suggest that primary school student ability in geometric transformations (translation, reflection, rotation) is synthesized b twelve distinct components of ability, four for each of the three geometri transformations, namely "Recognising image", "Recognising transformation' "Identifying parameters", and "Constructing image". In all three cases, the abilities c "Recognising the image" and "Recognizing the transformation" seem to share som common characteristics of thinking and form a common factor, which we consider to b "Recognising properties". From a mathematical structure perspective, "Recognisin properties", "Identifying the parameters", and "Constructing the image" for eac geometric transformation seem to form the components of "Translation ability' "Reflection ability", and "Rotation ability". Moreover, these findings suggest that th three geometric transformations of translation, reflection, and rotation seem to share th same structure of ability, as assumed in the study of Yanik and Flores (2009).

Furthermore, the findings of this study suggest that the three factors of ability in th three geometric transformations, namely "Translation ability", "Reflection ability", an "Rotation ability" seem to form a higher order ability, which is considered to b "Transformational Geometry Concepts Ability". This provides evidence for the positio that the structure of students' transformational geometry ability can be viewed from mathematical perspective, as has been previously noted by Moyer (1976) and Les (1978). According to these researchers, transformational geometry ability can be viewe from both a mathematical structure perspective and a cognitive structure perspective Moreover, they suggest that these two structures may not be in accord. Our preliminar results empirically confirm that the structure of ability in transformational geometr concepts can be viewed from a mathematical perspective. However, further research i needed in order to investigate and confirm the existence of a cognitive structure aspec and moreover to compare the possibility of these two being in accord.

The findings of the present study are important both for teaching as well as for assessin ability in transformational geometry concepts. It puts forward the importance o teachers to consider all the components that synthesize this ability when designing thei instruction. This would provide students the experiences to develop all these differen abilities of transformational geometry concepts within an effective instructiona program on transformational geometry concepts. The findings are also important for research in mathematics education, since they reveal the multiple facets of transformational geometry ability, which can be further investigated in order to reach a deeper understanding for each of these sub-abilities. Moreover, research could address the development of an instructional program which would take into consideration all the factors of the proposed model in order to develop primary school students' ability in transformational geometry concepts. Furthermore, it could also investigate the potential of such an instructional program to increase students' ability and understanding of transformational geometry concepts.

Further research can also examine the developmental structure of transformational geometry ability. Moreover, it can focus on finding effective ways of teaching for promoting all the components of transformational geometry ability and on developing and evaluating an effective sequence for teaching all components in primary school education.

ACKNOWLEDGMENTS

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Originality and Beauty of Arithmetic Reasoning

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> A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. Such experiences at a susceptible age may create a taste for mental work and leave their imprint on mind and character for a lifetime.
>
> George Polya, 1988, p. v.

ABSTRACT: *This paper presents the results of a teaching experiment carried out on twelve year old students. An open-ended task was given to them and they hadn't been taught the algorithmic process leading to the solution. More specifically, the problem of banknotes of piggy-bank was posed to the students. The formal solution to the problem refers to a system of two linear equations with two unknown quantities. In this mathematical activity students worked cooperatively. They discussed their discoveries in groups of four and then presented the final answers to whole class. This paper describes the characteristic arguments that represent certain different forms of reasoning that emerged during the process of justifying the solutions to the problem. The findings of this study show that within an environment conducive to creativity, which encourages collaboration, exploration and sharing ideas, students can be engaged in developing multiple mathematical strategies, showing beauty and elegance.*

Key words: *Mathematical reasoning, Arithmetic reasoning, Open problem solving, Whole class discussion, Justification, Conjecturing, Operative proof.*

INTRODUCTION AND THEORETICAL PERSPECTIVE

In the following literature review, we will describe briefly the reasoning as a part of mathematical thinking and a class discussion activity. In addition, we will examine the role of reasoning in the Greek curriculum and the correspondent mathematical practices in the junior high school.

Mathematical Reasoning

Generally, researchers concur that reasoning and proof form the foundation of mathematical understanding and that learning to justify is fundamental for the development of mathematical knowledge (Stylianides, 2007; Mueller et al., 2010). Reasoning is to mak arguments in order to explain a fact. The crucial goal of all sciences is to observe an explain phenomena (Gale, 1990). In mathematics, the explanation that indicates *why* statement is true is called proof. Argumentation consists of one or more logically con nected arguments. An unproven proposition that is believed to be true is known a a conjecture (Houston, 2009). Reasoning is not necessarily based on formal logic an proofs. It can be seen as thinking processes, as the product of these processes, or as bot (Bergqvist & Lithner, 2012). Mathematical reasoning is a part of mathematical thinkin it's a process of coordination of data, ideas and beliefs in order to make inferences abou the truth (Leighton & Sternberg, 2004).

Formal proofs are mainly confirmations of implications that require rigor language an deductive reasoning. However, not all students are mature enough for it. Thomso (1996) defines mathematical reasoning as a "*purposeful inference, deduction, inductio and association in the areas of quantity and structure*" (p. 267). In addition, "*mathe matical reasoning is an evolving process of conjecturing, generalizing, investigatin why, and developing and evaluating arguments*" (Lannin et al., 2011, p. 10). In the lev of primary school and junior high-school education, students deal with informal proo in the form of drawings, visual arguments or numerical data, where inductive, abductiv or plausible reasoning play a vital role.

Arithmetic Reasoning

Mathematical reasoning includes many forms of reasoning, such as algebraic, statistica geometric, combinatorial, and arithmetic. Arithmetic reasoning is connected to the un derstanding of the meaning of natural numbers and the relationships between them (Ko syvas, 2009; Kosyvas, 2010a). According to Piaget, using numerals correctly doesn guarantee complete understanding. Achieving profound understanding of arithmeti relationships depends on the growth of both logic and the students' maturation (Piage 1941). Arithmetic reasoning is an essential mathematical ability of using multiple repre sentations of numbers flexibly, grasping properties or patterns, as well as possessing th meaning of different arithmetic operations and skills in mental computation and estima tion. Moreover, it is the fundamental essence of doing mathematics and using it in socia life. Without arithmetic reasoning there is no mathematics, as it helps students to unde stand why arithmetical relationships exist. This is critical for the development of deeper understanding of mathematics. Principally, arithmetic reasoning is used to solv problems. The justification ensures that the solution is correct.

Reasoning and Class Discussion

Furthermore, Yackel and Hanna (2003) describe reasoning as an activity of the clas community in which students interact with one another in order to solve problems in supportive environment. Teachers should therefore create the appropriate environmen in which problem solving can flourish. Other researchers believe that participating i discussions concerning mathematical ideas in the classroom community of learne leads to mathematical reasoning (Balacheff, 1991; Stylianides, 2007). Many studies show that students are able to deal with challenging problem solving situations that offer them important opportunities such as sharing of ideas, exploration and collaboration (Cobb, et al., 1992; Harel, 2007). In a supportive and cooperative environment, students have the chance to formulate and examine conjectures, to make and refute claims, to investigate and describe various ways of generalisations, as well as to create and validate arguments and justifications (Lampert, 1990; Lannin et al., 2011). Teachers should encourage students to reveal their ideas in public, to explain and give evidence of their claims and to build arguments that are convincing to their classmates (Yackel & Hanna, 2003; Mueller at al., 2010; Potari, 2010).

Greek Curriculum and Practices of Reasoning

In Greece, the authors of school textbooks that are used in the three classes of junior high-school since 2007, despite their differences, point out that the improvement in teaching mathematics is related to the quality of learning opportunities offered to students. Mathematical content is organized such that the "activities" are its central axis. Those activities are mainly problems in a context familiar for the student (real life related or mathematical) and hopefully, through solution procedures, lead them to introduce new mathematical knowledge. They aim to engage students in learning and particularly in the deployment of reasoning and communication using mathematical language. Unfortunately though, many teachers give less importance to "the activity of students" and more priority to their own activity. As if this was not enough, the positive attitude that some teachers maintain towards the activities is rather artificial and it is not connected to their everyday teaching practice (Dimitriadou et al., 2009; Verykios, 2009). Sometimes, teachers make interventions with detailed instructions, boring explanations or complete presentations of the solutions to the activities on the blackboard, maintaining the traditional teaching model and pretending to be in a greater disposition to communicate with the students. Often, they insist on common pedagogical practices, gained by the application of the former curriculum, and don't want to move forward to an innovative practice.

Moreover, in the new official curriculum, it is recommended that teachers use real situations, as well as original or unusual problems that mobilize students' abilities so they can make sense of concepts. In this perspective, the intention is to encourage teachers to promote active learning and innovative practices. In teacher books the following is mentioned: *"Activity aims at encouraging collaboration in learning and working in small groups..."* (Argyrakis et al., 2007, p. 9). "*The teacher should encourage students to adopt "active methods" of learning. In this way, learning activities that consist of investigation and working in small groups are an important tool*" (Vandoulakis et al., 2007, p. 32). Also, "*Our students were assigned to open activities instead of exercises of 2 or 3 minutes, which is a step towards the transfer of learning responsibility from the teacher to the students*" (Vandoulakis et al., 2007, p. 33). Usually, teachers deal with habitual problem solving activities. They present every solution on the blackboard as a linear process of steps and emphasise on getting the right answer. Most importantly, alternative instructional approaches are not largely spread in everyday practices an open problem solving activities are marginal in common class practices, as teacher usually avoid them (Kosyvas, 2010b). Teachers struggle to cover the curriculum an they claim that it is ineffective to change their practices and offer students ample tim for exploration and reinvention. The textbook is the main source of exercises and prob lems and many instructional practices has an excessive dependence on them. Teacher are therefore encouraged to use various materials and organize "effective instructiona practices".

OUR STUDY

We will describe and analyse a specific instructional practice with an open-ended prob lem for Greek junior high-school. The following questions will be answered:

(a) What levels of reasoning do students use when justifying their solutions to the prob lem?

(b) How do students form arithmetic reasoning when generalizing and investigating th reasons?

(c) Can students refine their strategies and build new arguments, more beautiful or ele gant?

Method

We adopted the qualitative research paradigm that consists of the design experimen (Cobb et al. 2003; Gravemeije & Eerde, 2009) and the participant's observation methoc A teaching experiment was conducted during the school year 2008-2009 in the firs class of the first experimental junior high-school of Athens, as part of a larger study o students' mathematical reasoning. The sessions were videotaped and some selected epi sodes of the data are presented here, as evidence of students' arguments and justifica tions. Data analysis is qualitative and we examine the development of reasoning durin the activity (Erickson 1986; Cobb et al. 2003; Collins et *al.* 2004; Kosyvas & Baralis 2010).

The problem is the following:

Problem: *The students of a class have collected in their piggy-bank 120 euros. They have 15 banknotes of 5 and 10 euros. How many banknotes of each sort are there in their piggy-bank?*

The posed problem is an everyday life problem, so little knowledge is needed. It is ex pected that the twelve-year-old students would solve it with arithmetic and not wit algebra, as they hadn't yet acquired knowledge of this type.

Twenty-six twelve-year-old students took part in this experiment. They mainly origi nated from medium socioeconomic environment. The participants worked in small het erogeneous groups with a very good pupil, two in medium level of ability and one weak pupil. These groups were equivalent in mathematical abilities. We used two school periods giving students sufficient time to work with minimal teacher interventions.

In detail, the work of the class is organised in three phases:

Phase of individual understanding of the problem (10 minutes): Students read carefully the problem (it was shared on a sheet of paper for any group) and were asked to start the investigation of the problem. The teacher-researcher passed successively from all groups and discussed the problem, making sure not to turn the open reflexive situation in a closed. In case that many students have comprehension difficulties a short clarification discussion was organised throughout the class.

Phase of cooperative research and redaction of slides (about 30 minutes): After the students have understood the formulation of the problem and they individually elaborated strategies, conjectures and possible solutions, they were involved in the group work, where research is a team-work. They examined all individual ideas and searched new strategies and solutions. The members of each group were invited to agree on the solution of the problem, which they wrote on a slide. In the group, students shared and compared their findings. Control and evaluation of the solutions occurred first in the small group. Each team must provide a written explanation of the solution.

Phase of class discussion (60 minutes): Each projected slide is presented to the whole class for critical thinking. Each representative presents the work of the team and other groups through their representatives weigh the pros and cons and specify if their team agree or disagree on the proposed solution. During this phase, reasoning took place in front of in the whole class and students were encouraged to think and express their personal ideas, control hypothesis and arrive at conclusions, acceptable by the class. The teacher is not the one who validates what is right and what is wrong. The whole class discussion is the final collective sieve that picks holes the arguments. So students are involved in the discussion and ought to explain their options each time.

RESULTS AND DISCUSSION

Generally, the problem piqued the students' curiosity, motivated their investigation and made them enjoy doing mathematics. Students were able to connect familiar mathematical concepts in a new and unusual way that surprised them. In this open situation that we will expose, students participated actively in the mathematics classroom discussions, making their ideas public and displaying arguments that are convincing to their classmates. The problem was a suitable opportunity that caused great excitement and promoted the revisiting and the reconstruction of previous knowledge, in order to build new arguments.

This study is limited to a brief description of the most important students' discoveries that emerged during cooperation in small groups and discussion among all students of the class. These conditions allowed students to manifest different types of arithmetic reasoning that we will present below. The following three levels remind us of the analo gous Van Hiele levels of geometric reasoning.

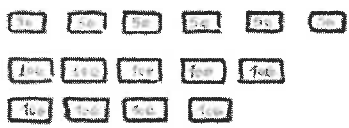
Level 1 – Holistic Reasoning or Insufficiently Justified Explorations (Trial an Error, Answers without Cohesion, Intuitive Level)

At first, correct holistic discoveries are summed up, as well as empiric chaotic invest gations or processes without a concrete goal. We collected responses "by trial and erro without sufficient justification or other statements without logical cohesion that lead t erroneous problem solutions or unsuccessful strategies. Often, students are misled b their previous experience, misinterpreting the problem as a situation of division or pro portionality and making up conceptions which are pitfalls that can trap them. Moreove they don't consider whether their answers satisfy the required constraints. The solutio to the problem is reduced to the discovery of a pair of numbers that verify the equatio $5x+10y=120$ without checking that $x+y=15$ or the opposite. Some students immediatel guessed the solution and gave a correct answer without being able to convince the classmates that their choice is correct. The following dialogue is characteristic:

M.: There will be 6 banknotes of €5 and 9 banknotes of €10 in the piggy-bank.

Teacher: Why?

M.: We have found the correct result. The solution is shown in the figure. Look at it!



Teacher: Could you write your justification on the slide?

M.: We designed our solution and we are sure that is correct!

Teacher: How would you explain it ?

M.: (Silence).

Teacher: How did you find the solution?

D.: By chance!

These students produced a holistic iconic solution to the given problem. This semioti representation shows both the 15 banknotes (separated into two groups: 6 banknotes o €5 and 9 banknotes of €10) and the total of €120 and it is a useful visual guide for mak ing the verifications of the calculations. The image persuades students who need to se the banknotes designed giving them evidence (Hanna, 2000).

The above dialogue indicates that some students predicted the answer, since the successful discovery of the problem isn't related to other guesses or checks. Of course, there is coordination between figural representation and arithmetic result. Despite the logical consistency, there is lack of valid justification. Those students that didn't have a method or a ready algorithm examined arithmetic trials and, without checking their validity, they "fall by chance" to the right solution. Their holistic discovery is represented by an image.

Level 2 – Organised Arithmetic Explorations, Hypotheses and Justifications (Descriptive Level)

The strategies of students are systematic. Usually students create tables on which they place their observations in organized structures. They write data (complete recordings or limited lists) and they choose the numerical solution that satisfies the two conditions of the problem. We will mention one of them.

Group D: The solution of the group was given by a table, different to the complete tables of groups A and B which presented all the possible combinations of banknotes. Here, the total of €120 ($5x+10y=120$) is maintained consistent and the combinations of banknotes of 5 and €10, which form a sum of 15 banknotes ($x+y=15$) is researched. The justifications are described in the following dialogue:

P.: In our group we made the hypothesis that the €120 of the piggy-bank are all banknotes of €5, as showed in the table.

S.: If this was true, we would have 120:5=24 banknotes of €5. However, in our piggy-bank there are 15 banknotes in total, not 24.

D.: You say that there are 15 banknotes that make up €120. I don't understand why we want 24 banknotes, when not all banknotes have a value of €5!

P.: I don't claim that they really are 24. Well, we pretend there are 24 banknotes of €5! We know that this is not true. For this reason, we decide to reduce the banknotes by two in continuous stages. If from 24 banknotes of 5€, we remove two banknotes of €5 and we replace them by a banknote of €10, we'll have €120 again ($120-2\times5+1\times10=120$), but 23 banknotes ($24-2+1=23$). By continuing the same way we create the following table.

Table 1

Replacements Table of the Group

REPLACEMENTS TABLE OF GROUP D				
Replacements numeration	Number of banknotes of €5	Number of banknotes of €10	Total of the piggy-bank in €	Sum of banknote
1.	24	0	24×5+0×10=120	24+0=2
2.	22	1	22×5+1×10=120	22+1=2
3.	20	2	20×5+2×10=120	20+2=2
4.	18	3	18×5+3×10=120	18+3=2
5.	16	4	16×5+4×10=120	16+4=2
6.	14	5	14×5+5×10=120	14+5=1
7.	12	6	12×5+6×10=120	12+6=1
8.	10	7	10×5+7×10=120	10+7=1
9.	8	8	8×5+8×10=120	8+8=16
10.	6	9	**6×5+9×10=120**	**6+9=15**

The previous table shows a linear process for the solution that is characterized by analy sis and description. The most common strategy used is the organization of particula cases by data lists or tables (Allen, 2001). In the previous solution we should point ou the importance of conjecturing as part of the arithmetic reasoning process. The emerge tentative: "*Well, we pretend there are 24 banknotes of €5!*" leads to an erroneous hy pothesis. This false statement was a heuristic mathematical process, linked to furthe investigation, as involves justification and validation steps. In the previous table, w observe that students, without using visual images, try to make successive replacement by removing two banknotes of €5 each time, and replacing them with a banknote c €10. In this way, they increase the plurality of banknotes of €10, while the piggy-ban always has the same total of money. The solution would be found when the sum c banknotes will become 15, i.e. 6 banknotes of €5 and 9 banknotes of €10. While ex changing arguments, a student was able to find another original solution (see level 3).

In addition, during class discussion, students formulate arithmetic hypotheses by cor structing appropriate explanations. These pursuits of explanations, whether they concer something that works mathematically or not, they support the development of a justif cation. However, their solutions aren't the shortest possible. The strategies "invariabl plurality of the piggy-bank banknotes" and "invariable sum of the piggy-bank money were revealed. These findings are consistent with other similar problem researches c "erroneous hypotheses" (Porcheron & Guillaume, 1984; Silver et al., 1990; Sauter, 1998 ; Pluvinage, 2008). The students' strategies lead to two types of tables:

- In the first type, which was mainly observed in groups A and B, students maintained the sum of the piggy-bank banknotes invariable (x+y=15) and successively formed sums of 15, which they checked, while trying to create the number 120 (5x+10y =120) progressively. Students examined the reasons for which their arithmetic conjectures were valid or invalid.
- In the second type of table (Group D, see the above table) students formed compositions of the number 120 and then they controlled them to see if they form the sum 15. In this case, replacements are counted: we have repeated replacements, where two banknotes of €5 are replaced by a banknote of €10 successively. In this way, the plurality of banknotes of €10 increases, while the piggy-bank always has the same total amount of money.

Although organized counting with all the specific data of the table takes time, it represents arithmetic relationships, which have a significant educational importance. Arithmetic operations of the table are fundamental and recurring, and arithmetic investigations play a vital role in comparison with the development of convincing arguments and the justification. The construction of a table is a scientific method that requires accurate and organized verification and raises students' confidence. The presentation of the table leads to reflection and promotes subtle observation of structures and relationships.

Refuting and validating claims, is very important, as it produces true and false arithmetic results, preparing the evolving process of conjecturing and generalizing. A child's justification can support an empirical reasoning process with proof-like arguments (Yackel & Hanna, 2003). In this case, the use of the method of trial and error is not arbitrary, occasional or unexpected. We reformulate briefly the process of student adaptive quantitative reasoning that we found in groups B and F:

> If all the 15 banknotes of the piggy-bank were only banknotes of €10, then the total sum of money would be €150. Similarly, if they all were banknotes of €5, then the piggy-bank would have €75. However, since the content of the piggy-bank is €120, we should try appropriate intermediate combinations, such as:
>
> With 11 banknotes of €10 and 4 banknotes of €5 we find €130.
>
> With 7 banknotes of €10 and 8 banknotes of €5 we find €110.
>
> Further trials approach the number 120:
>
> With 10 banknotes of €10 and 5 banknotes of €5 we find €125.
>
> With 8 banknotes of €10 and 7 banknotes of €5 we find €115.
>
> Finally, trying 9 banknotes of €10 and 6 banknotes of €5 we find €120.

In this double direction process, students examine specific examples; develop refined trials and specific relationships that lead to inferences. Each new trial is justified, as it is formed based on the improvement of the previous one. Gradually, the "error" is corrected, while the trial and error method approaches more and more the expected resul For this reason, it is better to refer to "multiple successive trials", "successive corre tions" or "successive approaches" (Polya, 1962). This evolving reasoning is a valuabl learning practice, which enhances the students' deep understanding and leads them t clarify what is true or false and learn from their mistakes.

Level 3 - Reasoning on Arithmetic Relationships (Informal Deduction)

Students, in both their small groups and within the larger class community, justify the conjectures and develop their strategies by inventing brief, beautiful solutions that ar characterized by reformulated and revised reasoning and original organization of logic arguments. Three perfect solutions that fit into this category (Kosyvas, 2011) were ob served. We will expose a selected dialogue presenting the work of group D.

A.: Looking at the table attentively, we observe that we could have found ho many replacements are necessary from the beginning, without building table, and thus, save time.

Teacher: What do you mean? Could you explain it?

A.: Well, each replacement reduces the banknotes by one and we want to reach 1 banknotes, starting with 24 banknotes. Right?

Teacher: Continue!

A.: To find how many replacements are necessary, we will make a subtractio From 24 we remove 15, i.e. 23, 22, 21, 20, 19, 18, 17, 16, 15. Thus, w find 9 replacements (He is counting with his fingers).

Teacher: Continue!

E.: Why? I don't understand!

Teacher: Understanding requires that you pay attention!

A.: So the replacements that are necessary are 24-15 = 9. I think it is better to calcu late by heart rather than count them one by one.

M.- K.: Or counting with the fingers. I don't agree! We need the entire table. Suc as the one we created.

Teacher: Everyone has his own way. Respect it. Please, don't interrupt. Continu A!

A.: The table is useful but it requires much time to create it. Thus we see that after replacements 15 banknotes will exist in the piggy-bank with a total sur of €120. We have discovered that with 9 replacements we will have re moved 9×2=18 banknotes of €5 in total. Then, we will have 24-18= banknotes of €5 and finally we will have 9 banknotes of €10, since 15- = 9.

P.: The table is useful. If we hadn't created this table ...

A.: The table helped us to find a shorter solution!... It's funny! ...

.....

A.: Our solution is the best. If the piggy-bank had, for example, 100 banknotes, of €5 and €10, and so there was a large amount of money of €950, then would you create a table?

M.- K.: Then no... (silence).

A.: It is impossible! With our idea we can solve the problem for 100 or 1000 banknotes in the piggy-bank. Although the numbers of the problem can be completely different, the fundamental idea remains always the same.

M.- K.: I am not sure if I could think of your solution.

The above justification was repeated by another member of the group and became acceptable, evoking a reflective look at their practice for some of their classmates. At this level the relationships between numbers consist of an object of reflexion. It is clear that the constitution of the table gave the student the idea of the general arithmetic relationships' description that help his flexibly reforming strategy and lead her to a previous simple, synoptic and insightful solution, which is also applied to large numbers. Subtle observation of the systematic organization of special cases in the table contributed to the omission of unnecessary calculations and to the discovery of important hidden regularities, using mental arithmetic and reasoning abstractedly and quantitatively. The condensation of the solution demonstrates a deeper understanding.

The previous reasoning leads to a new kind of informal proof that, according to Wittmann (2009), could be called "operative proof". This proof does not refer to an algebraic description of symbolic objects within a systematic-deductive theory. On the other hand, it is founded on "concrete" arithmetic data ordered in a table that incorporates fundamental mathematical relationships and regularities in a logical way. In this case, inductive reasoning is useful in order to discover properties from arithmetic phenomena that allow operations (Polya, 1962). These operations are generally applicable and independent of the specific objects to which they are applied. So, it is not from particular cases that the generality of a pattern is derived, but from the operations on objects or on arithmetic relationships.

As a result, a discussion is needed for the choice of the best solution. Students looked at a problem from different angles and perspectives, inventing their own methods; some of them appealed to them aesthetically. Certain students felt that it is better to use analytical methods by creating tables, where they can examine many examples (perhaps all) and choose the correct solution. Others believed that it was better to use methods of developing and evaluating arguments mainly via arithmetic operations that are logically chained. Most importantly, with original reasoning, the solutions are brief and elegant. It is remarkable that the dynamic investigation of the problem lead to new derived solutions and beautiful justifications. Students expressed feelings of delight, joy and aesthetic pleasure while talking about their work. The above strategy is refined; it is n based on chance or on full census of many cases that are boring and unnecessary ar that can be and make the interest disappear. Solutions integrated at this high achieve ments level were observed in three groups (due to lack of space, they are not included i this article). These strategies use empirical evidence and can be implemented succes: fully in other similar problems concerning large numbers (Panagakos, 2004). They hav a generalization character, since the arithmetic reasoning on a specific refined solutic can be regarded as a special case. As a result, the close relationship between arithmet: and algebraic reasoning becomes clear. Algebraic thinking is not automatically co nected to using letters (Radford, 2008). These composed solutions include sophisticate methods that are based on justification and develop a generalized, arithmetic, or almo: algebraic reasoning. *Although the numbers of the problem can be completely differen the fundamental idea remains always the same.* This feature is crucial in compariso with the ineffective tables.

In addition, since the students' arithmetic reasoning is direct and eloquent, it has an ac vantage compared to the mechanical use of formal algorithms, which often lead to th loss of sense of important ideas. Using letters as abstract symbols and representing th unknown variables of the linear system is an evolution of the informal strategies. Thi method requires the translation of everyday language in algebraic language. It can b applied successfully for both small and large numbers. It doesn't require the inventio of an original strategy and thus it is effective for a variety of similar problems. This i the next stage for students when they attend the 3rd year of Greek junior high-schoo Finally, it is possible to further generalize the problem using parametric variables. Thi generalization is taught in senior high-school. The almost exclusively abstract, symboli and formal approach to algebra, in which variables are defined as letters that stand fc numbers, can cause many cognitive and affective difficulties (Tall, 2004).

CONCLUSIONS

> Thus a teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his student in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking.
>
> George Polya, 1988, p. v.

In this teaching experiment the majority of the 26 students that were involved, use various ways of reasoning, showed a relational understanding of the problem, experi encing a sense of success from their discoveries. Reasoning was more important tha using previous knowledge in order to solve this problem. Students of different mathe matical levels had the opportunities to be engaged persistently, to make sense of thei strategies, to build and to extend their approaches as well as to acquire rich and mean ingful experience. The task was open-ended, challenging, and thoughtful and it allowe dynamic inquiry for multiple entry points, representations and strategies for solutions. Taking this into account, three different levels of reasoning were presented: the intuitive level with visual or arbitrary explorations, the descriptive level with organised arithmetic relationships and the refined reasoning on the previous relationships followed by the generalisation of the solution (informal or operative proof). Students worked together questioning each other's ideas. They apply previously acquired knowledge and skills, thought with visual images, constructed tables, observed and organised particular cases enhanced and explained arguments that vary in completeness and elegance, identified and grasped general relationships. Among the solutions, there were three creative and inventive ones, which are unique and original and they also focus on arithmetic and not on formal algebraic reasoning.

As a result, students don't use formal procedures that aren't connected to the real world. Algebraic generalization includes only certain aspects of the specific, while, in this work, some interesting advantages emerged, such as properties and patterns, as well as beauty and elegance. These aspects usually remain in the darkness. However, they appeared in original students' answers in the mathematics classroom, favouring a great experience. With the use of abstract algebraic techniques and ready-made mechanical rules, the intrinsic elegance and surprising beauty of these arithmetic solutions would be lost. The more the students learn about the originality of arithmetic reasoning, the more they are engaged in the creative mathematical endeavour. The justification practice pervaded learning mathematics. Justification is a fruitful, complex and intellectually demanding learning practice that requires a consistent knowledge, as well as an understanding of how ideas are logically connected. As a result, arithmetic reasoning and generalizations are closely dependent on each other and this broadens their innovative thought and promotes mathematical creativity.

In addition, the findings of this research lead to the conclusion that during problem solving activities in the classroom, there shouldn't be given priority to the formal algorithmic procedures, which are usually recommended in textbooks, but in the deployment of original ideas and informal students' strategies. Generally speaking, textbooks are full of exercises, but there is lack of real problems. In addition, by offering ready-made solutions, textbooks focus on the "how" and not the "why" of mathematical thinking and they therefore shouldn't be the unique source of the didactic material. Most textbook activities don't request justification and they end up being unsuccessful. It is clear that the engagement and perseverance of the students to problem solving are inherent from the way of the class management by the teacher (Potari & Jaworski, 2002).

Students can find original solutions, invent multiple strategies, build erroneous hypotheses, try and examine ideas, prove their validity, think, justify and develop the responsibility of individual and collective work. In addition, they learn to listen, to share ideas during open discussions, to judge the ideas of their classmates and to support their own arguments, to reflect upon their previous justifications and to amend them. Similar attitudes have been reported in other researches (Brown & Walter, 1983; Silver et al., 1990; English, 1997). All these proofs that the problem studied is suitable for the collaborative investigation and the development of students' mathematical creativity and reasoning.

We suggest that the strands of open-ended tasks, that elicit reasoning and build unde standing and communicating, can be integrated into regular mathematics instruction all grade levels. They serve to engage students in doing mathematics and to build co rect and original arguments, which play a vital role in developing a mathematicall powerful world.

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Investigating Mathematical Creativity from a Cognitive Perspective

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ABSTRACT: *The present study examines the effect of a number of cognitive factors on mathematical creativity, such as mathematical abilities, intelligence, working memory, speed and control of processing. Specifically, the aim of the study was threefold. Firstly, to assess whether a theoretically driven model fits the data of the study. Secondly, to investigate whether different groups of students can be traced across mathematical creative ability and thirdly, to investigate which factors cause differentiations on students' mathematical creativity. The study was conducted among 359 elementary school students in Cyprus, using four instruments. Data analysis revealed that students' mathematical creativity may be affected by mathematical abilities. Intelligence, working memory, speed and control of processing, were not found to predict mathematical creativity. Furthermore, the participants could be grouped in four distinct categories according to their mathematical creativity. These categories of students presented statistically significant differences across specific mathematical abilities and intelligence.*

Key words: *Mathematical creativity, Mathematical ability, Intelligence, Memory, Processing.*

INTRODUCTION

Mathematical creativity is considered as an essential ability that may and should be enhanced in all students (Pelcszer & Rodríguez, 2011). Therefore, international organizations, contemporary curricula and researchers stressed the importance of developing students' creative thinking in order to be congruous with the continuous changes and challenges of current society (e.g. Lamon, 2003; NCTM, 2000). In an attempt to enhance creative ability in all students, further research is needed, regarding the factors that differentiate creativity across students.

Recently researchers examined in which way several cognitive factors may affect general creative ability. Among these factors, intelligence (Sternberg & O'Hara, 1999), prior knowledge (Sheffield, 2009), memory (Geary & Brown, 1991) and information processing (Sternberg & O'Hara, 1999) were included. The present study purports to examine whether the abovementioned cognitive factors may affect students' creative ability in the domain of mathematics, as well. In particular, we first intend to examir whether a theoretically driven model, concerning the prediction of mathematica creativity from several cognitive factors, fits the data of the study. Moreover, we wi investigate whether participants may be grouped according to their mathematica creative ability. Finally, we will examine the discrepancies among these groups c students, in order to shed light on the reasons that differentiate students' mathematica creativity.

THEORETICAL FRAMEWORK

Creativity is commonly considered as a psychological construct that refers to th production of something new and novel, breaking away from existing pattern (Sternberg & Lubart, 2000). Over the past years, researchers investigated th relationship between creativity and several cognitive and psychological factors, such a intelligence, memory, processing, prior knowledge and abilities (e.g. Sternberg & O'Hara, 1999; Sheffield, 2009).

In regard to the relationship between creativity and intelligence, research studie focused either on investigating whether there is a relationship between the two concept or on examining the construct of this relationship. As for the relationship betwee creativity and intelligence, incongruent results have been proposed. On the one hand, statistically significant relationship was found between the two concepts (e.g. Ripple & May, 1962; Srivastava & Thomas, 1991), whereas on the other hand creativity wa found to be independent from intelligence (e.g. Getzels & Jackson, 1962). Recentl researchers adopted the first perspective and therefore the research interest focused o the structure of the relationship between creativity and intelligence. In particula Gardner (1993) proposed that intelligence constitutes a superset of which creativity is subset whereas Sternberg and Lubart (1995) considered intelligence to be a subset c creativity. Furthermore, other researchers considered intelligence and creativity a overlapping sets (e.g. Sternberg, 1985) or even as disjoint sets (e.g. Torrance, 1975).

Moreover, content knowledge has been reported as the variable that contributes mor than any other variable to students' creativity (Sak & Maker, 2006). Content knowledg constitutes the backbone on which new information will be organized (Sheffield, 2009 and provides the foundation on which individuals may be based on, in order to develo new ideas (Weisberg, 1999). Particularly, if an individual knows well how things wor in a field, he/ she is able to propose new ideas beyond stereotypes without concentratin on the manipulation of basic skills (Sternberg, 2006; Weisberg, 1999). In Weisberg' words (1999) "Rather than breaking out of the old to produce the new, creative thinkin builds on knowledge" (p. 226).

Apart from content knowledge, the importance of memory in creative thinking is als emphasized by a number of researchers (e.g. Guilford, 1962, in Mann, 2006; Sternber & O'Hara, 1999). According to Cowan's definition (1999), "Working memory refers t cognitive processes that retain information in an unusually accessible state, suitable fo carrying out any task with a mental component" (p. 62). Although creativity is usually considered as a spontaneous action of producing a novel idea or action, the ability to recall knowledge and processes and use them appropriate in a specific context for new potentials is necessary. Thus, flexible thinking and appropriate switching between conceptual systems characterize the way in which creative thinkers process information (Sternberg & O'Hara, 1999).

Taking into consideration the abovementioned literature review, we may conclude that there are numerous studies which aim to investigate the effect of cognitive factors on creativity. However, there is a lack of corresponding research on domain specific creativity, such as mathematical creativity. Furthermore, these studies examined the relationship between cognitive factors and creativity in isolation; in other words no attempt has been made to ascertain the effect of a combination of cognitive factors on mathematical creativity. Due to this fact, the first aim of the study is to assess whether a theoretically driven model, that assumes that several cognitive factors may predict creative ability in mathematics, fits the data of the study. The second aim of the study is to trace groups of students that differ across mathematical creativity. Finally, the third aim of the study is to investigate which factors cause differences in students' level of mathematical creativity.

METHODOLOGY

Sample

The sample for this study consisted of 359 4th, 5th and 6th graders from eight elementary schools in Cyprus. One hundred and forty three students attended 4th grade, while 118 and 98 students attended 5th and 6th grades, respectively. All students attended average public schools in Nicosia, in urban and suburban areas. The only requirement for a school to be used in this study was the existence of a computer lab. This requirement was due to the fact that the instruments were presented and solved in electronic form.

Instruments

Each student completed four instruments: (a) the Mathematical abilities instrument, (b) the Mathematical creativity instrument, (c) the WASI Matrix Reasoning Scale (Fluid intelligence instrument), (d) the Working memory, speed of processing and control of processing instrument. The WASI Matrix Reasoning Scale was administered in a paper and pencil format, whereas the rest of the instruments were administered electronically. What is followed is a brief description of each of the instruments.

Mathematical Abilities Instrument

Given that mathematical ability is not a uni-dimensional entity rather it is a multidimensional construct (Krutetskii, 1976), the mathematical abilities instrument includes 29 tasks measuring the following abilities (examples of tasks are presented i Pitta-Pantazi, Christou, Kontoyianni, & Kattou, 2011): manipulation of quantitie (quantitative ability), causal relationships (causal ability), visualization and spati reasoning (spatial ability), processing of similarities and differences (qualitative ability inductive/deductive reasoning (inductive/deductive ability).

Mathematical Creativity Instrument

For the design of the mathematical creativity test, we found guidance from previou research. It appears that previous studies used open-ended problems or multipl solutions tasks and assessed students' creativity based on the fluency, flexibility an originality of their solutions (e.g. Kattou, Kontoyianni, Pitta-Pantazi, & Christou, 201 Leikin, 2007). Fluency refers to the number of correct responses that students presente For flexibility, the different types of responses were measured. Originality is calculate by comparing a student's solutions with the solutions provided by all students and th rarest correct solutions received the highest score. The mathematical creativit instrument includes five open-ended multiple-solution mathematical tasks in whic students were required to provide: (a) multiple solutions, (b) solutions that wer different between them and (c) solutions that none of their peers could provide.

Fluid Intelligence Instrument

Fluid Intelligence is measured using the subtest Matrix Reasoning Scale from th Wechsler Abbreviated Scale of Intelligence (WASI) (Wechsler, 1999). The WAS Matrix Reasoning Scale provides a measure of nonverbal fluid abilities using 32 task for students of 9 to11 years old and 35 tasks for students older than 11 years. There ar four types of items: pattern completion, classification, analogy and serial reasoning.

Working Memory, Speed and Control of Processing Instruments

The working memory and processing instrument includes two activities. The firs activity measures an individual's ability to remember a figure when it appears on th screen for a short period of time and to distinguish it among other similar figures. In th second activity the individual has to focus on the form of stimuli presented and press th correct keyboard button according to the stimuli. The items that the stimuli appears i the same direction as the keyboard button that had to be pressed, measure speed o processing. The items for which the keyboard arrow to press is inconsistent with th part of the screen that the stimuli appears, measure control of processing. Each activit is timed and the student has to respond rapidly and correctly.

Scoring and Analysis

The items of the mathematical abilities instrument were marked as correct (1) o incorrect (0). Students' creativity was assessed based on the fluency, flexibility an originality of their solutions (Leikin, 2007). Specifically, we performed the following steps: (a) For the fluency score we calculated the ratio: number of the correct mathematical solutions that the student provided, to the maximum number of correct mathematical solutions provided by a student in the population under investigation. (b) For the flexibility score we calculated the ratio: number of different types of correct solutions that the student provided, to the maximum number of different types of solutions provided by a student in the population under investigation. (c) For the originality score, we calculated the frequency of each solution's appearance, in relation to the sample under investigation. A student was given the score 1 for originality if one or more of his/her answers appeared in less than 1% of the sample's answers. Correspondingly, a student was given a score of 0.8 if the frequency of one or more of his/her answers appeared in between 1% and 5%, a score of 0.6 if the frequency of one or more of his/her answers appeared in between 6% and 10%, a score of 0.4 if the frequency of one or more of his/her answers appeared in between 11% and 20%, a score of 0.2 if one or more of his/her answers appeared in more than 20% of the sample's answers. Three different numbers (fluency, flexibility and originality scores) were calculated for each student, indicating the score in each mathematical creativity task. The total fluency, flexibility and originality scores were obtained by adding the respective scores across the five creativity tasks (Kattou et al., 2013).

For the scoring of the Fluid Intelligence instrument we followed the directions provided by the manual of the Wechsler Abbreviated Scale of Intelligence (Wechsler, 1999). For the working memory, speed of processing and control of processing instruments, we calculated students' reaction time. This was calculated by dividing the total time needed for the tasks to be completed by the number of correct responses of each individual. As a result, three reaction times were calculated for each participant.

Data Analysis

The objectives of the analysis was first to articulate and empirically test a theoretical model that addresses the effect of cognitive aspects, namely mathematical ability, intelligence, working memory, speed and control of processing on mathematical creativity (see Fig. 1). Secondly, it was our aim to trace groups of students that differ across the components of mathematical creativity and thirdly to examine which cognitive factors differentiate these groups of students.

In regard to the first objective, Confirmatory Factor Analysis (CFA) was conducted in order to investigate the fit of the theoretical model to the data of the present study, using the statistical modeling program MPLUS (Muthén & Muthén, 1998). It is important to note that the CFA gives some indices of goodness of fit for the model in which the evaluation of models are based. For the purposes of the present study, goodness of fit was based on three fit indices: the comparative fit index (CFI), the ratio of chi-square to its degree of freedom (x^2/df) and the root-mean-square error of approximation (RMSEA). According to Marcoulides and Schumacker (1996), for the model to be confirmed, the values for CFI should be higher than 0.90, the observed values for $x^2/$(should be less than 2 and the RMSEA values should be close to or lower than 0.08.

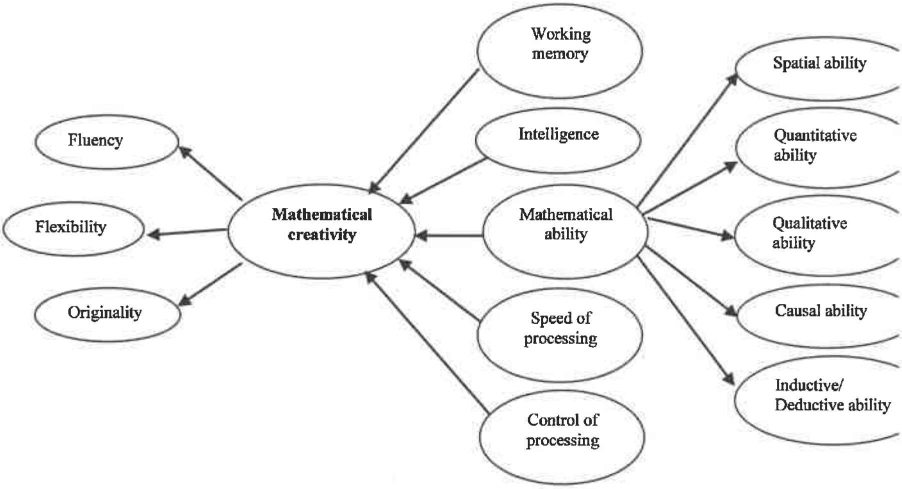
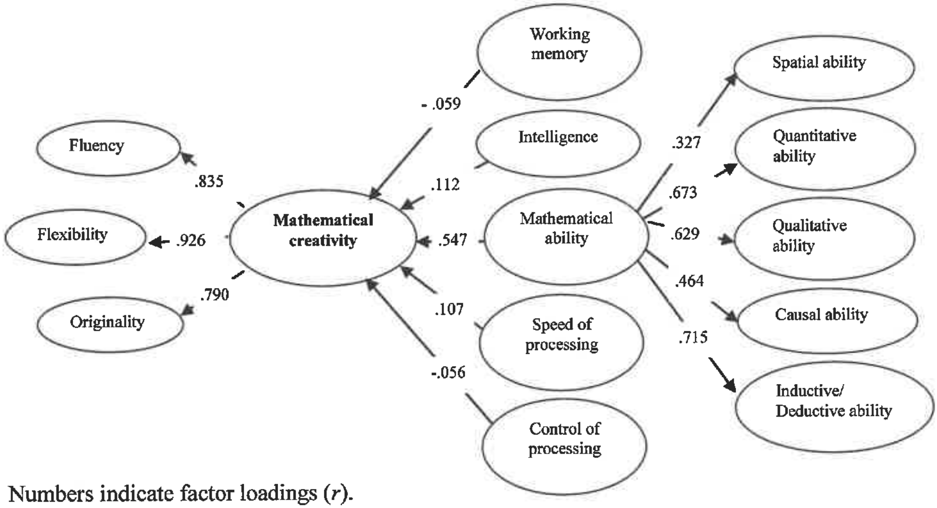


Figure 1. The structure of the proposed model.

As for the second objective, latent class analysis was used to explore whether there wer different categories of students in our sample whose achievement could vary accordin to mathematical creativity. Once the latent class model was estimated, subject classified to their most likely class by mean of recruitment probabilities. For th accomplishment of the third objective, analysis of variance (ANOVA) was conductec in an effort to investigate differences between the groups of students on mathematica abilities (spatial ability, quantitative ability, qualitative ability, causal ability, inductive deductive reasoning ability), intelligence, working memory, speed and control c processing due to their different degree of mathematical creativity (fluency, flexibility originality).

RESULTS

The first aim of the study concerns the extent to which a theoretically driven model fit the data. Data analysis revealed that the theoretical model matched the data set of th present study and determined the "goodness of fit" of the factor model (CFI=0.991 x^2=16.845, df=10, x^2/df=1.684, RMSEA=0.044). Figure 2 presents the structura equation model with the variables and their indicators.



Numbers indicate factor loadings (*r*).

Figure 2. The confirmation of the structure of the proposed model.

According to the model, mathematical creativity consists of flexibility (r=.926, p<.05), fluency (r=.835, p<.05) and originality (r=.790, p<.05), whereas mathematical ability consists of five distinct components: (a) inductive/deductive ability (r=.715, p<.05), (b) quantitative ability (r=.673, p<.05), (c) qualitative ability (r=.629, p<.05), (d) causal ability (r=.464, p<.05) and (e) spatial ability (r=.327, p<.05). The structure of the proposed model also addresses that mathematical ability may affect mathematical creativity (r=.547, p<.05). On the contrary, the loadings of intelligence (r=.112, p>.05), working memory (r=-.059, p>.05), control (r=-.056, p>.05) and speed of processing (r=.107, p>.05) were not statistically significant, indicating that they have not effect on mathematical creativity.

Concerning the second aim of the study, latent class analysis (LCA) was used, testing the model under the assumption that there are two, three, and four groups of subjects varying on their performance on mathematical creativity instrument. The best fitting model with the smallest AIC (1610.68) and BIC (1680.58) and the largest entropy (0.821) indices (see Muthén & Muthén, 1998) was the one involving four groups. Taking into consideration the average group probabilities as shown in Table 1, we may conclude that classes are quite distinct, indicating that each class has its own characteristics.

Table 1

Average Group Probabilities by Group

Groups	Low mathematical creativity	Average mathematical creativity	Above average mathematical creativity	High mathematical creativity
1 (N=22)	0.902	0.098	0.000	0.000
2 (N=148)	0.035	0.904	0.061	0.000
3 (N= 149)	0.000	0.070	0.888	0.042
4 (N=40)	0.000	0.000	0.101	0.899

From the information presented in Table 1, it can be deduced that students can b grouped in four distinct levels of mathematical creative ability; Group 1 (N=22) consis of low mathematical creativity students, Group 2 (N=148) consists of averag mathematical creativity students, Group 3 (N=149) consists of above averag mathematical creativity students and Group 4 (N=40) consists of high mathematic creativity students. The means and standard deviations of each of the components c mathematical creativity and mathematical abilities tests are presented in Table 2.

Table 2

Means and Standard Deviations of Students' Performance in Mathematical Abilitie and Mathematical Creativity Components

Groups		Flu.	Flexib.	Origin.	Spatial Ability	Quantit. ability	Qualit. Ability	Ind./Ded. ability	Causal ability
1	Mean	0.291	0.738	0.827	0.727	0.909	0.909	1.648	1.084
	S.D.	1.944	0.285	0.429	0.827	1.109	0.868	1.197	0.978
2	Mean	0.984	1.421	1.647	1.114	1.188	0.959	1.879	1.466
	S.D.	0.324	0.240	0.491	0.897	1.055	1.045	0.957	0.903
3	Mean	1.688	2.013	2.449	1.567	1.865	1.595	2.551	1.735
	S.D.	0.401	0.252	0.494	1.155	1.245	1.239	1.006	0.859
4	Mean	2.430	2.560	3.440	1.775	2.600	2.375	3.125	2.012
	S.D.	0.460	0.271	0.538	1.206	1.428	1.213	0.775	0.866
Total	Mean	1.392	1.750	2.127	1.351	1.607	1.376	2.281	1.615
	S.D.	0.659	0.522	0.823	1.069	1.279	1.226	1.070	0.910

Table 2 presents that students in Group 4 are the high mathematical creative students, since they outperformed students in Group 3, Group 2 and Group 1, across fluency, flexibility and originality. Students in Group 3 have above average mathematical creative ability, in all the aspects of the concept, and furthermore they outperformed their counterparts in Groups 2 and 1. In addition to this, Group 2 students also outperformed students in Group 1. It is important to note, that students with high score across mathematical creativity components have also high score across all the aspects of mathematical abilities test. Similar patterns of performance exist between the components of mathematical creativity and mathematical ability across the remaining groups of students (Groups 1, 2, 3).

From the information presented in Table 3, it can be deduced that the four groups of students yielded by the different degree of mathematical creativity reflect differences on students' fluid intelligence, working memory, speed and control of processing. The mean of fluid intelligence increases from Group 1 to Group 4, indicating that as the fluid intelligence is increased the degree of mathematical creativity is higher. On the other hand, the means of working memory, speed of processing and control of processing decrease from Group 1 to Group 4. Since ability in these three instruments was based on reaction time, the most successful group (Group 4) was the one with the smallest mean of reaction time and the least successful the one with the larger mean of reaction time (Group 1). In other words, the smallest reaction time needed to respond to these instruments the more likely is that the individual will be mathematically creative.

Table 3

Means and Standard Deviations of Students' Performance in the Intelligence, Memory, Speed and Control of Processing Instruments

Groups		Fluid Intelligence	Working Memory	Speed of Processing	Control of Processing
1	Mean	0.530	2.097	1.089	1.252
	S.D.	0.190	2.125	0.285	0.636
2	Mean	0.599	1.552	1.021	1.157
	S.D.	0.194	2.714	0.249	0.344
3	Mean	0.712	1.380	1.026	1.096
	S.D.	0.147	0.872	0.277	0.404
4	Mean	0.775	1.436	0.964	1.026
	S.D.	0.110	2.050	0.239	0.288
Total	Mean	0.661	1.502	1.021	1.123
	S.D.	0.182	2.028	0.262	0.389

A one-way ANOVA was used to test for statistically significant differences among th four groups of students, as it is presented in Table 4. The differences between th groups of students across spatial, quantitative, qualitative, inductive/deductive an causal abilities, as well across fluid intelligence are statistically significant, wherea working memory, speed and control of processing cannot differentiate students mathematical creativity. Given this fact, we can assume that although there ar differences in students' performance on working memory, speed and control o processing these differences are not statistically significant.

Table 4

ANOVA Analysis Indicating Differences among the Groups of Students

	df	F	p
Spatial ability	3	9.703	0.000
Quantitative ability	3	20.415	0.000
Qualitative ability	3	19.937	0.000
Inductive/Deductive ability	3	24.045	0.000
Causal ability	3	5.985	0.000
Fluid Intelligence	3	21.817	0.000
Working Memory	3	3.511	0.466
Speed of Processing	3	1.135	0.335
Control of Processing	3	2.271	0.080

Furthermore, post-hoc analysis took place, aiming to investigate which variables lead t discrepancies among the groups of students. Scheffe post-hoc comparisons (Table 5) o the four groups indicated statistically significant differences between low creativ students (Group 1) with above average (Group 3) and high creative students (Group 4) In particular, the differences of the groups across spatial (Group 1 and 3: p=0.006 Group 1 and 4: p=0.002), quantitative (Group 1 and 3: p=0.007, Group 1 and 4 p=0.001), inductive/deductive (Group 1 and 3: p=0.001, Group 1 and 4: p=0.001), an causal (Group 1 and 3: p=0.017, Group 1 and 4: p=0.002) abilities, as well across flui intelligence (Group 1 and 3: p=0.001, Group 1 and 4: p=0.001) were statisticall significant. By comparing students qualitative ability, we noticed that that low creativ students had statistically significant difference with high creative students (Group 1 an 4: p=0.001) but not with above average creative students (Group 1 and 3: p=0.076) Comparisons between low creative students (Group 1) and average creative student (Group 2) did not reveal any statistically significant difference on these cognitiv aspects, indicating common performance.

Table 5

Comparing the Performance of the Four Groups of Students on the Mathematical Ability and Intelligence Tests

		Group 2	Group 3	Group 4
Group 1	Spatial	.443	.006*	.002*
	Quantitative	.787	.007*	.001*
	Qualitative	.998	.076	.001*
	Causal	.314	.017*	.002*
	Inductive/Deductive	.782	.001*	.001*
	Intelligence	.360	.001*	.001*
Group 2	Spatial	-	.003*	.005*
	Quantitative	-	.001*	.001*
	Qualitative	-	.001*	.001*
	Causal	-	.079	.008*
	Inductive/Deductive	-	.001*	.001*
	Intelligence	-	.001*	.001*
Group 3	Spatial	-	-	.076
	Quantitative	-	-	.008*
	Qualitative	-	-	.002*
	Causal	-	-	.381
	Inductive/Deductive	-	-	.013*
	Intelligence	-	-	.214

* Statistical significant differences, $p<.05$

On the comparison of average creative students (Group 2) with above average creative students (Group 3) and high creative students (Group 4), statistically significant differences were presented across spatial (Group 2 and 3: $p=0.003$, Group 2 and 4: $p=0.005$), quantitative (Group 2 and 3: $p=0.001$, Group 2 and 4: $p=0.001$), qualitative (Group 2 and 3: $p=0.001$, Group 2 and 4: $p=0.001$), inductive/deductive (Group 2 and 3: $p=0.001$, Group 2 and 4: $p=0.001$) abilities and across fluid intelligence as well (Group 2 and 3: $p=0.001$, Group 2 and 4: $p=0.001$). Regarding causal ability statistically significant differences were observed between Group 2 and Group 4 ($p=0.008$) but not between Group 2 and Group 3 ($p=0.079$).

As for the differences between above average creative students (Group 3) and hig creative students in mathematics (Group 4), the analysis revealed statistically significar discrepancies across quantitative ($p=0.008$), qualitative ($p=0.002$) and inductive deductive ($p=0.013$) abilities. The two groups behave similarly in regard to spatia ($p=0.076$) and causal ($p=0.381$) abilities, and have got similar degree of flui intelligence ($p=0.214$).

DISCUSSION

In a changing and challenging world, education should not only focus on increasin students' knowledge but it should empower students' ability to approach problem creatively (Best & Thomas, 2007). Indeed, according to Hunsaker (2005), creativ thinking provides students with a mechanism to manage change and challenge Therefore, the necessity of enhancing students' creative ability is obvious, leading th research community to investigate the reasons that some students are more creative tha others.

For the accomplishment of this objective, the examination of the effect of severa cognitive factors, such as intelligence, prior knowledge/abilities, memory an information processing on creativity were examined in the context of mathematics More specifically, the objectives of the present study were firstly to assess whether theoretically driven model fits the data. Secondly, to investigate whether participant may be grouped according to their mathematical creative ability and thirdly, t investigate whether mathematical abilities, fluid intelligence, working memory, spee and control of processing have the potential to differentiate students' mathematica creativity.

In regard to the first aim of the study, a theoretical model was conceived whicl integrated several cognitive factors as predictors of mathematical creativity. Th analysis of the data verified that the fundamental components of mathematical creativit are fluency, flexibility and originality (e.g. Kattou, et al., 2013; Leikin, 2007) Additionally, mathematical ability in terms of spatial, quantitative, qualitative, causal inductive/deductive abilities may predict mathematical creative ability. In contrast other factors as working memory, intelligence, speed and control of processing canno affect mathematical creativity. To this end, it can be assumed that mathematical abilit is one of the components that contributes to the development of mathematical creativity This is in accord with other researchers who claimed that content knowledge is th variable that contributes more than any other variable to students' mathematica creativity (Sak & Maker, 2006).

Furthermore, four different categories of students can be identified that varied acros mathematical creativity. Group 1 reflects students with low mathematical creativity Group 2 reflects students with average mathematical creativity, Group 3 reflect students with above average mathematical creativity and Group 4 reflects students witl high mathematical creativity. Between these groups of students appeared statistically significant differences across all mathematical abilities and fluid intelligence.

In a deeper analysis of the differences between the groups of students, we observed that the four groups of students who varied in the degree of mathematical creative potential represented three distinct groups of students varying in mathematical abilities. More concretely, students with low and average mathematical creativity behaved similarly and they were distinguished from above average and high creative students, across spatial, quantitative, qualitative, inductive/deductive, and causal abilities, and fluid intelligence as well. Taking into consideration the abovementioned result, one may conclude that for above average creativity, above average abilities are prerequisite. Indeed, if an individual has strong cognitive background he/she is more able to combine and/ or edit ideas in order to propose something novel in a specific domain (Sternberg, 2006; Weisberg, 1999).

Furthermore, high creative students were differentiated from above average students due to their ability in manipulation of quantities (quantitative ability), in processing similarities and differences (qualitative ability) and in inductive/deductive reasoning (inductive/ deductive ability). With respect to quantitative ability, if an individual is able to manipulate basic mathematical knowledge and skills, such as numbers and operations, he/ she is focus on proposing new ideas beyond stereotypes without concentrating on the manipulation of basic skills (Sternberg, 2006; Weisberg, 1999). In regard to qualitative ability, Haylock (1987) proposed that "creativity includes the ability to see new relationships … and to make associations between possible unrelated ideas" (p. 60), justifying that the ability to see patterns and relationships between numbers and figures leads to the emerging of creative thinking. Finally, inductive/deductive ability is incorporated in the definition of mathematical creativity given by Ervynck (1991). As Ervynck (1991) stated "mathematical creativity is the ability to solve problems … taking account of the peculiar logico-deductive nature of the discipline" (p.47).

With respect to intelligence, analysis revealed that this cognitive factor differentiated low and average creative students from above average and high creative students, creating two groups of students. This result proposes that a high degree of intelligence appears to be necessary for above average creativity but not a sufficient condition for high creativity (Torrance, 1975).

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Teaching Non Euclidean Geometries Through Drama in Education: Creativity for Enhancing Mathematics Teaching

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ABSTRACT: *The present paper describes experiences through a teaching experiment where drama-based teaching of the process of axiomatic definition of Euclidean and Non-Euclidean Geometries was interrelated to the history of Euclid's 5th postulate in an 11th grade students. Our main focus was to explore the students' understanding of the three systems of Geometry as axiomatic ones and to challenge their stereotypical beliefs around mathematics as a static body of knowledge. The use of ethnographic research techniques (i.e. participant observation and interviewing) helped us to gather empirical evidence concerning students' experiences. Our research offered considerable evidence for the effectiveness of drama techniques as an approach, which could help for the forming of creative and innovative practices in a class where students' mathematical activity can be active and cooperative.*

Key words: *Mathematics education, Non Euclidean geometries, Drama in education, Creativity, Interdisciplinarity.*

INTRODUCTION

Researcher: If someone would suddenly come into Geometry class what could he/she observe?

Student (11^{th} grade): A teacher who talks all the time; either he is drawing geometrical figures on the blackboard or trying hardly through several ways to keep us awake. The half of the class, since they are majoring in theoretical sciences are not engaging, but they talk amongst themselves, send messages, roll cigarettes, read magazines, solve crosswords!

The above dialogue is typical amongst secondary school students in Greece who often perceive the teaching of Geometry as a very abstract and boring subject. Around Geometry as a mathematics curriculum subject, there are disagreements regarding its necessity for teaching, priorities for didactic objectives and content as well as the suitability of teaching methods at different levels of education. The 'New Math' reform in mathematics education resulted in the nearly disappearance of Geometry content a an autonomous subject in most curricula worldwide. However, Geometry according t the International Commission on Mathematical Education (1995, p.52) is an importan domain of knowledge that enables intuitive thinking, theoretical proof and relatednes to the physical space(s) we live, and is regarded, perhaps, as the most intuitive an closer to the 'real' world domain knowledge and as a 'precise' analytical tool tha supports understanding of space and spatial relations whilst at the same time tends t enhance human efforts for describing and interacting with the space in which we liv (ICMI, 1995).

However, it has been argued that learning geometry is not an easy process (Clements & Battista, 1992) and part of students' difficulties have been attributed to the 'structure' o a traditional Geometry curriculum (Freudenthal, 1971), as well as, to the way geometr becomes taught through mainly talk and chalk and abstract problem solving methods For this reason, mathematics education researchers often raise questions regarding th quality and purposefulness of content and methods of its teaching. At the same time they identify a need to search for alternative didactic approaches and innovativ practices so that the students can be engaged actively and creatively in classroon lessons. Drama in Education is, as shown by recent and older research (Andersen, 2004 Somers, 2001; Wagner, 1999; Wooland 1999; Heathcote, 1984; Bolton, 1986) a innovative mediating tool for learning and sets the framework for the students t become active thinkers, through experiential processes that subsequently foster thei creativity. For this reason, the use of Drama in Education is proposed in our research, a an innovative approach for the teaching of Geometry. Specifically in mathematic education, drama techniques have been used to construct a context not only fo cognitive development in mathematics but also for raising students' awareness o mathematical connections to their physical and social environment (Chronaki, 1990 Stathopoulou, Chronaki & Kotarinou, in press)

CREATIVITY IN MATHEMATICS EDUCATION

Since the 1950s there has been a strong concern that education should prioritize th development of creativity and from the middle 1990s till now, there has been a growin recognition from policymakers and commentators alike, that learner creativity is a extremely important aim for education (Craft, 2001). Major studies, reports an initiatives across a range of countries have advocated for the increased focus o teaching for creativity and they argue for the arts as having an essential role in thi process (Davis, 2007). There is a considerably robust body of research concerning th concept of creativity, which has been described in varied ways. A common thread i that creativity '*is a personal activity intended to produce something new*' (Bolden et al 2010, p. 144). Halpern (1984, pp. 344) stated that '*creativity can be thought of as th ability to form new combination of ideas to fulfill a need*'. Perkins (1984) also ha stressed the notion of new and unique combinations of ideas as a defining characteristi of creative thinking but emphasized the importance of generating a product (mental o physical) that fulfills a specific function in an appropriate, yet unique, way.

Some researchers -adopting a platonic or essentialist point of view- envision creativity as something that one possesses that depends solely in the individual, despite research efforts for indicating that appropriate educative experiences can increase creative production (Sternberg & Lubart, 1992). Contrary to such a static view, Plucker and colleagues (Plucker et al., 2004 p. 90) define creativity as the '*interaction among aptitude, process, and environment by which an individual or group produces a perceptible product that is both novel and useful as defined within a social context*'. Underscoring the role of social context they adopt a perception about the nature of creativity as something that is non-static, liquid, and subject to change. Csikszentmihalyi (2000) through a 'systems approach' tries to describe creativity as an interaction between an individual, the domain, and the field in which creativity occurs (Csikszentmihalyi, 2000; Sriraman, 2009, p. 17). The 'systems approach' focuses on the social and cultural dimension of creativity, instead of simply viewing creativity as an individualistic psychological approach. Thus, according to Csikszentmihalyi, creativity is not just a mental activity, but above all a cultural and social event (as cited in Makel & Plucker, 2008. p. 249).

Sternberg and Lubart (2000) define creativity as the ability to produce unexpected original work, which is useful and adaptive. Regarding creativity's dimension of usefulness Sriraman, (2009, p.14) notices that mathematicians do not consider the results of creative work as having useful implications applicable to real world situations at all times: *Mathematicians would raise several arguments with this definition [of creativity], simply because the results of creative work may not always have implications that are ''useful'' in terms of applicability in the real world....hence, I think it is sufficient to define creativity as the ability to produce novel or original work* (pp. 3–4). Beghetto and Kaufman (2009, p. 40) place the question: *What level of novelty and meaningfulness is necessary for something or someone to be considered creative?* They notice that in some cases, for example speaking about Einstein there is no matter. This kind of creativity in the literature is referred as big-C (creativity) in opposite array of little c creativity. Big-C creativity is eminent creativity such as the creativity of Freud or Picasso (Gardner, 1993) while little-c is everyday creativity—creativity that people can perform on a regular basis (Makel & Plucker, 2008, p. 251), a type of creativity accessible to almost everybody (Runco & Richards, 1998). Beghetto & Kaufman (2007, p. 73) introduce one more aspect of creativity: 'mini-c creativity'. They define mini-c creativity as the novel and personally meaningful interpretation of experiences, actions and events. Their definition of mini-c incorporates both the ''personal'' (Runco, 1996, 2004) and ''developmental'' (Cohen, 1989) aspects of creativity.

What is then that differentiates mini-c creativity from Big–c and little-c is the intrapersonal dimension: *both little-c (or everyday) creativity and Big-C creativity rely on interpersonal and historical judgments of novelty, appropriateness, and lasting impact* (Beghetto & Kaufman, 2007, p. 73). According to Vygotsky (1967), as cited in Beghetto and Kaufman (2009), "*any human act that gives rise to something new is referred to as a creative act, regardless of whether what is created is a physical object* *or some mental or emotional construct that lives within the person who created it and i known only to him"*(p. 41). Adopting the idea of mini-creativity, children are bein creative in the mathematics classroom when they produce something new to themselve (Haylock, 1987; Bolden et al, 2010, p. 144). Specifically, Beghetto and Kaufman asse that everyone has some creative potential—even if that potential has not yet bee manifested. Under such an argumentation mathematics' educators have to striv towards re-activating their students' creativity. It is, thus, necessary to start recognizin cultivating, the creative potentials of all our students. And this could be possibl through, for example, finding suitable problems, and encouraging students to take th intellectual risks necessary for sharing their unique mathematical insights an interpretations (Beghetto & Kaufman , 2009, p. 41). Our research, as part of thi discussion, focuses towards exploring the extent to which the use of Drama i Education techniques could support the production of innovative practices i mathematics education where students' creativity can be re-activated and, consequentl their learning of Geometry can be enhanced.

Drama in Education and Creativity

The use of drama techniques in education has been proposed among others, as framework that offers innovative ways for teaching varied disciplinary curriculum area and as creative ways for the students to reflect on broader learning issues (social political). Drama in Education can be a highly structured pedagogical procedur utilizing specific rituals and techniques of dramatic art aiming to focus participants attention towards *the process* of participants' experience and not on the final produc (Alkistis, 2000). As a result, these drama perspectives do not sharply distinguis between actor and audience; the learner is both participant and observer, playing a rol while interacting with others in role (Andersen, 2004, p.282). Drama in Educatio constitutes an embodied and experience-based approach to teaching and aims a collaborative, active learning (Alkistis, 1998; Alkistis, 2000; Somers, 2001; Wagne 1999; Wooland 1999) while providing the child the opportunity to increase participatio to develop understanding and curiosity, and to enhance the ability to express ideas, self consciousness and skills in team-work (Alkistis, 2000). The main advantages of usin drama in teaching varied curriculum areas, the effect on students' intrapersonal an interpersonal skills, the teacher-students relationship, as well as, the cognitive an metacognitive domains.

It has been argued extensively that drama techniques stimulate the imagination an support the development of creative thinking (Annarella, 1992; Bolton 1986; Kelne 1993; Morris, 2001: cited in Duatepe & Ubuz, 2004a) and critical thinking (Bailin 1998), foster divergent and reflective thinking (Andersen, 2002; Neelands, 1984) develop problem solving ability (Bolton, 1985), foster decision making skills, serve a methods to foster metacognition in classroom learning (Andersen, 2002; Johnson, 2002 and act as a catalyst in interdisciplinary projects (Somers, 2001). Regarding creativity drama provides the individual with the opportunity to play with objects in new an unusual ways and to participate in any kind of activity within a social context. As such it involves interaction, imagination and symbolic transformation similar to the creative process. Overall Drama in Education, when successfully used, directly impacts on students' creativity by means of integration or intervention strategies, transformation, imagination, flexibility and affective processes (Dürdane, 2009).

Regarding the teaching and learning of mathematics, although relatively limited, the research of the influence of introducing drama in the teaching process is very encouraging in students' understanding and retention of mathematical notions (Saab, 1987; Omniewski, 1999; Fleming et al.; 2004; Duatepe & Ubuz 2004b; Kotarinou & Stathopoulou, 2008), in creating positive impact in their attitudes towards mathematics (Kayhan, 2008; Duatepe, 2004a) and in offering to students opportunities for reflection on their social interaction in mathematics (Chaviaris & Kafousi, 2010). The significantly better performance was attributed in Duatepe's (2004) research to the potential of the drama based teaching to make learning easy and understanding better by: a) supporting active involvement, b) creating group work environment, c) giving opportunities to improvise daily life examples which were meaningful, more logical interesting and familiar for them, d) giving opportunities to communicate, e) creating excitement during drama based activities f) providing visualization, g) providing long lasting learning.

Specific techniques, such as the 'as-if' approach, can create the context for teaching a concept, an idea or an event and offer opportunities for exploring mathematics in a variety of historical, social, political and cultural contexts (Gadanidis, 2005; Kotarinou, Chronaki & Stathopoulou, 2010; Lawrence, 2000; Holden, 2002; Ponza, 2000a; Gensrick, 1992; Pennington & Faux, 1999). Drama techniques help students to experience the dimension of mathematics as a cultural construction as well as the aspect of mathematics as a socio-political 'tool' (Kotarinou et al, 2010) and also seem to be a suitable framework that offers ways for integrating Mathematics with varied disciplinary areas of curriculum (Stathopoulou et al, 2012). According to Gadanidis (2012, p.20) *''working in small groups, students script dramatic dialogues to share their learning. This sharing may be seen as a form of community service which enhances the sense of audience, motivating and giving purpose to students' learning; creates school-community links by opening public windows into school learning; and provides an opportunity for students to give voice to their mathematical identity''*

RESEARCH PARTICIPANTS, SETTING AND METHODS

Empirical data for the research presented in this paper arose from our endeavors towards exploring the contribution of drama in the formation of such a teaching that fosters students' participating and creativity in teaching Geometry in high upper school. This paper focuses on accounting on the impact of Drama in Education-based teaching of Geometry in students' learning of geometry and developing of creativity. The research participants included a group of 26 eleventh grade students and their secondary school teachers from varied curricular areas (Mathematics, Physics, Drama, Language etc.) who all came together to work for this particular project. The research setting and location of our study was an urban elementary school in the greater area of Athens, an took place at the academic year of 2010-11 whilst its duration was four months and wa based on weekly and daily meetings.

In terms of the research methods used, we designed and implemented a interdisciplinary didactical intervention, based on a teaching experiment methodolog (Cobb et al, 1991, see also Chronaki, 2008 for an overview of its potentiality). Th teaching experiment focused on a detailed design of the teaching of the axiomatizatio of Euclidean and Non-Euclidean geometries as well as the history of Euclid's fift postulate through using Drama in Education techniques. Utilizing ethnographic researc techniques as part of the teaching experiment implementation helped us gatherin research data: a) participant observation of students' mathematics classes in thei ordinary lesson and participatory observation during the teaching experiment fo exploring the way Drama in Education techniques affect the instructional classroon norms and practices, b) questionnaires concerning students' beliefs about Geometry before the teaching experiment, c) students' interviews (two months later) regardin mathematics achievement and retention of knowledge of each student, their image o Geometry as the result of the teaching experiment experience, as well as the reasons o students' motivation and active participation through specific acts in the teachin experiment. All students' presentations were videotaped and recorded regarding th proper use of mathematical notions in their dialogues, while some episodes of students group work were analyzed regarding the role of Drama as a mediating tool for th negotiation of meaning and the development of understanding.

Designing a Geometry Project: 'Is our World Euclidean'?

The teaching experiment entitled: "*Is our world Euclidean?*' was carried out by th researcher in teaching role (first author in this paper) as part of her engagement i participant-observation, in 25 teaching periods, during seven weeks, in Geometry History, Language, Literature and Ancient Greek Language classes. Our teachin experiment was designed to introduce students to an epistemological rupture in th history of mathematics, assuming that this would have the ability to challenge students beliefs about Geometry and generally their image of mathematics as a science o absolute truth. The discovery of non-Euclidean Geometries is such a rapture in th history and evolution of mathematics, through the separation of reality fron mathematical space and through the conscious realization that mathematical structures in their role as models, are the new mediating artifacts to explore space (Hegedus & Moreno-Armella, 2011). As characteristically Hegedus and Morena state '*Witl Euclidean ontology a mirror was placed between the world and mathematics. No Euclidean Geometry broke the mirror*' (Hegedus & Moreno-Armella, 2011, p. 379)

Through the design of the present teaching experiment for accessing the axiomatizatio of Euclidean and Non-Euclidean Geometries as well as the history of Euclid's fiftl postulate we aimed: a) to motivate and actively engage all the students of a mathematic class through Drama, b) to develop students' understanding of the axiomatization o

Euclidean, Hyperbolic and Elliptic Geometry and the general concept of the axiomatic foundation of a science, c) to challenge students' stereotypical believes around Geometry encouraging them to develop a critical stance towards mathematical knowledge, as absolute, objective and infallible. Specifically, the following stages were encountered as entries to the teaching intervention;

- an introduction to the topic, a lecture enhanced with digital projection was provided by the teacher/researcher
- subsequently, the students were asked to work in teams, using appropriate bibliographical recourses such as digital or haptic material and books for them to acquire suitable knowledge regarding their presentations
- a summing up activity by the teacher/researcher ensued where there was ample chance to discuss ideas at the public space
- the teams prepared their presentations with drama conventions
- after rehearsing students performed their presentations
- a concluding and reflective session followed, while at the end of the teaching experiment an entire class period was dedicated for the same reason.

The project consisted of the following units:

1. *Euclid's Elements and the axiomatization of Euclidean Geometry* (6 hours): The main teaching aims of this unit of the teaching experiment were: a) the students understanding Euclidean Geometry as an axiomatic system, b) to clarify different concepts as: definition, 'common notions', postulate, proof. The relation between Euclid's axiomatic foundation of Geometry and Aristotle's 'Logic' was studied by students in small team-work as well as the definitions, the Postulates and Common Notions from the Book I of Euclid's 'Elements'. In the Ancient Greek subject class, we gave students, excerpts from Euclid's original texts and encouraged them to read and translate the text from ancient to new Greek language. Then, during four teaching periods, students divided into six bigger groups, worked with the help of a worksheet and appropriate literature in order to understand Euclid's definitions, 'common notions' and postulates and to present them with DIE techniques. During the Geometry class, students in groups were asked to answer questions concerning the mathematics of the text, after having studied some relevant excerpts of the book '*The Historical Roots of Elementary Mathematics*' by Bunt Jones and Bedient (1976). Then students prepared their presentations concerning the postulates, the common notions and some definitions of Euclid's axiomatic foundation of Geometry and made their performances using drama techniques as 'role-playing', 'reportage', 'alter-ego', 'interview'. As we realized from students' answers during reflection, some students had not yet understood the axiomatic foundation of Geometry by Euclid, so we used the drama technique 'teacher in role' for a short recapitulative presentation of the same subject. At the beginning, for the educational role of Euclidean Geometry to be emphasized, an excerpt from Einstein's '*Autobiographical Notes*' was read, from a teachers' book authored by

Thomaides et. al (1999) and then the researcher in the role of Euclid and the math teacher in role of Hilbert talked about each one's foundation of Geometry. The researcher 'out of role' addressed the students with open questions and it was identified how the majority of students had understood the five postulates set by Euclid, and through this intervention they were able to develop a broader understanding concerning his role for establishing the foundations of Geometry.

2. *Euclid and the historical, cultural and political frame of his era* (4 hours): The main aim was for the students to understand mathematics as a cultural and human creation whose development is influenced by the political, economic and cultural environment. In order for the students to know the historical context in which Euclid's Elements were written, a digital presentation was held by the researcher in History class, concerning Alexandria in the Hellenistic period, while excerpts from Denis Guedj book '*The Parrot's Theorem*' were read concerning the history of this era, as well as the reasons for the blossoming of Mathematics in this historic period and area. Knowing that dramatization is an important tool in the repertoire of a teacher for humanizing and contextualizing the development of mathematical concepts (Hitchcock 1996a; 1996b; Ponza, 2000a, 200b), the chapter 'Euclid's conceit' from J. P. Luminet book '*Euclid's bar*' (*Le Bâton d'Euclide)*, presenting the mathematician Euclid and his era, was read expressively by some students in Literature class, while some scenes of the same chapter concerning differences in thought between Pythagorians and Euclid, as well as historical anecdotes about Euclid were dramatized.

3. *'History in shadow': the controversy of Euclid's Fifth postulate till 18th century (5 hours)*: The purpose for all the activities of the unit 'History in the shadow', was the students to get to understand through the history of contestation of the fifth postulate, how an intellectual problem —in our case a mathematical problem— can de a challenge for mathematicians to solve it, for almost 20 centuries. Through students' acquaintance with this problem, especially with the ongoing and unsuccessful attempts to solve it, we also aimed to challenge student's stereotypical images of Mathematics as an absolute and certain body of knowledge. At the same time our cognitive goals were the students to know various propositions equivalent to the 5th postulate characterizing Euclidean Geometry and also to know the logical error of Petitio Principii (circular argument) an error that students do quite often in their proofs. In the mathematics class, the history of the fifth postulate as one of the five set by Euclid and the challenging of it as an independent one of the other postulates, by mathematicians of classical Islam and by the West mathematicians until 19th century was presented. More specific the unsuccessful efforts of Arabs mathematicians Thabit ibn Qurrah, Al-Haytham, Omar Khayyam and Nasir al-Din al-Tusi as well as Saccheri and Lambert to prove the famous Euclid's fifth postulate were presented, while their errors in all these efforts –using equivalent to the 5th postulate proposition to prove this postulate— were interpreted. During the history class the development and the reasons of development of Mathematics in the Islamic world, from the 8th to the 13th century AD, were discussed. For this reason, a short extract from the Denis Guedj book *"The Parrot's Theorem"* (pp. 269-272), which refers to the "House of Wisdom" in Baghdad and its role in the collection and

translation of the work of the ancient Greeks, was read. A combination of 'Shadow Theatre' and 'role playing' was used for presenting all these unsuccessful efforts as well as the errors made at the proofs and students prepared their performances after having studied relevant excerpts from the books: *'A history of mathematics'* by Boyer and Merzbach (1997) and *'Great moments in mathematics'* by H. Eves (1983).

4. *János Bolyai, Lobachevski, Riemann, the founders-creators of non Euclidean geometries (3 hours)*: As the biographical allusions serve the purpose of humanizing concepts (Ponza, 2000a) students in Greek language class studied the biographies of the two latter mathematicians from the book *'Men of Mathematics'*, a book on the history of mathematics written by the mathematician E.T. Bell and quotes from letters from Gauss to F. Bolyai and between father and son Bolyai. The aim of students' studying the biographies of the founders of non-Euclidean geometries was for them to change their stereotypical image of mathematicians as people disconnected from social life and problems, and on the other hand to make students aware of Mathematics as a historically evolving human creation. Through these biographies, we wanted students to understand that mathematicians are people who are not cut off from the social environment, they participate in it, they are facing personal problems, they are persons with special appeal in Mathematics who also spend a lot of time studying the works of other mathematicians of the past. Regarding the Lobachevski biography copies of the book *'Men of Mathematics'* were given, specifically from the chapter which refers to this mathematician. This biography is written in a very lively manner, giving at the same time a very vivid picture of Tsarist Russia. For presenting Lobachevski, the Drama convention (Role-on-the -wall) was used. The students were asked to write within the outline of the figure, ideas and feelings that according to the various elements of his biography Lobachevski himself might have and outside the outline to write their own feelings and thoughts about him. During the second class period students were asked to read the biography of Riemann, from Boyer and Merzbach book and to underline the fundamental points of it, as well as the points they impressed them. To present the second biography the Drama technique 'a portrait comes to life' was utilized. One student took the role of Riemann's portrait and was asked to come alive to answer the questions and queries of other students. The third teaching period the technique 'Letters' was chosen, in which three students-in-role of Gauss, Farcas Bolyai (father) and Janos Bolyai (son) read three letters of the correspondence between them concerning their work with the 5th postulate, from Mankiewicz (2002) book *'The History of Mathematics'*. Two more DIE techniques as the 'conscience alley' and 'conflicting advice' followed. The conscience alley' concerned Janos Bolyai struggle about to continue or not working on 5th postulate[1]. At the crucial moment he had to take his decision, Bolyai walked between two rows of students who gave him different

[1] In this technique, the participants divided into two parallel rows facing each other form a corridor. A person who faces a dilemma passes through this alley, while every person of the alley gives him advice on what decision to take. These pieces of advice which may be contradictory, form an inner dialogue, which may be formed in the conscience of a person called to take a difficult decision. (Avdi & Hadjigeorgiou, 2007, page 92).

pieces of advice on the decision he had to take. Bolyai's conflict between to prove th fifth postulate or to replace it with another non-equivalent to this proposition wa attributed by the technique 'conflicting advice', i.e. by 'voices in his head' which gav him conflicting advice. Students took the role of 'voices' arguing in favor of one or th other mathematical approach he had to follow. The scene ended with Gauss reading hi letter to Farcas Bolyai, in which he refers in a very typical way to the work of his so Janos. This letter highly embittered and discouraged Janos regarding his mathematica work. At the end we mentioned the example of Immanuel Kant, the dominar philosophical physiognomy of the century who preceded the discovery of nor Euclidean Geometry, whose ideas about space and Geometry acted as an obstacle on th acceptance of non-Euclidean Geometry and some others attribute the non-publication c Gauss's ideas on non-Euclidean Geometry just in his unwillingness to conflict with Kar (Davis, 2007, p. 133).

5. *Hyperbolic Geometry and Poincare model* (6 hours): This unit aims were to enabl students: a) perceiving the axiomatic foundation of Hyperbolic Geometry, b) perceivin the role of the postulates in every axiomatic system, c) redefining Euclidean Geometry d) perceiving the role of a model in mathematics, e) to gain a deeper understanding c Geometry by comparing the similarities and differences of Hyperbolic Geometry wit the Euclidean. In Modern Greek Language class (3 hours) a digital presentation wit historical data, key concepts and theorems of Hyperbolic Geometry, elements of th Poincaré model, as well as works of Escher was conducted by the researcher. A discussion with students followed about the notion of an axiomatic system, about it consistency, the independence of the axioms, and the meaning of a model of a axiomatic system. Students in groups studied excerpts from the chapter 'Platterlan from Ian Stewart book *'Flatterland'* concerning Poincaré model of Hyperboli Geometry, in order to prepare a radio show with the same name issue. The team studied the chapter of the book and prepared their own texts for the radio broadcasts i 3 teaching periods. In Geometry class (1 hour) students used ICT (Interactive Jav software 'NonEuclid' by J. Castellanos, Joe Austin, Ervan Darnell, and Maria Estrada for visualizing the Poincaré model, the axioms and basic concepts of this non-Euclidea Geometry. The students worked on computers using worksheets, in groups of two (groups) or three (3 groups). They explored the model by drawing points, lines, segment angles and perpendicular to a given straight line. They, also, measured segments an angles and they wrote their comments about the construction of cycles, line segments c equal length and of the measurement of the sides and angles of a triangle. Finally th axiomatic foundation of Hyperbolic Geometry was held through the model. The Drama technique 'radio broadcasts' was used for presenting 'Hyperbolic' Geometry an its Poincaré model. The radio broadcasts were presented from behind a screen so tha the students not be seen by the audience.

6. *Spherical Geometry and the axiomatization of Elliptic Geometry (1 hour)*. Student studied the axioms and the basic notions of Spherical and Elliptic Geometry throug 'Lénárt sphere' and other haptic tools, as table tennis balls (Lénárt, 1996). A role playing activity was used for the evaluation of their developing knowledge.

7. *The film* (during the class breaks): A documentary film entitled 'Our lives with Euclid' was created around these drama-activities with the students in the role of narrators, who wrote their own texts of the narration after having studied the relevant literature. A student as a cameraman filmed all the narrations and two others wrote and performed the music for the film. Students were taught the program Windows 'movie maker' for making their own montage of the film, but there was no available time and the montage was made by the researcher. The DVD with the film was distributed to all students.

In the following section, the outcomes of implementing the above teaching experiment will be discussed in relation to its impact on students' experiences in geometry in four interrelated areas a) students' experience in geometry, b) retention of knowledge c) epistemological beliefs, and d) creativity

Drama-Based Teaching and Students' Experiences in the Geometry Project

Through the observation and the interviews it appears that the innovative Drama based teaching motivated students' active participation in the teaching experiment and helped for new practices to be formed in a class of active students who cooperated towards the development of mathematical knowledge. All of the 15 interviewed students answered that Drama in Education has been the main motivation for the active participation of all students in the teaching experiment.

Tony: [...] the presentation for me, sparkled my interest ...(Tony, interview, 11-06-11)

Sofi: If we just studied or just wrote, we would have been bored to death and even more. (Sofi, interview, 05-06-2011)

Nicky: [without the presentations] I do not think that anyone would be interested. (Nicky, interview, 09-05-2011)

Nick: No, I think we would have idled, even more. (Nick, interview,18-06-2011)

Students' responses emphasize that it was the activities through Drama techniques that differentiated the whole project.

Chris: I believe that the presentations were the most interesting part. (Chris, interview, 03-06-2011)

Tzina: This was pretty nice, because when you just write it will not be very different, because who you will see it, but when you do a skit is more interesting, more beautiful and time passes more pleasantly. (Tzina, interview, 15-04-2011)

By these students' answers we may conclude that students' presentations through DIE techniques were the main motivation for the students to participate actively to the teaching experiment. The innovative character of teaching are one of the reasons the students highlight about the important role that Drama played in the teaching of

Geometry. Teaching Geometry with the use of drama was something new and differer for the students, which motivated them and piqued their interest.

Virianna: [...]it was something new, it was so very different for everyone, for m personally who I have no particular relation with theater and everythin around it, it was very original and it was so very different for everyone because something similar has not been tried before.(Virianna, intervie 18-06-2011)

Mary: I beleive that everyone will remember it. It was something out of the norma routine. (Mary, interview, 14-06-2011)

The contribution of DIE in the whole teaching experiment was decisive in motivatin students' participation experiment and as we will see had an impact in students experiencing geometry, in their retention of knowledge and in their epistemologica beliefs.

I) The Impact of Drama in Students' Experiences in Geometry

In our didactical experiment besides students' motivation and students' well-being, w aimed for encouraging students' understanding of mathematical concepts such as th axiomatization of Euclidean and Non-Euclidean Geometries. Students' performanc (and comprehension through performance) of these concepts was evaluated via relevan questions in specifically organized interviews and through analysis of the enacte dialogues through the teaching experiment. Analysis of the dialogues in students' dram performances suggests that students conceived the mathematical concepts that the were asked to present, integrating them correctly in their performances. Th presentations for the axiomatic foundations of Geometry by Euclid had no math error (except a few, which did not affect the understanding of these concepts by the audience and gave a complete picture of the topic. Here are some quotes from the dialogues a they appeared through their performances.

Mary: And finally what do they stand for, all these 'common notions'?

Chris: All these 'common notions' serve for comparison. The comparison coul not be made without such common notions. (Mary and Chris, Role playing presentation, 16-03-2011)

Excerpt 1. A dialogue amongst two students in role. Mary a femal student in role of a reporter in a TV show interviewed Chris a mal student in role of a Mathematics' History researcher.

Virianna: If these are the fundamental concepts that Euclid defines, how do w know that they really exist?

Tony: This question Euclid has answered it with the postulates. (Virianna an Tony, Role-playing presentation, 16-03-2011)

Excerpt 2. A dialogue amongst two students in role. Tony a male student in role of a teacher of mathematics and Virianna a female student in role of his student.

Different comments on Euclid's five postulates were presented as thoughts of Euclid by the student in role of Euclid's Alter-ego. (A student in role of Euclid is seated in a chair and reads his postulates in Ancient Greek Language while Joanna as his Alter-ego, standing behind him, explains to the audience the reasons of putting these postulates.)

> *Joanna: So with the first three postulates I want to ensure the existence of the two fundamental concepts of line and circle. The fourth postulate refers to that all right angles are equal. I thought this postulate, because I want the value of these angles to remain constant regardless the position they have.*
>
> *The last two postulates do not demand the existence of a fundamental concept as for example the line or circle. They are used so we can accept some of their properties. At this point, however this question arises: since according to Aristotle, these postulates are neither common concepts, since they refer only to Geometry, nor specific notions that define the fundamental, why did Euclid put these postulates?* (Joanna, Drama presentation, 16-03-2011)

As far as the presentations regarding the attempts for proving the 5th postulate all teams included in their dialogues the propositions (equivalent to 5th postulate) that were used to prove it. This is a quote of their presentation of Ibn al-Haytham efforts using Shadow theater techniques.

> *Chris: Ibn* al-Haytham *or Alhazen ... assumes the existence of a quadrilateral with three right angles...and examines the cases the fourth angle to be acute, right or obtuse... He excludes the cases of acute and obtuse angle considering parallel lines as equidistant. This is equivalent to the fifth postulate.* (Chris, Shadow theater performance, 01-04-2011)

The teams in their dialogues generally refer to the logical mistake made in the procedure of the proof, i.e. the circular argument, but they do not formulate it in all cases with absolute precision and clarity.

Following the analysis of radio broadcasts texts that students prepared about Hyperbolic Geometry, we observe that two teams did a full description of the Poincaré model, referring to how the Poincaré disc, straight lines, angles are defined and also referring to the apparent decrease in the length of segment and the alternative 5th postulate. The remaining teams referred to only a few concepts, those who they considered as important or they have impressed by.

Eva: *I was impressed that in Platterland*[2] *there are only two dimensions an that while from far away the country seems to have a certain extent actually reaching there you realize that it is infinite.*

Angela: *It is noteworthy that Platterland is a circular disk without circumferenc and that points are defined just inside.* (Eva and Angela, radio broadcast 04-04-2011)

Excerpt 3. A dialogue amongst two students in role. Eva a female student in role of a Journalist in a Radio broadcast and Angela a female student in role of a Mathematician Guest in the broadcast.

About the straight line in hyperbolic geometry, Angela in the same broadcast explain the definition of a straight line in Poincaré disk.

Angela: *In Euclidean Geometry what we call a straight line, in their own world w perceive it as an arc that intersects the circumference at right angles* (Angela, radio broadcast, 04-04-2011)

The same student regarding the measure of an angle in Poincaré model says:

Angela: *The measure of an angle corresponds to the Euclidean measure of th angle formed by the tangent line at the top of the two intersecting curve that form the angle.* (Angela, radio broadcast, 04-04-2011)

Maria a student in role of a "resident" of Platterland participating in the same radi broadcast gives the following response about the maintenance of the measure of the lin segment despite its apparent decrease when removed from the center of the disc:

Maria: *The objects of Platterland as they remove to infinity, they shrink, althoug this can not be perceived by a person moving in Platterland, because a the object shrinks, at the same time the meter shrinks too. So in ever measurement we have the same result.* (Maria, radio broadcast, 04-04 2011).

II) The Impact of Drama on Students' Retention of Knowledge.

The students' answers in the interviews, two months after the end of teachin experiment, also indicates that Drama based teaching had positively affected students learning and retention of the axiomatization of Euclidean and Hyperbolic Geometry.

Peter: *The way of teaching was interesting ... because during this teaching method the student obtains more accurate knowledge. It is not that he just learn something that he would forget later, it would stay longer in his mind* (Peter, interview, 11-06-2011)

[2] Platterland in Ian Stewart's book Flatterland, is a land having the shape of Poincaré model o Hyperbolic Geometry.

Zoe :We remember many things from what we did in this way, while when we were in class waiting for the bell rung as to get out of the class, we had completely forgotten everything.(Zoe, interview, 14-06-2011)

In the questions 'How was Geometry founded by Euclid?' or 'What did Euclid essentially offer to geometry', to 20 students two months after the teaching experiment, we had the following results: four students were able to give us an adequate synopsis of how Euclid founded Geometry, while most (all ten of them) remembered that Euclid organized the existing knowledge and put some fundamental concepts and postulates for the axiomatic foundation of Geometry.

In the interviews, 17 students responded about the controversy of the fifth postulate. From their responses we conclude that nine students were able to give us a good overview of what Arabs and Sacchieri on the one hand and Lambert on the other tried to prove. The majority also referred to 'circular reasoning' a logic error of mathematicians who tried to prove the fifth postulate, a concept which had impressed the students.

Sixteen students in the interviews were asked questions about hyperbolic Geometry. It seemed that students were so impressed by Poincare disk that they identified it with hyperbolic space. The majority of students responded that they were impressed by the shape of the lines in Poincaré disk (11 replies).

Vicky: First of curves were considered as straight lines. For those who were in hyperbolic world it seemed like a normal straight line, it was just a matter of how you see things. See them from outside or inside? It's completely different what we call in Euclidean Geometry straight line and what we call in hyperbolic Geometry straight. (Vicky, interview, 14-06-2011)

The second thing that impressed them was the apparent change in size.

Tony: As we went towards the periphery of Platterland (the Poincaré disk) the segments of the straight line seemed to become smaller. The size seemed to be diminished but at the same time the measure was diminished too. Thus they were considered to be equal with the rest segments. (Tony, interview, 11-06-2011)

III) The Impact of Drama on Students' Epistemological Beliefs

The teaching experiment led students to a broader perception of the nature of mathematics and challenged their stereotypical beliefs around Geometry and more generally around Mathematics as a science of absolute truth. Through students responses in the interviews we tried to detect the "image" of mathematics that they formed throughout this experiment. Students experienced a diachronic evolution of Geometry that was connected with the historical, social, economic and political conditions of each era.

Stefan: We realized that in order for science to be evolved, there needs to b evolution in all other areas first, starting with financial areas and the cultural ones. (Stefan, interview, 14-06-2011)

Tony: The historical connection of mathematics is very important. Mathematic seemed like coming from outer space. We had just been told the 1 and and the numbers and the calculations, all ended there. (Tony, intervie 11-06-2011)

Through the efforts of proving Euclid's fifth postulate and through the mathematicians biographies, students perceived mathematics as a historically evolving human creatio and also realized that other cultures also, have contributed to what we define as Wester mathematics.

Vicky: Previously, I thought that the knowledge of mathematics came from Go from outer space. I couldn't believe that someone thought of it. Basicall I was thinking about that, but I was wondering: how can one think of it I was telling myself he was born that way. But eventually, through this realized that all these people make research, read, learn. (Vick interview, 14-06-2011)

Students emphasized the continuous and strenuous effort that was made b mathematicians of the wider Islamic world and later by Renaissance mathematicians, t prove the fifth postulate.

Sofi: I remember that the whole thing gave them a hard day. I remember this, th they (these mathematicians) resulted in mistakes, many mistakes, but, it okay, one learns from mistakes.(Sofi, interview, 05-06-2011)

and highlight the dedication of Bolyai, Lobachevski and Riemann in mathematics an the perseverance in their objectives.

Mary: I saw that these people in a great part of their lives were dealing wit mathematics and they had set a target, trying to reach it. It was ver interesting, all this effort to prove something, really interesting. (Mar interview, 14-06-2011)

Active involvement in the teaching of non-Euclidean geometries provoked students perception about mathematics as a science of the absolute truth. Their involvement i this procedure made students to perceive Mathematics as corrigible and as a creatio under constant negotiation, modifying thus their epistemological beliefs abou mathematics and provoked the dominant belief that Euclidean Geometry is the onl model that interprets and represents our real world, shaking thereby and other certaintie

Stefan: Certainly the plasticity of mathematics emerged and the way mathematic are created and changed depends on the needs of the mathematician, o the scientist and of the human being generally. It is clear th mathematics is a complex notion, which is not restricted to only one wa of understanding reality.... (Stefan, interview, 14-06-2011)

Angela: Finally there are and alternative views and we cannot say which is absolutely right and which not. (Angela, interview, 08-06-2011)

All these answers offer considerable evidence of the effectiveness of the teaching experiment to challenge the dominant image of students regarding mathematics, to give students a humanistic image of mathematics and so to modify their epistemological conceptions about mathematics.

IV) 'Teaching Creatively' and 'Teaching *for* Creativity'

In NACCCE (National Advisory Committee on Creative and Cultural Education) report (Robinson, 1999) creative teaching is defined in two ways: first, teaching creatively, and second, teaching *for* creativity. 'Teaching creatively' means teachers using imaginative approaches to make learning more interesting, exciting and effective while, 'Teaching *for* creativity' means forms of teaching that are intended to develop young people's own creative thinking or behavior. We claim that by this didactical experiment (teaching Geometry through Drama in Education techniques) we managed at the same time 'Teaching creatively' and 'Teaching *for* creativity'.

Concerning teaching creatively, Drama based teaching is a creative way of teaching. The researcher had to cultivate her own creativity and find activities that would interest and would activate students and also had to find a subject from the curriculum that would be interesting and exciting for them. Field research answered the questions we had set but also revealed and new questions. One of them concerned 'Teaching for creativity', creativity as a general educational value. The only previous research concerning the affect of Creative Drama on students' creativity in mathematics classroom is the one of Saab (1987). The research showed that experiences of 6th grade students with Creative Drama/Mathematics activities caused a significant increase in levels of mathematics achievement in regard to Mathematics computation, while these experiences were not shown to significantly affect their levels of creativity. However, results of the comparison of subgroups means indicated that female experimental subjects performed significantly higher in levels of creativity as compared to male experimental subjects. *Teaching for creativity* in our teaching experiment consisted of: a) encouraging young people to believe in their creative identity, b) identifying young people's creative abilities, and c) fostering creativity by providing opportunities to students to be creative. In the beginning of our research we didn't have research questions concerning students' creativity, so we did not use any relevant instrument to assess students' levels of creativity. As a result to find out if creativity was enhanced through Drama activities, we analysed students' drama presentations, students' writings and students' interviews.

Both, from the responses we got through the interviews with the pupils and from observation during the whole process, it seemed that the pupils became creative within a cooperative context and through their engagement in new teaching practices. The pupils' creativity was captured in their resourcefulness in drawing the 'dramatic context' and the dialogues of their presentations, in their artistic expression, and in general in the cultivation of skills that surpassed their till then identity as learners c mathematics. The students in their responses emphasized that Drama fostered the: creativity which gave them the opportunity to express themselves in their own way an for everyone to bring out his special abilities and talents.

> *Virianna: And (it was) very creative and everyone could show his abilities, th different ones that everybody has.*(Virianna, interview, 18-06-2011)

The students had to be creative, to find the way they had to present the notions for th other students to understand them.

> *Vicky: It was very creative because we had to invent things, how to present it t others to understand it, what to say, how to say it.* (Vicky, interview, 14 06-2011)

The students most of the time chose for their presentations an imaginative dramati context, to pique the audience's interest and to have a good aesthetic result and create original dialogues some times with humor for them to entertain.

> *Chris: Several times we tried not to do it in a simple way as the other groups dic i.e. simply copy words from the books, we tried to say something more t do something different.* (Chris, 03-06-2011)

Students used brainstorming for the selection of the appropriate Drama convention an the creation of dialogues for their presentations.

> *Chris: Several ideas were dropped and several times we combined them and so w came to a result* .(Chris, 03-06-2011)

The notion of imagination is crucial to the creative experience. As Vygotsky (1998 refers *"Everything that requires artistic transformation of reality, everything that i connected with interpretation and construction of something new, requires th indispensable participation of imagination."* Students activated their imagination t create for the first time a 'dramatic context' for their presentations. Some examples ar the following:

- A modern mathematics classroom with the presence of Euclid who is invisible to th students of the skit, but is visible to audience.
- Euclid in Ancient Alexandria library reads the five postulates and his Alter eg comments them.
- The shades of Omar Khayyam and Euclid discuss about the fifth postulate.
- A 'theater company of spirits' presents the work of Al Haytham.
- A journalist who wants to write an article on Al Tussi, he sees him in his drean talking about his work.
- A radio broadcast with guests some residents of Platterland (the disc Poincaré) wh talk about their country.

From students' descriptions it seemed that 'Drama in Education' gave them the incentive to create texts of 'creative writing' about mathematics issues. Some of the dialogues of the happenings were very original, and a couple of texts were written in 'automatic writing'[3] in which students felt free to write and express themselves without commitments to specific form or content. As an example of this writing we present students' dialogues about Alhasen's [4] effort to prove the 5th postulate. For this presentation students chose techniques of shadow theater in which only the students' shadows could be seen.

Zoe: We are a group called 'forgotten spirits' and we will present Al Haytham's work.

Chris: I' m terrified. Help me overcome my fear. I haven't slept in a vertical position for three days.

I' ve been sleeping on the floor, nose down, horizontally.

John: And, how are you feeling?

Chris: As the 5^{th} postulate. Al Haytham or Alhazen (like the old Alcázar football ground in Larissa[5], like Abou Diaby[6] of Arsenal or like a yellow submarine of Villarreal[7]) assumes the existence of a quadrilateral with three right angles.

John : A Lambert quadrilateral

Chris: and examines the possibilities of the fourth angle being right or obtuse or acute.

John: Only this?

Chris: Of course not. He rejects the obtuse and acute angle hypothesis assuming parallels lines as equidistant. This is equivalent with the 5^{th} postulate.

John: Finally, the others who called me a locus were right.

Chris: Indeed; I admit being a point moving like a multicolored dolphin (never let alone to relax) in the emerald ocean of love and real geometry, so as to keep equal distance from line to line, from road to road, from heaven to heaven, from purple to red and from the board straight into the

[3] Automatic writing has been a key element of the poetic movement of surrealism.

[4] Al Haytham or Alhazen (965-1040), constructed a quadrilateral with three right angles and considered that he had proved that the fourth angle is also a right angle and as a result of this the 5th postulate. Alhazen for proving this proposition assumed that the locus of the point that moves in a constant distance from a straight line is necessarily a straight line parallel to the first, a proposition which is equivalent to the fifth postulate (Boyer et al, 1997).

[5] A Greek city.

[6] A French footballer who plays for English club Arsenal.

[7] Villarreal, is a Spanish football club with the nickname *El Submarino Amarillo* (Yellow Submarine).

swimming pool. (Zoe, Chris and John, shadow theater presentation, 01 04-2011)

As a result students felt that the skits were their own creation.

Tony: Without Drama presentations this would not be a part of us. Through Dram we prepared it, we presented it; we felt it as our thing. (Tony, intervie 11-06-2011)

Drama cultivated students' creativity and this Drama creative character motivate students in mathematics learning and enhanced mathematics teaching.

Tatiana: Especially when it is a skit, it is more creative and generally it's like game, and through the game, the child and the student does mor things.(Tatiana, interview, 07-06-2011)

CONCLUSIONS

Geometry, synonymous with Euclidean Geometry, as taught in school is an object tha is usually not attractive to students as it emerged from their answers. The presentatio of Euclidean Geometry as an integrated productive system, which students are invited t learn just leaves them little room for creative engagement and it develops further th impression that it is an out of human creation.

Focusing on concepts of not Euclidean geometries using Drama in Education technique created a new dynamic to the teaching of mathematics and seemed to act as a stimulu that triggered the students and changed their perception of mathematics and thei teaching. Through role playing, dramatization and other Drama techniques we were abl to discuss concepts and historical events of mathematics that are not often discussed i class context, broadening with this way students' perceptions about the nature o mathematics. Students, through the collaboration and the experiential processes tha Drama offered, succeeded to understand the essential elements of the axiomati foundation of Euclidean and Hyperbolic Geometry. The analysis of the texts prepare by the student groups for their presentations indicates that they conceived th mathematical notions that they had been assigned to present while the interviews, nearl two months after the presentations also indicate that Drama based teaching ha positively affected students' retention of this knowledge. This finding related wit students' achievement supports the ones' of previous studies (Kotarinou et al, 2008 Duatepe, 2004; Omnieski, 1999; Saab, 1987) which provided evidence of the efficienc of drama based teaching in facilitating understanding of mathematics concepts i primary and high school level.

Through this interdisciplinary project students had the opportunity to be taugh meaningfully mathematics. Students had the opportunity in language class to write text which included mathematical discourse, while in History class for the first time the experienced mathematics as a cultural issue. Our teaching experiment pointed ou Drama's effectiveness in integrating different subjects of the curriculum. This findin also agrees with the results of Stathopoulou et al (2012) project with 10th grade students about the Greek tradition of Xysta.

Interdisciplinarity through history of mathematics can reveal this wider aspect of mathematics as a cultural activity; as a human activity both done within individual cultures and also standing outside any particular one. This interdisciplinary approach through the history of 5th postulate was the key role for the historical and social context in which mathematical concepts were created to be illustrated and provided students the opportunity to broaden the idea they had formed of the nature of mathematics. Drama provided students the incentive to work while through drama techniques students were able to experience all these dimensions of mathematics, not only mentally but also emotionally and physically. This modification of students' epistemological conceptions about mathematics supports the same findings of a previous research of the same authors (Kotarinou, Chronaki and Stathopoulou, 2010) concerning the establishment of the 'meter' as a unit of length measurement.

Drama techniques were an alternative teaching approach in fostering students' creativity by providing them opportunities to be creative and in encouraging young people to believe in their creative identity. This finding of the study related with 'teaching for creativity' supports the findings of previous studies, not in mathematics classrooms, (Annarella, 1992; Bolton 1986; Kelner, 1993; Morris, 2001: cited in Duatepe & Ubuz, 2004a) that drama techniques stimulates the imagination and support the development of creative thinking. Through students' discourse describing aspects of teaching techniques, it was revealed that the Drama in Education techniques have acted as an effective mediating tool to develop students' creativity. As Tzina, one of the students, said: *Well. Until now, Geometry was completely outside my interests, and will still be if they teach it the way they do. Because learning something by heart is not nice, while if there is a story it becomes a little more interesting, while if you do a sketch it is nicer and it becomes a more interesting lesson. and we feel more creative!!!!. you learn more through this way.*(Tzina, interview, 15-06-2011)

Our research offered considerable evidence of the effectiveness of the use of Drama techniques as an alternative approach in the creation of the appropriate learning conditions where all the students participate in teaching/learning process. Interdisciplinary projects as described above with theme from History of mathematics through Drama conventions can promote participatory performance and give a context that offers innovative and creative ways for the students to modify their epistemological conceptions about mathematics.

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Geometrical Meaning

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ABSTRACT: *The mathematical objects, despite the fact that are being constructed in the mind of the mathematician, are absolutely objective entities. Mathematical objects exhaust their being in what they present; their being is their seeming. This is the reason why, their subjective genesis notwithstanding, they go beyond the subjectivity of the original act of their construction and acquire an objectivity that traverses all their occasional expressions. This means that the mathematical objects retain unaltered their meaning so that despite the endless series of objectifications and subjectification, the modern mathematicians know that they talk about the same things as the ancient mathematicians. The present paper aims to examine the factors that render this phenomenon possible. I argue that the reason that we can talk of as single Geometry despite the different forms that Geometry has assumed in the course of history is that all mathematicians can and do reactivate the same primary and primordial meanings of Geometry through the same actions that the ancient mathematicians did. These meanings remain unchanged because Geometry at a last analysis emerges from our actions on the surfaces and forms of the objects in the environment. These actions remain unaltered through the centuries no matter how the surrounding world has changed.*

Key words: *Mathematical objects, Representation of space, Actions on objects.*

INTRODUCTION

The mathematical objects are idealized entities and, thus, absolutely objective, in the sense that they are completely free from any empirical subjectivity despite the fact that the act of their creation takes place in the mind of the mathematician. The mathematical objects are what they seem to be. It is to this factor that one must take recourse in order to explain how it is possible to overcome their subjective genetic act and acquire an objectivity that traverses all their occurrent expressions through time. The mathematicians ponder on the products of the conceptions/constructions of other mathematicians and produce themselves other constructs that, in their turn, are objectified and become available to the examination and scrutiny of other thinkers. There is, therefore, a congruence of subjectivities that are objectified and of objectivities that are subjectified in order to be objectified again. The problem is to explain how the mathematical concepts retain their meaning unaltered through this continuous and inexorable series of objectifications and subjectifications, so that the mathematicians know that they all talk about the same thinks and, therefore, there is a genuin communication and not some form of less or more radical incommensurability. Th problem, in other words, is to explain how the meaning of say, geometry, remains th same in the passage of time so that we can all talk about "the" geometry despite all th new forms that geometry has acquired in the passage of time.

Explaining how this is possible is the focus of this paper. In the first section, I argue tha we can talk of one and only one geometry through its various historical forms because i is possible to reactivate its primary meanings through reenacting the actions of th ancient geometers. These meanings remain unaltered because geometry is constructe through our actions on the forms and surfaces of objects. These forms remai unchanged through the centuries no matter how the environment changes.

In section two, I take a brief recourse in the Aristotelian conception of mathematics wit a view to show that for the ancient Greeks the perception of space is fundamental no only on the construction of geometry (something) that is expected after all), but also fo the construction of arithmetic. For the ancient Greeks, arithmetic was a metrica geometry. This is important not only because it suggests that arithmetic was born on th same basis as geometry, but also because it underlines the fundamental role of th perception of space in the genesis of arithmetic, which, in its turn, signifies th importance of our actions on the spatial features of the objects in the environment.

In the final section, I claim that our actions on the spatial features of objects are prior t all other actions because our perception of space precedes all other cognitive functions Furthermore, this is typical both for adults and newborns, which suggests that th perception of space is genetically prior of all other cognitive capacities and probabl selected and hardwired in the course of our evolution in our cognitive architecture Moreover, the perception of space rests on a form of representation, to wit, analog representation that is distinct from symbolic representation in that it represents space i a direct way in which the represented elements are similar to the represented one and thus, the representation acquires its meaning through a direct mapping to th represented domain rather through a convention, the way symbolic representations do If all these are on the right track, then the capability of producing and reproducing th primordial geometrical meanings through repeatable and unaltered in their essenc activities becomes less mysterious.

REACTIVATING GEOMETRICAL MEANINGS

To answer the first question, it would be useful to take a brief recourse to the history o mathematics in order to discover the origins of geometry, or rather, the fundamenta first action that gave meaning to a series of geometrical activities. That is, one mus attempt to reactivate the original meaning of the first abstract geometrical ideas b returning to the perceptual structures and the activities of the ancient geometer. T achieve this, one should ensure first that there is something that is as present to us as i was to the first geometer and it is by acting on that something that the ancient geomete founded the geometrical ideas. Otherwise, the original foundational act will remain permanently hidden in the past and no historical research could ever unearth it.

That which is always presented first and on which everything else is grounded is the present. All of our activities stem from and return to the world that surrounds us as an open horizon of realities and infinite possibilities. We live and think always situated in the present world and we are conscious that we are living in this world. This knowledge and this way of being aware of the world is not something that we learn through teaching or experience. On the contrary, this awareness is presupposed by any sort of learning to be possible; it is a prerequisite for having the ability to learn. This means that every question is raised and every inquiry is undertaken always within the confines of the present world. This is something that remains unaltered since the beginning of the world and, therefore, marks the activities of both modern and ancient geometer.

However, it is also obvious that the modern world that surrounds us and is presented to us is different from the world that surrounded the ancient geometer. Consequently, the common ground on which the original geometrical activity took place and which we seek to unearth and on which we will attempt to base our understanding of the original meaning of the abstract geometrical ideas should be investigated underneath the variability in the mode of presentation of the world to the corresponding geometers. In other worlds, we should search for an element that remains forever unaltered underneath of all the presentational differences of the ever-changing world.

To delimit the problem space, we have to think that geometry is related to space, the forms of the objects in space, their shapes, the transformations of these shapes, especially when all these concern measurable magnitudes and units of measurement or are related to practical activities. The common element that we are looking for should be sought in exactly this characteristic of geometry, that is, in its inextricable and essential relation to the spatial forms of the objects in our environment. Once we have determined that we constant that our world, exactly like the world of the ancient geometer, consists of objects that have a concrete existence (in contradistinction with abstract objects such as justice). These objects have spatial forms (shape, volume, spatial location) and properties that are related, or better, are carried as predicates by, these spatial forms (colors, weight, temperature, etc.) Moreover, our world, like the world of the ancient geometer constantly requires of us to act on it. Some among these activities are exercised on the objects and require the transformation of the spatial features of the objects (such as the smoothing of the corners of a stone or a piece of wood so that we could sit on it, or the sharpening of the angles of a piece of stone or wood in order to construct a sharp tool). Among these practical needs one finds the need of a king to distribute land among his subjects in an equitable way, an activity that presupposes measurement units so that the king could measure the size of the land to be distributed. The measurement units are also required for construction, opening roads, counting distances etc. All these activities or rather measurement techniques are parts of a tradition or a practice that precedes the geometer and which the geometer knows when he starts thinking in order to invent geometry.

The term 'to invent' initially strikes as paradoxical. Why 'to invent' and not 't discover'? After all, the material on which the geometer ponders is, as we have seer given beforehand in the traditional practices. Although this is true, one must bear i mind that the geometer starts with a set of determined and delimited practices an attempts to envisage theoretically the pure spatial forms of the objects in the world. I the mind of the geometer, the existing as necessarily finite spatial forms and thei transformations are conceived as limits set along a horizon of open and, thus, infinit potentialities (Husserl 1939, p. 385). It is at this point that the geometer produces b means of a purely intellectual act idealizations (e.g., circularity) and classes of idealize objects (circles), The geometer mentally produces the form of the spatial forms o objects by abstracting away the material element of these objects. This way, h overcomes the limits set by the present world and by the various forms in it and open the horizon for the infinity of forms in all possible worlds.

This is what renders the mental construction of the geometer diachronic despite the fac that the conditions or circumstances that made possible the mental construction wer entrenched in the actuality of the then-present world in which the geometer livec exactly as they are entrenched in the actuality of the now-present world in which th modern geometer lives. What the geometer receives from the cultural tradition of its er is a material structure consisting in a set of material practices. What he endeavors t achieve is to denude this structure of its material and circumstantial surroundin material so that the structure emerge in its pure form, a form that objectifies it since i relieves it from its historical and, consequently, subjective mantle and reveals it perennial objectivity.

To the extent that this structure is not bound by a specific historical era and is, in thi sense, objective, it is possible for the modern geometer to reactivate it or reconstruct i through a similar mental act so that he too could grasp the meaning of the geometrica ideas in exactly the same way as the ancient geometer did. Note that any understandin of a geometrical structure presupposes the activation of a mental activity that starts fron a material presence and cannot be understood in a medium devoid of any materiality exactly as it was conceived in a material presence that the mind of the geomete eventually overcame. In other worlds, one cannot understand geometrical pure form starting from a world of Platonic ideas. The reason that makes possible the activation o the same meanings despite the variability in the starting material origins (the differen worlds) is two-fold. First, underneath the variability in the cultural forms of materia expressions, the same set of ideas is to be found. Second, the activities on the materia objects are the same across the centuries; perceiving spatial forms and transformin them to our needs. Although the techniques of transforming surfaces nowadays ar much more efficient than they were 4000 years ago, the transformation itself remain the same since it is determined by the form of the surface to be transformed and th form of the surface that is the target of the transformation.

A BRIEF HISTORICAL EXCURSION

There is an adage that comes from combining History and Philosophy of Science, Developmental Psychology, and Didactics of Sciences, namely that, ceteris paribus, ontogenetic development corresponds to some extent to phylogenetic development. In our case, this means that children's development of the conception of arithmetic from analog magnitudes to highly abstract mental entities should be somewhat reflected in a similar development in mathematical thought in the history of mathematics. In other words, there should be a progression in the history of mathematics from thinking of numbers as magnitudes that closely represent something in the world to considering them as abstract entities that form their own system whose structure is not constrained in any way from the structure of the world. In what follows, it is argued that indeed this is the case in moving from the ancient Greek conception of number to the corresponding Cartesian conception.

A conception, quite widespread, of ancient Greek mathematics considers them to be a form of "geometrical algebra," which uses geometrical resolutions and formulations to attack genuine algebraic problems. In other words, the Greeks practiced algebra in a geometrical dress (Gaukroger, 1989; 1992). This view has been criticized by Klein (1968), on the basis that algebra, as we conceive it today, namely the study of certain abstract entities called numbers, was far from the Greek mathematicians' mind. The reason is that the Greeks had never conceived of numbers as abstract entities. To see why, one has to take recourse to the Aristotelian conception of number, a conception that was predominant in Western thought for about 1,500 years, and from which Western thought was gradually emancipated through the work of Descartes and Vieta.

According to the mathematical tradition that Aristotle codified and elaborated on, numbers and shapes are properties. As such, they must be instantiated in something, which carries them, in the same way as there is no "red" by itself but only red things. Things come in numbers and shapes, and thus sensible numbers and shapes are instantiated in real things. But the numbers and shapes of mathematics are constructs of the mind. In that sense they must be instantiated in something that exists in pure thought. Thus, numbers and shapes, as mathematical objects, or better properties, are instantiated in a special kind of matter, which, for Aristotle, is "noetic matter." The term "noetic" is used in contrast to the term "sensible". The former means that the objects of mathematics are immutable, indestructible, and have no independent existence. Sensible things, on the other hand, exist independently, undergo motion, that is, changes, and can perish. Noetic matter is acquired through a process of abstraction from sensible matter. Thus, noetic numbers and shapes result by means of an abstraction from sensible numbers and shapes, that is, from the numbers and shapes of sensible objects.

In the case of shapes the result of such an abstraction is easy to conceive. In the end, the noetic matter in which shapes are instantiated is "a substratum of indeterminate extension characterized solely in terms of its spatial dimensions: length, breadth, and depth." (Gaukroger 1992, p. 100) Similarly, and this is important for our purposes, the noetic matter of numbers is spatial dimensions endowed with measure units, namely, line lengths, areas, or volumes. Thus, whereas geometry operates with lines, algebr operates with line lengths. Line lengths (also volumes and areas) are defined b Aristotle as "pluralities" measured by a unit or a "one." Thus the division of a lin length yields discontinuous parts, the units or the "ones," which, by being units c "ones," are indivisible. A geometrical line, on the other hand, is continuously divisibl in smaller lines. Hence, algebra deals with discontinuous magnitudes and geometry wit continuous magnitudes.

The number in this picture emerges as being synonymous to the line length in which i is instantiated. Algebra is effectively "metrical geometry." As such, it deals with line and planes not as lines and planes but as unit lengths and unit areas. As Gaukroge (1992, p. 101) remarks, all ancient Greek arithmetical propositions are explicitly state in terms of line lengths, the latter substituting for numbers. The reason is that number are line lengths. This shows the predominance of a "spatial intuition of numbers. Numbers are irrevocably related to spatial dimensions.

The implications of that are conspicuous throughout Greek arithmetic. If numbers ar always numbers of something, then when one multiplies numbers also multiplies th things in which these numbers are instantiated. And since this something is spatia dimension, line lengths in particular, when one multiplies two numbers one multiplie two line lengths. Naturally the result is not a line but an area. The square of number a i a square with side a and area a^2, the product of number a times number b is th rectangle ab, and of course, since there are no things with more than three dimensions no more than three numbers can be multiplied together.

The aim of the brief historical overview was to show that for the ancient Greel mathematicians, as for all of us judging from our discussion of the cognitiv underpinnings of arithmetic, numbers are intuitively numbers of something. I particular, due to the predominant role of spatial information in individuating objects numbers related to spatial dimensions, mainly the line length. In short, our historica excursion confirms "the spatial intuition of numbers."

The lasts stop in our excursion back in time is the person who radically changed ou conception of mathematics eventually emancipating us from the Aristotelian ties, that is Descartes. As it is usually the case, people who radically change the course of huma thought are very much originally indebted in the tradition that their work will eventuall overthrow. So was with Descartes. Popular wisdom has it that Descartes replace geometry with mathematics by algebrizing geometry and that he sought to reduc geometry to algebra. This means that Descartes tried to replace the analo representations that befit geometry with the symbolic abstract representations o algebraic numbers breaking thus abruptly with the Greek tradition. As with most piece of popular wisdom, this view is far from the truth. Contrary to what is widely believed Descartes' aim was not to algebrize geometry. Instead, as we shall see in a while Descartes sought to solve algebraic equations using geometry and more specifically lin analogies.

This attempt presupposes a natural correspondence between numbers and lines. Indeed, as Grosholz (1991, p. 20) remarks, for Descartes straight lines, which figure in the list of material simple natures, are the best geometrical proxies for numbers, since points have no dimensions and cannot participate in mathematical operations, surfaces are extended to many directions, and curves are not easily measurable, since they cannot measure themselves. The line length becomes the same as number. As geometry operates with lines, so arithmetic operates with line segments. Thus arithmetic amounts to metrical geometry and "although it deals with lines, planes, etc., it deals with them not qua lines and planes but qua unit lengths and unit areas, or sums or products of such unit lengths and areas." Gaukroger (1992, p. 101) It follows that the Cartesian conception of numbers does not diverge from the Greek conception of numbers.

The close relationship between numbers and unit lengths has far reaching consequences in Descartes' thought and especially in his thinking that his famed method of scientific inquiry, namely analysis and synthesis, can be applied equally to all scientific fields. Studying Descartes, one comes across several themes of the Cartesian thought about the method that suggest that the analogy between the method as applied to physics and mathematics is much more than structural. One of these themes is the role of proportional ratios, or compositions, in problem solving, whether it be in mathematics or in natural sciences. The process of finding the appropriate compositions that would explain the natural phenomena and the role of mathematical proportions is discussed in an excellent way by Sepper (1996). Sepper shows that the foundation of the Cartesian method lies in Descartes' work with his mechanical 'proportional, or mesolabe, compass', which was initially used for finding mean proportionals, and whose use was extended by Descartes to the solution of cubic equations. As Gaukroger (1995, p. 99) remarks, Descartes saw that the key to the operation of the compass is the manipulation of proportional magnitudes, and that the compass could be used to resolve any questions regarding, or being reducible to, proportional magnitudes.

The Cartesian Physicomathematics is the project of applying geometrical analysis to problems construed in a mechanistic and corpuscular understanding of things and events. Proportional relations apply to problems in natural philosophy too. As Sepper (1996, p. 78) notes it is not hard to see how Descartes might have believed that he had found a way to unite both these methods, the causal (which is the method applied to the natural sciences) and the proportionate (which is the method applied to mathematics). The kinds of devices he imagined, which consisted of interconnected moving lines, effect an outcome in principle that can be causally translated to real machines in the physical world: all these motions and effects are interconnected by proportional relationships. All complex proportions, in turn, can be constructed from simple proportions. This means that the comparisons involved in problems in natural philosophy can be expressed in differences in magnitudes that are easily imagined, and that Cartesian mechanics is foremost the study of the dynamics of physical objects as they are interconnected in a network of proportional relations and proportional motions.

It seems that the main theme underlying Descartes' belief that proportions is the key to the solution of all problems concerning magnitudes is Descartes' conception of analogy of nature and of the psychology of perception, especially in *Le Monde*. The relationa mathematics (that is, the mathematics of proportional relations) that Descartes sought t apply is (Sepper 1996, p. 100) "grounded in an ontological epistemology of analogy tha implied similarity between different levels of being." Since 'phantasia' and objects bot participate in the same simple natures of extension and motion in similar ways, th figures imagined participate in these simple natures wholly, and thus, the representin figure units, the figures as they are depicted in phantasia, and the objects in the worl that produce them are all of the same nature. This is why their relations can be describe by a system of proportional relations. So, we see here reemerged in Descartes' though the theme that we pointed out in the beginning of our discussion, namely that arithmeti through geometry is grounded on our perception of the world.

Descartes thoughts of motions and curved lines are closely interconnected. Descartes work with his mesolabe made him believe (Gaukroger, 1995; Sepper, 1996; Shea 1991), that the motions of real bodies can be described by means of algebraic curves Imagine, for instance, a pen fixed at some point of the mechanical compass while it i expanding. The path traced is one (Sepper, 1996, p. 230) "that a point would follow if were subjected to forces of collision or pressure from immediately surroundin particles." But then, it is also true that the complex geometrical curves are derived fror the motions of real points subjected to real forces. In fact, Descartes (AT. X. 157; CSM 3: 2) had used the notion of motion of the compass to classify the geometrical problem into those that can be solved by means of a single motion, those that can be solved b motions subordinated to one another, and those that are solved by means of tw independent motions. But the motion of the compass with a pen attached at some poir traces a path that a point would have followed if subjected to various forces. Thus, th motions of points subjected to forces may seem to determine the classification o curves. Here is another reason for Descartes' claim regarding the universality of hi method; geometry itself is modeled upon motion. A similar thread of themes ha emerged in research in Mathematics education where understanding geometrical figure is tied either to spatial visualization abilities that consist, amongst other things, in th ability to understand movements in three dimensions, rotations of objects, relativ change of position of objects, and the ability to manipulate and transform a figure int other figures (McGee, 1979), or to the so-called operative apprehension (Duval, 1995 that encompasses the various ways a figure can be modified.

THE PRIORITY OF PERCEIVING SPACE

In a series of experiments, Uller et al. (1999) show that very small children do not coun objects in a visual scene and then codify the result of their numbering, the cardina number of the set of objects, by means of a symbol that they store in memory. Wha they do is to represent the objects in the visuals scene by building models of the object and by updating the contents of these objects as they change. The objects in thes models are represented by means of storage files or object files (Kahneman & Treismar 1994). A set of two objects is being represented as "O_i O_j", where O_i and O_j are th files that contain information about the two objects. This representation contains the information that there exist two discrete objects and that these are the only objects in the scene. If after some time an activity that is undetectable by the child takes place and only one object appears in the scene, the child constructs a model that contains only an object file, namely "O_k". If two objects appear but the one is different from the object in the first encounter, then the child builds the model "O_k O_l". The new and the old models are compared by means of a process that detects one-to-one correspondences between the object files in the two models. In the first case, such a correspondence does not exist between the models "O_i O_j" and "O_k" and the child gets surprised, as evidenced by the experimental results. It seems, therefore, that the children represent objects in the form of object files and using these files they build models of the visual scene. Perceiving an event as expected or as unexpected and the subsequent reaction is based on a process that detects one-to-one correspondences between the compared models. If such a correspondence exists, and in view of the fact that the child does not expect objects to simply disappear without a reason, the event is expected. Otherwise, it is unexpected. It is important to remind the reader that if this theory is correct, the children do not store symbols (that correspond to cardinal numbers) and they do not count objects in sets and compare the symbols that counting produces.

Uller et al. (1999) did not examine the sort of information that the object files contain. In what follows, I will present some evidence suggesting that the object files contain primarily spatio-temporal information that is extracted perceptually from the visual scene. The role of attention in parsing a visual scene and in individuating the objects in it is well known. Most attentional models construe attention as a mechanism that enhances processing of information coming from those areas or objects in a visual scene that are important for the needs of the viewer at that time. In the case of spatial attention, attention enhances processing of information contained in some specific areas of the visual field. In addition to spatial attention, there is another sort of attention that focuses on objects as individuals in the visual scene and not on spaces; this is the so-called object/feature-centered attention. (Scholl et al., 2001). In this case, the limitations imposes by the operation of attention concern the number of the various objects that the visual system can process simultaneously. Object-centered attention allows the representation of objects as bounded, solid individuals that persist in space and time.

Think of two identical red squares places on two different positions. To the extent that all their features are identical, the only way one could consider them to be separate objects is by considering their different spatio-temporal histories. This presupposes that there exists a mechanism that is sensitive only to spatio-temporal information and not to anything else and which can detect the two squares and construct two different representations for them, which contain only spatio-temporal information. This mechanism files the objects in the visual scene as separate entities that persist in space and time, and allows their tracking as they move in space. This mechanism is the object-centered attention.

Xu and Carey (1996) showed that 10-months old infants can use spatio-temporal information to 'infer' the existence of objects hidden behind a scene, but they cannc use to the same end other featural information although they can certainly see the othe features (color, shape, etc.) of the objects. This means that the objects are individuate by indexes or files that are initially devoid of any featural information. Twelve-mont olds can use both sorts of information to 'infer' the presence of objects behind a screer The same studies showed that the spatio-temporal information overrides conflictin featural information. Using two objects that differed both in their sensible features an in category (a yellow duck and a white truck), the researchers noticed that the infant did not get surprises (they did not look longer) when the truck disappeared behind th on end of a screen and the duck appeared from the other end, but they were surprised i nothing came out from the screen. This suggests that there is a mechanism sensitive t the spatio-temporal history of objects and which allows the infants to parse objects in visual scene and track their movements, whereas the mechanism that detects th features of the objects and uses them to represent the objects either does not exist fror the beginning, or if it does exists (as it is probably the case) its function is supercede by the mechanism that detects spatio-temporal information.

Spelke et al. (1995) draw similar conclusions from their studies and argue that th infants differentiate qualitatively identical objects based on spatio-temporal informatio and that they use this sort of information to differentiate the objects and also to perforn operations of elementary arithmetic on the objects. Kahneman et. al. (1992) showed tha the features of an object may change while the infant still represents the object as th same object as before. In other words, it is not a new object but the same object witl different features.

Wynn (1992; 1995) studies the elementary arithmetic operations of infants. Infants lool more, that is, they get surprised, when mathematical principles like $1 + 1 = 2$, ή $2 - 1 =$ 1 are violated in the experimental conditions, but they do not get surprised if thes principles hold while the objects change suddenly their features while they are hidde from view for a while behind a screen. This suggests that the infants expect changes o the features of objects provided that the basic arithmetic principles do not get violated Object-based attention that focuses on individuals supports the prerequisit representations, since it can explain the experimental results by providing th mechanism that ensures that objects are perceived as the same numerically object despite featural changes.

Finally, experiments in which the participants are asked to follow the simultaneou movements of several objects (MOT experiments) by Pylyshyn and Storm (1988) witl adults support the same findings. In these experiments, the participants must follow number of identical objects that move with independent motions amongst other object distractors that also move. The targets were designated as such by means of a cue. Th participants can follow up to 5 objects. Since both targets and distractors are identica and their motions random, the participants could succeed in following their movement only if they had tagged the targets from the beginning and could track their movements Success in MOT experiments presupposes that the participants focus their attention t spatio-temporal information and not to the features of the objects or even to the positio of the object. This explains why changes in features and changes in location do not disrupt the object tracking (Pylyshyn 2001; Scholl et al 2001). Pylyshyn claims that attention catalogues the targets by attaching to them tags that the participant can then track.

All the aforementioned studies underline the priority of spatio-temporal information in parsing a visual scene and individuating its objects. The perceptual system opens files for some of the objects (those objects that are behaviorally relevant). Once a file has been opened and assigned to an individual on the basis of spatio-temporal information, the features of the objects can be added.

The studies we discussed cover two general fields, namely, the development of object representation in infants and the object-based attention that individuates objects. The evidence shows a convergence of the relevant findings in two the areas. Specifically, (a) both infants and adults individuate objects based primarily on spatio-temporal information, (b) the representations on which the individuation of the objects is based survive the occlusion of the objects behind a screen, (c) the representations based on spatio-temporal information genetically precede and override both featural representations and semantic representations, and (d) in both fields featural information is used for object individuation only when spatio-temporal information is inconclusive. This evidence shows that the representations constructed by the infants and the representations that are built based on object-based attention are identical.

The primacy of the system that processes spatial information over the systems that process featural information is independently supported by findings that serious damage, even during early infancy, in the brain areas that process spatial information leaves serious impairments in spatial processing, unlike damages inn, say, areas that support language and which are almost completely reversible. (Stiles, 1995). This suggests that the neuronal system that implements spatial processing is much more genetically prespecified, and thus innately determined, and less plastic than the phylogenetically much younger linguistic system. The reason suggests it self. The spatio-temporal capabilities and, consequently, the neuronal brain mechanisms that implement them are much older phylogenetically than other capabilities and mechanisms that we have developed through evolution, because they are much more essential to our survival. Moreover, once they proved successful they have remained unaltered and have become integrated in our brains. It makes sense that nature has seen to it that we come equipped with these mechanisms that function before other mechanisms, and that the brain structures that support them are much less plastic than other structures.

Finally, the representations of space are distinct in that they are analog representations, as opposed to the symbolic representations of, say, language. Fodor (2008) argues that perceptual representations are iconic and cannot recombine, whereas conceptual representations are discursive and can be recombined the right sort of way. The reason is that iconic representations have no canonical decomposition, that is, although they have interpretable parts, they have no constituent parts because they are homogeneous.

Specifically, discursive representations have canonical decomposition because they consist of distinguishable parts. Simply put, a representation is compositional if its syntactic structure is determined by the syntactic structure of its parts and the syntactic features that are used in the composition. Having syntactic structure means that some parts of the representation are constituents and others parts are not. "Φ", for instance, is a constituent of the representation "Φ(a)" but "Φ(" is not a constituent. In this sense discursive structures are not homogeneous. Iconic representations, on the other hand satisfy the Picture Principle, which states that if P is a picture of X, then parts of P are pictures of parts of X (Fodor, 2008). In this sense, iconic structures are homogeneous. But then, all the parts of a picture are among its "constituents" and, thus, an icon is compositional whichever way you carve it up, that is, no matter how you cut the picture you always get a picture of something. In such a system, it does not make much sense to stipulate the existence of constituents. To appreciate the difference between iconic and discursive representations consider that any part of the picture of the ocean is a picture of a part of the ocean, where as not any part of the discursive representation Φ(a) is a discursive representation of a part of Φ(a). So perceptual representations being pictorial are structurally unlike discursive representations.

It is also true that a discursive representation is compositional if its semantic content is also determined by the semantic content of its parts and, hence the preceding discussion about syntactic entities can be recast in terms of semantic entities. This is a result of the view that syntax parallels semantics in that there is a neat one-to-one correspondence between syntactic and semantic constituents, which, in its turn is required by Fodor's Language of Thought hypothesis. This is the thesis of semantic transparency. Clark (1989, p. 2) defines as semantically transparent those systems in which there is a clear mapping between states that are computationally transformed at the algorithmic level of description and semantically interpretable bits of sentences at the computational level. Classical systems that posit syntactically structured symbolic representations and whose computational operations apply to such representations constitute semantically transparent systems, in that the states that are computationally transformed are syntactically structured representations that are neatly mapped to semantically interpretable bits of sentences in virtue of their syntactical structure; syntactic facts are directly translatable to semantic facts.

The homogeneity of iconic representations entails that geometrical relations in the world are mimicked by geometric relations in the analog representations. Furthermore the causal relations in the world are mimicked by causal relations in the representation since representations of the side effects of a change are side effects of the representation of that change. In that sense, the updates of the consequences of a change are automatic and free. There is a further, perhaps deeper, sense in which the causal structure of the world is mimicked by the structure of our analog representations of it. It concerns the deep interrelationship between spatial representations, which is the sort *par excellence* of an analog representation, and the pattern of motions of objects in the world. We will discuss this relationship when we briefly examine Descartes' views on the relations between geometry, algebra, and the world in the last section. Pictorial representations are complicit, that is, neither explicit nor implicit. The latter distinction arises within propositional representations, where only explicit representations are available and immediately usable. Implicit representations must be drawn out and become explicit in order to be usable. However, once a representation is explicitly drawn out, it is no longer automatically updated when other representations change since explicit representations are quite independent of one another in that it takes an inference to update one in the face of a change in another. In complicit representations, on the other hand, everything is readily available and, also, when some aspect or other changes, anything else is updated automatically. (Haugeland, 1987, p. 91)

Let us take stock. In discursive symbol systems compositionality entails that M(s)=f {M(c1), M(c2) . . . M(cn)}, where M(s) is the meaning of *s*, *s* is a complex representation, and c1 ... cn are *s*'s constituents. In analog systems, which are not compositional since they do not have constituents, the meaning of the complex representation is not determined by the meanings of its parts but by the way it depicts the worldly state of affairs it represents, that is, by the fact that it mirrors its geometrical and causal structure. In other words, whereas in discursive symbol systems meaning comes from the combination of the meanings of symbols, in analog systems meaning comes from the represented world itself.

This is why in analog representations there exists an isomorphism between operations in the representing world and transformations in the represented world. An analog representation has an inherent non-arbitrary structure that governs how it operates and the relations between aspects of the analog representation are not arbitrary but are determined by the structure of the represented aspects of the world. Consequently, transformations in the represented world are mapped onto operations in the representing system. An example (Gallistel & Gelman 1992) of an analog representational system will clarify this point. Think of the ink patterns that constitute the numerons "1", "2", and so forth, by which we denote numbers 1, 2 etc. The physical operations performed on these ink patterns, the symbols, are not isomorphic to the operations of arithmetic, that is, to the operations of the symbolized system. Thus, while the arithmetic operation 1 + 2 yields 3, no physical manipulation of the ink patterns "1" and "2" produces the pattern "3". The same holds for any kind of digital systems that manipulate symbolic representations.

Consider now systems, histograms for instance that use magnitudes (heights of columns), as symbols, to represent numerosities. The operations on histograms are isomorphic to the system of arithmetic, insofar as the addition of numbers corresponds to the addition of columns of the histogram. To get the operation 1 + 2 = 3, one has to add the column for 1 to the column for 2, by placing on top of it. The systems that use magnitudes for representational purposes are called analog systems and perform analog computations.

CONCLUSION

The reason for which the geometrical meaning remains unaltered in time rests in th special nature of geometry. This nature is related to the way humans perceive th environment and build representations of the objects in it based primarily on spatic temporal information. Both the brain system that processes this sort of information, an the information itself remains essentially unchanged. The basis shapes, surfaces, an volumes are the same for us as they were for the ancient geometer. Moreover, th structural frame of our actions that transform them remain the same even though th transformative actions themselves have changed.

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On the Way to Identify Mathematical Giftedness

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ABSTRACT: *The current study focuses in the description and presentation of challenging mathematical tasks as a means to identify mathematical giftedness. More specific, we will present the theoretical background which led to the development and use of certain types of tasks for the purpose of capturing giftedness in mathematics. We focus on identification principles of mathematical giftedness, mathematical abilities and thinking processes associated to mathematical giftedness. Following identification principles, abilities and thinking processes reported in the literature to be associated with mathematical giftedness, we will suggest challenging mathematical tasks that could be included in the framework of an identification process intended to capture domain specific giftedness, in our case, mathematics.*

Key words: *Mathematical Giftedness, Creativity, Mathematical Abilities.*

INTRODUCTION

Research on gifted students listed a number of attributes associated with giftedness (Freeman, 1994; Cross, 1997; Sheely & Silverman, 2000). Giftedness is domain-specific (Csikszentmihalyi, 2000; Benbow, & Minor, 1990) and this finding supports arguments on the heterogeneity of the gifted population (Clark, 2002). Consequently, methods employed for the identification of mathematical giftedness should differ than those used for the identification of any other type of domain-specific or general giftedness, to capture the characteristics unique for this group. Research on giftedness in mathematics, reveals the unique mathematical abilities and thinking processes of gifted individuals in mathematics (Krutetskii, 1976). Still, cognitive development of the gifted is not fully understood (Steiner, 2006). Thus, there is a need to observe mathematical abilities and cognitive processes to better understand the construct of mathematical giftedness and inform identification processes to align with research findings.

Multidimensional definitions for giftedness proposed in recent years, have incorporated numerous components to describe the construct. Still, it is challenging to assess all the behaviors involved in these definitions with only one test (Salvia, & Ysseldyke, 2001). In response to previous identification processes through intelligence testing, several researchers disapprove their use as a sole means of identification (Dai, 2010; Lohman,

& Rocklin, 1995). Therefore, researchers should use a combination of instruments i order to collect information for a student (Coleman, 2003; Davis, & Rimm, 2004).

Taking this into consideration, this paper describes and suggests challengin mathematical tasks as a means to identify mathematical giftedness. Specifically, th scope of this paper is twofold. Firstly, to present the theoretical background that guide the development of certain types of tasks and an identification process for the purpos of capturing giftedness in mathematics. Secondly, to describe the identificatio instruments and suggested procedure to follow.

THEORETICAL BACKGROUND

The development of the identification instrument and identification process was base on research on identification principles of giftedness in mathematics, mathematica abilities related to mathematical giftedness and thinking processes associated t giftedness in mathematics. In this section, these three topics are discussed.

Identification Principles of Giftedness in Mathematics

There are a number of issues to consider during the design, selection an implementation of an identification process for giftedness in mathematics. For example a number of mathematically gifted students may not have demonstrated their talent yet Although their gifts are not yet manifested, these students have potential i mathematics. Prior research overlooked these students. One possible reason is tha mathematically gifted students that are not high achievers in mathematics emplo qualitatively different methods and thinking processes during problem solving i comparison to high achievers (Sowell, Zeigler, Bergwall, & Cartwright, 1990). For thi reason, this study demonstrated the danger of neglecting underachievers if identificatio relies solely on test results instead of problem solving abilities which are central t mathematics education and giftedness. As a result, special attention should be taken t ensure that mathematically gifted students that are not high achievers will not g unnoticed in any identification process for mathematical giftedness. This study o Sowell, Zeigler, Bergwall, and Cartwright (1990) addressed the need for future researc to focus on the identification of mathematically promising students that will g unnoticed unless they are identified and on thinking methods and quality of reasonin during problem solving.

Not only mathematically gifted students who have not yet demonstrated their potentia may be excluded from identification and provision, but also spatially gifted students More specific, since analytical tasks are overemphasized during teaching, spatiall gifted students may underachieve in school (Diezmann, & Watters, 1996). Moreover spatially gifted students are experiencing difficulties in verbalizing their reasonin (Diezmann & Watters, 1996). Since spatially gifted students perform best on spatia mathematical problems and when using spatial methods, such as diagrams an visualization, identification processes of mathematical giftedness should allow them demonstrate their ability, by incorporating spatial tasks. Similarly, creative tasks in the context of mathematics should be included as a part of identification procedures for mathematical giftedness. Creativity is connected to mathematical giftedness, since a mathematically gifted person is also creative (Sriraman, 2005).

Very often, computational accuracy is overemphasized in class against reasoning abilities associated with high mathematical ability (Ficici, & Siegle, 2008). Moreover, computational proficiency is commonly used as the decisive factor in student selections and not necessarily quality of thought (Johnson, 2000). Some mathematically gifted students may be good in understanding mathematical concepts but relatively weak in computations (Lupkowski-Shoplik, Sayler, & Assouline, 1994) and this may be proven perilous in identification processes. Miserandino, Subotnik and Ou (1995) noted that computational errors made by mathematically gifted children do not imply poor understanding, whereas Sheffield (1994) has also remarked that rapid and accurate computational ability is not a prerequisite or sufficient characteristic of mathematically gifted students. In conclusion, we believe that speed and computational accuracy should not be used as factors affecting decisions about selections of gifted persons.

To avoid misidentification issues as those described above, we should first take into consideration the specific characteristics of the target population (Rotigel, & Lupkowski-Shoplik, 1999), such as specific mathematical abilities and thinking processes.

Mathematical Abilities and Thinking Processes Related to Mathematical Giftedness

In the field of mathematical giftedness there are studies that describe differences between gifted individuals and the rest of the population with respect to their cognitive abilities and thinking processes. Our discussion will revolve around three major abilities; problem solving, spatial and creative abilities.

Problem solving abilities. We would argue that problem solving is central to mathematics education, as well as gifted education. As a result, various attempts have been made to identify the cognitive characteristics of mathematically gifted students, with respect to problem solving (Greenes, 1981; House, 1987; Krutetskii, 1976; Miller, 1990; Sowell, Zeigler, Bergwall & Cartwright, 1990; Waxman, Robinson, & Mukhopadhyay, 1996).

For example, Krutetskii's groundbreaking work (1976) dealt with a twelve year time of comprehensive experimentation with schoolchildren aiming to explore the nature and structure of mathematical abilities. According to Krutetskii (1976), mathematical giftedness is the "unique aggregate of mathematical abilities that open up the possibility of successful performance in mathematical activity" (p. 76), making a clear connection between mathematical giftedness and performance in problem solving.

During problem solving, numerous researchers observed differences between the cognitive and thinking processes employed by capable and average students (for example, Sowell, Zeigler, Bergwall, & Cartwright, 1990). Namely, Wolfle (1986 reported that gifted students are able to go beyond the answer to a particular problem arriving to extensions and generalisations. More specific, Krutetskii (1976) noticed tha capable students generalize mathematical content rapidly and broadly 'on the spot' with a minimal number of exercises. On the contrary, average students, establis generalized links progressively.

Gifted children outperform their non gifted classmates in strategic ability from an earl age (Robinson, 2000). In studies of strategy knowledge, gifted children seem to have larger and broader repertoire of strategies to use during problem solving (Carr Alexander, & Schwanenflugel, 1996; Jausovec, 1991; Montague, 1991). When gifte children are in front of a problem solving situation, they often understand better an faster which strategies are suitable for the specific instance, selecting only the strategie that have proven effective in the past (Steiner, 2006). In the same study the gifte children quickly relied more on higher level strategies than the average-ability childre that focused mainly on lower level strategies. However, although the gifted group use superior strategies, they failed to complete the task in a less time and significantly fewe trials. This is consistent with other views that gifted children take more time and car when planning problem solving (Shore & Lazar, 1996), resulting to more time require to reach a solution. The findings of this study also support the view against the use o speed as a decisive factor in any identification process of mathematical giftedness.

Not be mistaken with speed during problem solving, mathematically gifted student tend to pursue clear, simple and economical methods and solutions in problem solvin (Krutetskii, 1976). In accordance with Miller's findings (1990) that gifted student exhibit speed in learning, other researchers also argued that rapid informatio processing is evident in gifted children (O'Boyle, Benbow, & Alexander, 1995 O'Boyle, 2000; Singh & O'Boyle, 2004). Paradoxically, there are occasions were gifte children may be slower when providing lower-level classroom answers (Gross, 2004) A possible cause for this phenomenon could be that these children may interpret th question at a higher level than was intended, thus handling too much information at th time (Gross, 2004).

Spatial abilities. The relationship between spatial abilities and mathematical giftednes has been studied by a number of researchers. Mathematically gifted students diffe among others in spatial abilities from their average-ability peers (Benbow & Minor 1990; Sowell, Zeigler, Bergwall, & Cartwright, 1990). In another study, Block (1985 found that gifted 4 to 6-year-olds, based on their IQ, varied from average IQ student and chronological age mates with respect to spatial tasks of two- and three-dimensiona rotations, paper folding, and geometric cross sections.

Krutetskii's work also suggested that spatial abilities play an important role i mathematical giftedness (1976), commenting on three types of mathematically gifte persons; analytic, geometric or harmonic, according in the predominance of the verba logical system, the predominance of the visual pictorial system or the balance betwee the two respectively. Acknowledging that students differ in their type of mathematica giftedness has subsequent connotations both for identification and provisions. With the heightened emphasis on numbers and arithmetic, there is the potential danger of the gifted students with special abilities in geometric reasoning to be underprivileged with regard to gifted selections.

Creative abilities. Nowadays, creativity gained an important place within the context of giftedness and gifted education (Kaufman, Plucker, & Russell, 2012), with gifted children being characterized by high creativity (Kanevsky, & Geake, 2005; Geake, & Dodson, 2005). This is also the case in the area of mathematics. As examples of creative behaviors of mathematically gifted students, Wolfle (1986) pointed to the unique solutions developed to common problems by mathematically gifted students, whereas Greenes (1981) commented on their ability to interpret problem information in original ways. Miller (1990) emphasized gifted students' ability to work with mathematical problems in flexible and creative ways rather than in a stereotypic mode.

Leikin (2009) provided an operational definition of mathematical creativity and a method for its assessment based on fluency, flexibility, and originality, following Torrance (1974). Gil, Ben-Zvi and Apel (2007) defined fluency as the ability of producing many ideas, flexibility as the number of approaches observed in a solution and originality as the possibility of holding extraordinary, new and unique ideas.

There is also another aspect of creative ability in mathematical processes that was observed during problem solving. Gifted students show signs of flexibility of mental processes, meaning that they are able to switch rapidly from one operation to another or from one train of thought to another (Krutetskii, 1976; Miller, 1990). Not only gifted students are able to change their way of thinking by changing operations or thinking paths, they are also able to curtail the process of mathematical reasoning by eliminating intermediate steps according to each situation (Krutetskii, 1976) and reverse their mental processes (Kiesswetter, 1985; Krutetskii, 1976).

The qualities discussed in this section are exhibited in gifted students in ways that are different from average ability students. Thus, they could be used as indicators of giftedness in the context of a domain specific identification approach for giftedness in mathematics. Taken in mind the identification principles for the identification of giftedness, mathematical abilities and thinking processes of mathematically gifted students, we suggest types of mathematical problem solving activities to be used in the framework of an identification process focused in mathematical giftedness.

Mathematical Problem Solving Activities for the Identification of Mathematical Giftedness

Research has shown that gifted students may demonstrate gifted behaviors and engage in productive mathematical activity employing higher-level cognition, only when the problem solving task becomes sufficiently problematic (Diezmann, & Watters, 2002). Thus, challenging mathematical activities may provide the means to observe and identify mathematical giftedness.

Greenes (1997) provided a list of characteristics of problems and projects considered t be good challenges. Among the characteristics, Greenes pointed out that challengin mathematical problems should integrate the disciplines. Thus, the tasks may requir students to use concepts, skills, and strategies from the various content areas c mathematics (e.g., algebra, geometry) for their solutions or students may have to us information drawn from nonacademic areas, such as daily-living experiences c extracurricular activities. Another characteristic of a challenging situation is to be ope to interpretation or solution. In an open-ended problem, the answer is neithe predetermined nor known in advance. Typically, these problems call fo experimentation, data collection, and analysis. Among others, open problems with mor than one interpretation, require students to identify possible hypotheses, recogniz different problems formed according to their selected assumption, and then specify th problem conditions before proceeding to the problem solving process (Greenes, 1997) Challenging mathematical problems suitable for gifted students should entail th formation of generalizations. Reaching to generalizations is of vital importance t analogical reasoning, since they provide a means of categorizing problems with simila mathematical structures (Greenes, 1997). Problems that allow students to demonstrat their potential in mathematics should also allow the use of multiple reasoning methods.

To sum up, in the context of a domain- specific identification process, with emphasis o problem solving, spatial and creative mathematical abilities, with less emphasis in spee and computational accuracy and when students are provided with opportunities to solv challenging mathematical problems, they show several of the behaviors and abilitie mentioned.

Taken in mind the research findings, we propose that an identification process fo mathematical giftedness in the two upper grades of the elementary school shoul incorporate two phases. The fifth and sixth grade level is of interest to the researche because these students are judged old enough to have some insight into the way the acquire and process new information. At first, a test containing problem solving, spatia and creative tasks should be group administered to the indented population fo identification. Later, a number of students should be selected according to thei performance on this first test to proceed to the second phase. The second phase shoul focus on the observation of mathematically gifted students' mathematical reasoning an thinking processes, while dealing with challenging mathematical tasks. This is in accor with, Flavell (1976), who advocated that to determine successful problem solving, w should both look at the result and the way the individual monitors and evaluates his o her problem solving process.

The next two sections discuss and present activities that are part of the two instrument created: namely the test for measuring mathematical giftedness for group administratio and the test with mathematical challenging tasks for individual administration.

Activities for the Test for Measuring Mathematical Giftedness for Group Administration

As test tasks, we wanted to use non-routine mathematically rich problems. For this reason, we developed a pool of problems by drawing upon and extending problems gathered from various sources to create a new identification mathematical test these sources include among others mathematical journals, web pages and mathematical contests (e.g. Kangaroo mathematical contest items).

The problems challenge students' thought processes, whilst the solutions do not require mathematical concepts and skills that students have not learned according to the national curriculum. In addition, this test places a greater emphasis upon learning than on memorization. Less attention is given to computational accuracy and speed. The type of questions and tasks in the test were selected to be different from "everyday" mathematics questions, since in such a case, the responses of students tend to follow their teaching. If the tasks were similar to those taught in class, then students' performance would reflect much more on how and how much they have been taught, rather than mathematical reasoning in a novel situation. Thus, the tasks were designed and phrased in a specific way, as questions that are not directly taught in lessons to the specific age group. Although they draw on the elements of the curriculum for the age group, the questions are unlikely to have been rehearsed. In this manner, it is felt that responses to the problems are more likely to be a reflection of qualities in the student rather than the success of teaching. Moreover, we want to identify also promise and underachievers, so these students may be motivated to show their abilities when confronting novel challenging problems out of the level of ordinary teaching, not depending on the taught curriculum.

As mentioned before, it is our belief that tasks should not be curriculum oriented, allowing students to show potential and mathematical thinking than accumulated knowledge from teaching. Following research reported in the literature on mathematical giftedness, we propose tasks aiming to investigate students' mathematical abilities and require problem solving ability, spatial ability and creativity. The items investigating spatial ability are reprinted with permission, since they were used as a part of an instrument used in the context of a doctoral dissertation (Pittalis, 2008).

The majority of the questions, except for the spatial and creative ones, is presented in a "user-friendly" layout, including additional illustrative material, in a five multiple choice format and a box headed "Show your work". The option of presenting the five answer choices in each task is made after careful consideration. Our intent is to draw students' interest in responding the items of the questionnaire. Since the level of difficulty is high of the test due to its purpose and given that it will be administered to the general population, we want to avoid a large numbers of non-gifted students not approaching the items. The insertion of a "Show your work" area, is also very important, since our intent is not just to find students who respond correctly but we also want to observe their problem solving strategies and methods employed. A description of tasks that we consider appropriate for a test attempting to capture mathematica giftedness follows.

The first category of tasks included in the test for measuring mathematical giftednes for group administration, assessed *mathematical problem solving*. Figure 1 presents tw representative tasks from this category.

A ticket for a mini football match costs €12 and with the same price the player rents a pair of football shoes. If the player buys his own shoes, the ticket costs €5. If the price of the shoes is €20, what is the smallest number of matches a player should play in order to be better in terms of cost to buy his own shoes?

A. 3 B. 4 C. 5 D. 6 E. 7

Show your work.

While reading a book, Stella notices that if she multiplies the numbers of the pages in which the book is open in, the unit digit of the result is 6. The book page numbers are two digit numbers. If Stella adds the two page numbers, what is the unit digit of the result?

A. 2 B. 4 C. 6 D. 5 E. Other answer

Show your work.

Figure 1. Example of two mathematical problem solving tasks of the test for measuring mathematical giftedness for group administration.

The first task shown in Figure 1 requires students to first understand the problem, sinc it involves a real life situation. The student has to choose between buying a ticket for mini football match that includes entrance and renting football shoes, and buying ticket that is valid only for the entrance and buying his own shoes. Thus, the studen should decide the smallest number of matches a player should play in order to be better in terms of cost to buy his own shoes.

The second task shown in Figure 1 requires not only to reason mathematically in order to find the correct answer, but also to experiment with possible page numbers, after thinking about real life conventions. In particular, the task involves paging in books. Therefore, the student should think that the two pages will be two consecutive digit numbers, following daily life experiences with books. If this convention is not taken into account, then an incorrect response will be given. In this problem, students may again work in different levels, after they think that they are looking for two consecutive two digit numbers. For example, a student may choose the trial and error strategy, in order to find the pairs of two consecutive two digit numbers that when multiplied the unit digit of the product is 6. However, this strategy will be proved unproductive, since the person is probably going to get lost in this search. It is important here for students to understand that to simplify the task, it is better to concentrate on the unit digits of the two consecutive numbers. Thus, their goal now has become much easier. Students should find two consecutive one digit numbers whose product is 6. In a higher level, another student may immediately think that 2 and 3 as unit digits give a product of 6 and thus the sum of 2 and 3 is 5. However, this student has failed to think that there is another pair of two consecutive two digit numbers, whose product ends in 6. The two digit numbers could end in 7 and 8. Thus, the sum of 7 and 8 is 15, and in this case, the unit digit of the product is again 5.

It is important here to note that although in the last two cases we described, the students gave the correct answer, that is 5, the latter student has fully understood the problem and the number relations involved, in order to think that there are two possible pairs of unit digits of consecutive numbers that fit to the problem information. In the context of an identification process and also in classrooms, this example clearly illustrates the importance of not paying attention only to a student's response to a particular problem, but also to their solution strategies. For this reason, we decided to add, where appropriate, a text box with a prompt for students to write down their thinking, trials, even a diagram they might make in their attempt to solve any problem. In this way, we could come back later on and examine their solution strategies, both for the mathematically gifted and also for the remaining students. This has important information to give for both groups of students. In the case of mathematically gifted students, it could be interesting to look closely at their strategies and investigate issues such as if they prefer mental rather than paper and pencil strategies, if they prefer higher level strategies, if they arrive more easily to generalizations, if they curtail their reasoning rather than the non-gifted students and so much more. In the case of non-gifted students, students' notes and responses in these challenging tasks will also provide useful feedback for teachers and researchers. More specific, through the examination of notes and responses of non-gifted students, even students facing a lot of difficulties in mathematics, we will be given the opportunity to examine their solution strategies, their misconceptions, errors made and difficulties experienced, in an effort to find ways to complement our teaching in the future so us to minimize their difficultie and improve their problem solving abilities.

The second category of tasks included in the test for measuring mathematical giftednes for group administration, assessed *spatial ability*. These problems, dealt with the notio of paper folding, perspective and mental rotation. A sample problem of perspective i shown in Figure 2. Here, the student has to place himself in the position of the boy i the figure, in order to imagine the front view of the solid, according to the boy' perspective. This task along with other spatial tasks included, was selected after th recommendations of the research literature to provide spatial tasks. This way, spatiall gifted students are given the opportunity to show their abilities and we overcome th danger of neglecting this particular subgroup of the mathematically gifted population.

In this part, a boy is looking towards a solid. The dashed line represents the direction o his sight. The four images below the line present four images; one is the image the bo sees from his position. Circle the right image.

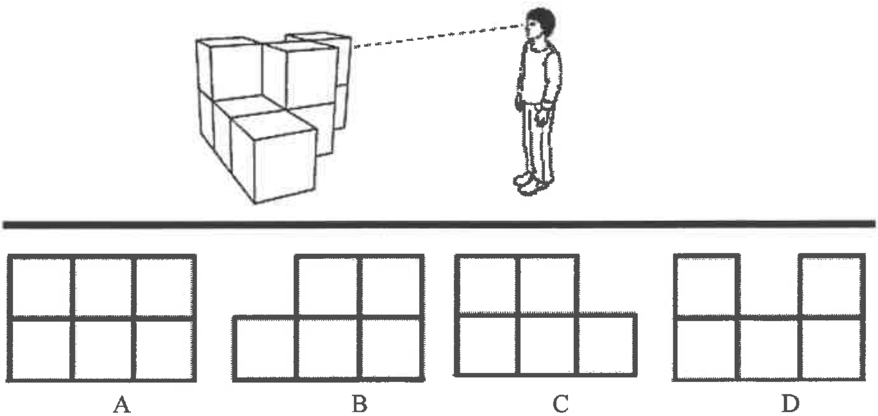

Figure 2. Example of a spatial task of the test for measuring mathematical giftedness fo group administration.

Figure 3 presents one sample problem from the *creative mathematical* tasks included i the test for measuring mathematical giftedness for group administration. This categor of tasks prompts students to provide multiple solutions, allowing us to assess thei fluency, flexibility and originality in mathematics.

Split each boot in four shapes with the same area in as many different ways you ca You may use four colours to show your answer.

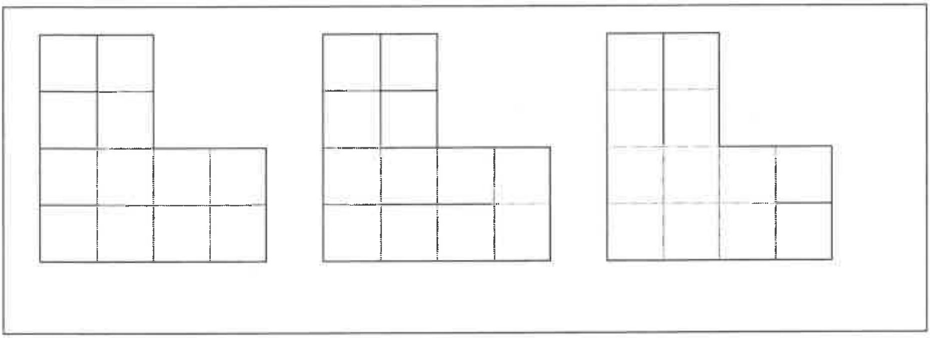

Figure 3. Example of a mathematical creative task of the test for measuring mathematical giftedness for group administration. (Note: Additional shapes are provided in the actual test)

This task appears on one page sheet with 12 boot-shaped figures in the actual test. Students are to be told by the test administrator that additional sheets with boot-shaped figures can be provided. There are numerous ways to proceed with this task and it is up to the student how many solutions he/she will come up with. In this task, persistence is an important factor since a student may divide two figures and then decide to proceed on to another task, while another student may keep dividing each boot-shaped figure until he/she fills two or even three sheets of paper. This is very important for the assessment of creativity, since fluency refers to the number of different correct responses provided by a student. In this task, we also measure a student's flexibility, in other words the number of different approaches observed in a solution. For example, our figures are divided into 12 squares and we want to create four shapes with the same area. The simplest way to go is to follow strictly the dotted lines and divide the figure into four shapes with an area of 3 squares each. There are many ways to provide a solution if staying on the path of the dotted lines, one of which is shown in Figure 4 (solution A). A different approach is also to use halves in combination with whole squares. An example of this approach is shown in Figure 4 (solution B). A student, who provides for example solutions of type A and B, exhibits a greater degree of flexibility than a student who provides solutions of type A exclusively. There are also other possible ways to proceed with this task, such as using fourths (Figure 4, solution C) or even eighths, combining them together or using exclusively one type, such as dividing all the squares into fourths and then drawing four shapes with area of twelve fourths. A greater number of different approaches observed in the solution results to a greater degree of flexibility. With regard to originality, this depends on the frequency of a proposed solution by other persons in the sample population. Hence, a solution of type A is expected to be proposed by the majority of students, while a solution of type B is more likely to have a smaller frequency rate in students' responses. As a result, a student who provides solutions of type B and C is expected to be graded with a higher mark of originality, in comparison to a student who provides only A type solutions.

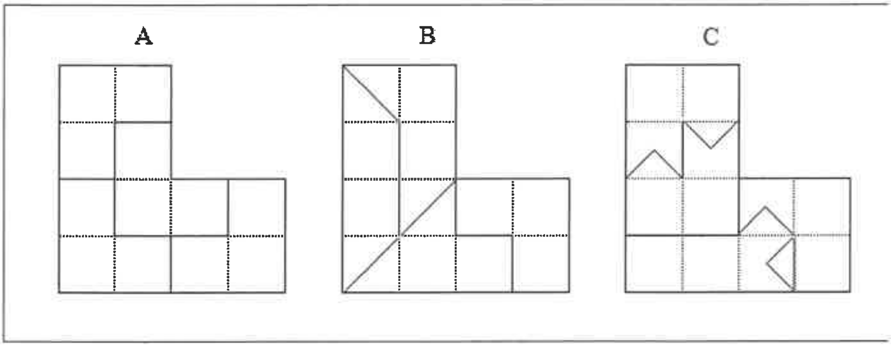

Figure 4. Example of various approaches in solving the mathematical creative task of the test for measuring mathematical giftedness for group administration.

In general, we selected this type of testing for several reasons. Firstly, the tasks include are designed in such a way as to focus on mathematical abilities associated wit mathematical giftedness rather than learned mathematical facts from specific curricula Secondly, this test is not likely to have a 'ceiling' for students at this age. Thirdly, it i cost and time effective because it is a group test rather than an individually administere test.

It is our belief that not only mathematically gifted students provide more correc solutions, but they also have qualitatively different thinking than non gifted individual or high achievers in mathematics. High achievers in mathematics may also provid numerous correct responses, but this does not alone pronounce them mathematicall gifted. In the context of this identification process, the second phase involves a secon instrument, now for individual administration. The aim now is to observe closel students believed to be mathematically gifted, in order to get a closer look in their wa of thinking while dealing with mathematical challenging tasks and examine the natur of cognitive processes, as suggested by the literature.

Test with Mathematical Challenging Tasks for Individual Administration

We believe that identification of mathematical giftedness is incomplete until th researcher has observed students while solving rich mathematical problems to examin their reasoning abilities, thinking processes and solution strategies. Thus, afte collecting and analyzing data from the first test which is group administered, a numbe of students are to be selected according to their performance. These students will hav the opportunity to complete another mathematical instrument at a later time.

This test is designed especially for the observation of students by the researcher whil engaged in challenging mathematical tasks in a one-to-one setting. Due to the fact tha this stage aims to study students reasoning and cognitive processes while problem solving, the selection of mathematical tasks is very important. In this study, the student should be encouraged to show his/her way of organizing relevant information, elucidate his/her conjecture and justify the provided response. Following recommendations from research literature (e.g. Koshy, 2001; Krutetskii, 1976), it is our intent to give students the opportunity to exhibit their abilities or potential through challenging problems posed to them and observing them through the process. We do not want to measure speed, so students are not to be rushed in completing the test in the shortest amount of time. Rather, we want them to take time to express their thinking strategies.

Figure 5 presents one sample problem from this instrument. The particular task is proposed in the webpage of NRICH program (http://nrich.maths.org/8065).

- Choose any number.
- Multiply the chosen number by itself.
- Subtract your starting number.
- Is the number you're left with odd or even?
- Can you prove that your result is always true and not just true for the particular number that you chose to start with?

Figure 5. A sample problem from the test with mathematical challenging tasks for individual administration.

The task shown in Figure 5 is a task that allows students to arrive to generalizations, demonstrate flexibility of thinking and even curtailment of reasoning. It also allows researchers to observe aspects of students' strategic ability, such as the repertoire of strategies, the different strategies that they may employ in order to reach to a solution as well as the possible ease in which they may possibly choose to move from unsuccessful strategies to alternative ones. The goal here is for students to understand that the result is always an even number and then try to explain why this happens. With this problem, students are given the opportunity to solve the problem using various solution strategies. A number of students may arrive to the proof immediately upon seeing the problem, thus demonstrating signs of curtailment and ability to generalize. Others may need to try numbers and then understand the underlying idea of the problem. A student may attempt to solve the problem mentally, while another student may attempt to try to write down different numbers, calculate the results and find the justification for the odd results.

There are a number of justifications that may be provided by students. Some students may try different starting numbers, then follow the procedure according to the instructions and write down the results. Then, they may observe that they are always left with an even number and try to explain why this happens. They may understand that with this procedure we have to subtract an even number from an even number or an od number from an odd number, and such subtractions always end with an odd numbe Others may focus on another relationship. Multiplying a number by itself and the subtracting the starting number from the product, means that we multiply the startin number with its precedent number. For example, if the chosen number is 8, the produc is 7 times 8. Since we have the product of two consecutive numbers, we conclude tha we always multiply an odd with an even number. As a result, the product of an even an an odd number is always an even number.

Taken altogether, the two instruments, proposed as a part of a domain-specifi identification process, complement each other in an effort to capture both mathematica abilities and thinking processes characterizing giftedness in mathematics.

DISCUSSION

The scope of this article was twofold. Firstly, we wanted to present the theoretica background that guided the design and development of challenging mathematical task to be used in an identification process of giftedness in mathematics. Our secon objective was to describe the identification instruments in combination with th identification process to follow.

Research on mathematical giftedness and identification of giftedness of mathematic served as our guide in the development of the identification process and instruments We decided to design a two phase identification process for the two upper grades of th elementary school, since a combination of tools should be used in order to gathe information for a student (Coleman, 2003; Davis & Rimm, 2004; Salvia & Ysseldyke 2001).

The identification process requires the use of two identification instruments; a test fo measuring mathematical giftedness for group administration and a test wit mathematical challenging tasks for individual administration. Our instruments aim t examine mathematical abilities and thinking processes associated with mathematica giftedness and taken as a whole reveal giftedness in mathematics. They do no emphasize speed or computational accuracy, since these have been proved to hinder th identification of gifted students who may be relatively weak in computations, a proposed by Lupkowski-Shoplik, Sayler, and Assouline (1994). Rather, the focus is o mathematical abilities and mathematical thinking processes.

The first instrument is intended to be administered to all the sample population of th two upper grades of the elementary school, in order to identify the students gifted i mathematics. The tasks included focus on mathematical abilities associated wit mathematical giftedness rather than learned mathematical facts from specific curricula Through this instrument we may observe the mathematical abilities of gifted students i mathematics in comparison to non mathematically gifted students. Such abilities hav been reported by numerous researchers and can help identify the mathematically gifte students (Krutetskii, 1976; Miller, 1990; Sowell, Zeigler, Bergwall, & Cartwright, 1990

Wolfle, 1986). In particular, this instrument assesses problem solving ability, spatial ability and mathematical creativity. The decision to include spatial tasks was based upon a number of studies illustrating the relationship between spatial abilities and mathematical giftedness (e.g. Benbow, & Minor, 1990; Block, 1985; Sowell, Zeigler, Bergwall, & Cartwright, 1990). Through the use of spatial tasks, we also allow spatially gifted students to demonstrate their ability, avoiding their possible misidentification, as shown in previous studies (Diezmann, & Watters, 1996). In regard to the inclusion of creative mathematical tasks, research has revealed the relationship of mathematical giftedness and mathematical creativity (Kattou, Kontoyianni, Pitta-Pantazi, & Christou, 2012; Sriraman, 2005).

In this paper, we argue that the quality of mathematical reasoning of mathematically gifted students is far more important than providing correct answers. We believe that gifted students in mathematics perform better and also use qualitatively different thinking than non gifted individuals or high achievers in mathematics. High achievers in mathematics may also provide numerous correct responses, but this is not a sufficient indicator of giftedness. What sets apart mathematically gifted individuals with high achievers is the quality of reasoning. The quality of mathematical reasoning is also supported by Krutetskii (1976), who pointed to the fact that mathematical giftedness is a special qualitative combination of abilities, unique for each person.

In this context, the second phase of the identification process involves another instrument, now for individual administration. The aim now is to observe closely students believed to be mathematically gifted as selected according to their performance in the first instrument, in order to get a closer look in their way of thinking while dealing with mathematical challenging tasks and examine the nature of cognitive processes, as suggested by the literature.

In contrast to the first instrument, the second instrument is completed only by students gifted in mathematics and provides an insight into students' cognitive processes during problem solving. Thus, this instrument allows the observation of processes as those supported by the research literature, such as the ability to experiment and quickly form generalizations, show signs of flexibility, curtailment and reversion of mental processes and mathematical reasoning (Kiesswetter, 1985; Krutetskii, 1976; Wolfle, 1986).

It is our belief that through this identification process, we may capture mathematical giftedness in a coherent manner and reveal both the abilities and thinking processes of mathematically gifted students as demonstrated through problem solving. Future research efforts should focus on the implementation of such identification processes, to identify mathematical giftedness and analyze the cognitive differences among gifted and non-gifted students in mathematics.

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Is it Important to Know Students' Perceived Value of Mathematics?

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ABSTRACT: *This study investigates the perceived value of mathematics in students from Greek Cypriot Gymnasium. It also examines individual differences in the perceived value of mathematics in relation to students' gender, grades, marks in mathematics, socioeconomic status (SES) and parent education (PE). The perceived value of mathematics is measured overall and with four dimensions: Interest, General Utility, Need for High Achievement and Personal Cost. A sample of 408 Greek Cypriot students from the three years of Gymnasium (mean age 13) completed the Mathematical Value Inventory (MVI). Results revealed that (a) girls value the General Utility of mathematics higher than boys; (b) boys evaluate Personal Cost of mathematics higher than girls; (c) the overall value of mathematics decrease with grade; (d) students with higher marks in mathematics show a higher overall value of mathematics; and (e) students with higher SES and students with higher PE show a higher overall value of mathematics. Implications are discussed for the Greek context. Also suggestions concerning intervention are given.*

Key words: *Perceived value of mathematics, Gender, Grades, Marks, Socioeconomic status, Parents education.*

INTRODUCTION

Mathematics is a core subject in the school curriculum all over the world and mathematical competence is considered to be a key skill for each potential employee in the present-day world-market. Moreover, research has highlighted the role of mathematical competence as a "critical filter" to effectively screen students for a prestigious career (Ma & Johnson, 2008; Shapka, Domene, & Keating, 2008). The educational system of each country is also monitored on the basis of the success of the students on international mathematical tests e.g., International Mathematics and Science Study (TIMSS), Programme for International Student Assessment (PISA). Given the importance of students' mathematical competence to employment, and high-status career, a large body of research has been developed concerning effective mathematical curricula for successful educational programs. For example, Cyprus recent educational reform reflects this new trend by fostering activity, problem solving in the students an focusing also in technology based learning in the new mathematical curricula (MOEC 2010). However, there is little attention paid to the perceived value that student attribute to mathematical literacy. According to Luttrell et al. (2010) students are trul mathematical literate, if they positively value mathematics. If students do not valu mathematics in a positive way, then any mathematical curricula or method will hav little impact on students' literacy.

The goal of the present study is to investigate students' perceived value of mathematic: As most of the studies are carried out with USA, the present study focuses in th unexplored context of Greek Cypriot Gymnasiums. Different factors are employec which might be related to the value that Greek Cypriot Gymnasium students attribute t mathematics. These factors are: students' gender, grades (i.e., years in Gymnasium) mathematical performance (i.e. marks), socioeconomic status and parent educatio Results provide important information to Cypriot researchers, policy makers an practitioners in the critical period of Educational reformation in Cyprus.

THEORETICAL FRAMEWORK

Eccles and her colleagues developed the Expectancy-Value Model to explain ho students' value system and expectancy beliefs affect their levels of engagemen persistence, and achievement in various subjects - such as mathematics (for an overvie see Eccles (Parsons) et al., 1983; Eccles, 2005). They also demonstrated that th subjective value assigned to a subject has several dimensions (Eccles & Wigfielc 1995). Specifically, the value assigned to a subject has the following 4 dimensions. (A Interest, which is defined by the importance that students attached to a subject becaus of intrinsic motivation and genuine interest for the subject (Deci & Ryan, 1985). (B General Utility, which is the importance students attached to a subject because of th usefulness of the subject to help them achieve short and long term goals (Husman & Lens, 1999; Kauffman & Husman, 2004). (C) Need for High Achievement, which is th personal value students attributed to the good performance of the subject an developing good conceptual understanding of the subject (Eccles et al., 1983; Lutrell e al., 2010). (D) Personal Cost, which is the negative aspects of dealing with the subjec such as performance anxiety and fear of failure; and the effort needed to achieve th objectives of the course (Lutrell et al., 2010).

Eccles' Model is receiving large research utilization, in the examination of the valu attributed by the students to mathematics and its relation to their future educationa choices (Eccles, 1983, 1984). Specifically, the overall value of mathematics predict good performance and high marks in mathematics (Berndt & Miller, 1990), intentions t enroll in mathematical courses (Meece, Wigfield & Eccles, 1990), number o mathematical courses taken (Simpkins et al., 2006; Updegraff et al., 1996), plans fo higher studies in the field of mathematics (Eccles et al., 2004) and expectations fo career related to mathematics (Watt, 2006).

Gender, Grade, Marks, SES, and PE in Relation to Valuing Mathematics

Great importance is given to the investigation of gendered choices and achievement in mathematics. Findings from international mathematical tests (TIMSS, PISA) showed that gender differences in achievement in mathematics, in different countries, and in different time-periods is somewhat mixed. In most countries boys and girls have similar results in the fourth and eighth year of schoolings; however, boys advanced in later school years (Eurycide, 2010). The National Center for Education Statistics (2008) reported that there are no gender differences in academic achievement scores of boys and girls' in mathematics. The most recent PISA report (OECD, 2010) shows that in many countries, the girls, on average, performed somewhat worse than boys in mathematics. This prevalence of boys is due to the very high levels of performance of a comparatively small number of boys. However, in other countries, such as Cyprus, this pattern is reversed: In TIMSS 1995 in advanced mathematics there is no gender difference between Cypriot boys and girls, while in TIMSS 2007 girls have higher achievement than boys from year eight (Eurycide, 2010). Exploring a country such as Cyprus where girls seem to outperformed boys in international mathematical tests (Eurycide, 2010) can shed light on the intervening factor of cultural context.

Although, girls and boys have similar abilities, as well as achievements and marks in mathematics, girls and women are underrepresented in studies and careers related to mathematics (Eccles, 1987; Jacobs, 2005; Watt, Eccles, & Durik, 2006; Zarrett & Malanchuk, 2005). Researches turn their attention in the perceived value of mathematics given by boys and girls in order to explain gender imbalance in career choices. Findings suggest that both boys and girls report similar overall value in mathematics (Eccles et al., 1993; Jacobs et al., 2002). However, when beliefs about the importance of mathematics are examined separately, findings have shown that boys attach greater personal importance in mathematics than girls (Meece, Wigfield & Eccles, 1990; Updegraff et al., 1996); boys also report higher expectations for success and abilities in stereotypically male domains (such as mathematics) than girls (Eccles & Wigfield, 2002; Skaalvik & Skaalvik, 2004). Similarly, recent studies that used person-centered approaches, showed that boys are more likely than girls to be overrepresented in the group that value mathematics the most (Chow & Salmela-Aro, 2011; Chow, Eccles & Salmela-Aro, 2012). Thus, the key mediator to gendered related differences in mathematics participation and career aspirations were found to be the different value that boys and girls attached to mathematics (Eccles, 1987; Watt, Eccles, & Durik, 2006).

Another issue that stems from the gendered related valuing of mathematics is to identify the specific age at which girls' interest in mathematics' declines (e.g., Watt, Eccles, & Durik, 2006). According to Linver et al. (2002) girls' interest in mathematics starts to decrease from middle childhood and continues to decline across high school, even when their marks are higher than boys'. On the other hand, boys maintained high level of intrinsic value for mathematics and high self-perception for their mathematical abilities throughout secondary school (Watt, 2004; Watt et al., 2006). However, other researchers suggested no age-related differences in both genders, showing that a children get older, the value they attach to mathematics is reduced (Eccles et al., 198: Eccles et al., 1993; Hyde et al., 1990; Jacobs et al., 2002). A recent study in a nor American sample showed that both boys and girls reported significantly bette performance in mathematics and a higher self-concept of mathematical ability whe they were at the age of 15, than when they were at the age of 16; with boys to hav higher self-concept of mathematical ability at 16 than girls (Sainz & Eccles, 2012). seems that both boys and girls are becoming less fond of mathematics as they get older.

Other factors that underpin the association of mathematics performance an mathematics valuing are: socioeconomic status and parent education. In th international studies (TIMSS, PISA), findings indicate that gender is less important tha socioeconomic status in predicting achievement. Controlled for gender and immigrar background, index of economic, social and cultural status explains about 5 - 25 % c variance and it is statistically significant in all countries (Eurycide, 2010) Socioeconomic status and parents' years of schooling were found to be relate indirectly to students' academic achievement through parents' beliefs and behavior (Davis-Kean, 2005). It is suggested that the amount of schooling that parents receiv influences how they structure their home environment, as well as how they interact wit their children in promoting academic achievement. Especially, mother's years o schooling were found to play a significant role for girls in regard to achievement i mathematics (Linver & Davis-Kean, 2005). By investigating the effect of each and o all above factors in relation to the perceived value of mathematics in students fror Greek Cypriot Gymnasiums we serve multiple purposes: extend the understanding o the impact of cultural factors; provide useful information in understanding mathematic underachievement; guide for effective teaching materials, methods and tools.

AIM AND HYPOTHESES OF THE STUDY

Our aim is to examine how students in Greek Cypriot Gymnasiums value mathematic and whether this value is associated with gender, grade, marks, socioeconomic statu (SES) and parent education (PE). Based on our review of literature, we propose th following five hypotheses:

H_1: There are no gender related differences in the students' perceived value o mathematics.

H_2: The higher the students' grade (years in Gymnasiums), the lower the students perceived value of mathematics.

H_3: The higher the students' performance (marks), the higher the students' perceive value of mathematics.

H_4: The higher the students' socioeconomic status (SES), the higher the students perceived value of mathematics.

H_5: Students, whose parents have a higher educational level, perceive higher value to mathematics.

In addition, the relationship between those individual variables and the overall value of mathematics will be explored.

METHOD

Sample

The study used a cross sectional survey with a representative sample of students from Greek Cypriot Gymnasiums. The sample consisted of 408 adolescents, 49.6% males and 50.4% females. Participants' age ranged from 12 to 15; they were students from the three grades of Gymnasium (45.9% from year 1, 29.6% from year 2, and 24.4% from year 3). The study includes students from 7 different Gymnasiums in 5 different provinces in Cyprus (36.6% from Paphos, 21.4% from Nicosia, 15.7% from Ammochostos, 13.5% from Limassol, and 12.8% from Larnaca). The socioeconomic status (SES) of participants was estimated on the basis of the father and mother's occupations showed that 30.2% of participants have a high SES, 33.7% have a middle SES and 36.2% have a low SES. Parent education (PE) was estimated on the basis of father and mother's education: 42.2% of participants have a high PE, 45.8% have a medium PE, and 12% have a low PE.

Procedure

To conduct the survey, official permission from the Centre of Educational Research and Evaluation in Cyprus (CERE) was obtained. Then, letters describing the purpose of the study were given, firstly to the head teachers of the schools to allow the conduct of the survey in their schools. Secondly, letters describing the purpose of the study and permission slips were given to parents. The survey was conducted anonymously and voluntarily to those students who had parental consent (2% of the parents did not approve the participation in the survey). The questionnaire was administered to 426 students (96% of the questionnaires were valid).

Measures

Mathematical Value Inventory (MVI, Luttrell, et. al., 2010)

The MVI is a self-report questionnaire that measures students' valuing of mathematical literacy with 4 dimensions: Interest, General Utility, Need for High Achievement and Personal Cost. MVI consists of 28 statements, 7 statements for each dimension. The scale of Interest included statements such as "I find many topics in mathematics to be interesting." The scale of General Utility components such as, "Understanding math has many benefits for me." The scale of the Need for High Achievement included statements such as “Earning high grades in math is important to me.” The scale o Personal Cost included statements such as “Math exam scares me.” The participant were asked to indicate agreement on a 5-point- likert scale (1 represents strongl disagree and 5 represents strongly agree). The questionnaire was translated into th language of Greek and back translated into English by professional linguists Additionally, the Greek MVI was given to 5 Gymnasium students, who discussed th content of each item. Small modifications were made on the translated MVI statement to be more clear and precise in the Greek language.

Demographics

Information was collected on gender, grade, marks on Mathematics, father and mother' occupation and education. The marks that students get in Gymnasiums are as follows Mark A (19-20 out of 20), Mark B (16-18 out of 20), Mark C (13-15 out of 20), Mark I (10-12 out of 20) and Mark E (1-9 out of 20 = failed).

The socioeconomic status (SES) of the participants was determined by both parents occupations (Nash, 1995). The classification of occupations was completed according t the International Standard Classification of Occupations (ISCO), i.e. 1-9 major group and 0 for armed force. Four more groups were added: unemployed (group 11) pensioners (group 12), dead (group 13), and housewives (group 14). The occupations o the students' parents were then collapsed into three categories of SES using th following categorization: (a) Belonging to high SES when at least one of the parents i in category 1, 2, and higher rank in group 0 and the other to any other category. (b Belonging to middle SES when at least one of the parents is in categories 3 or 4 and th other in any one of the lower categories. (c) Belonging to low SES when both parent are in category 5 or under and lower rank in group 0. Parent education (PE) wa estimated on the following categorization of parents' education. (a) High PE signifie that at least one of the parents holds a university degree. (b) Medium PE signifies that a least one of the parents has completed education up to Lyceum and the other has simila or lower education. (c) Low PE signifies that at least one of the parents has complete education up to the Gymnasium and the other has similar or lower education.

Analysis

The results are organized in three sections. Firstly, we reported findings of construc validity of the MVI. Secondly, we tested the gender differences, the grade differences the mark differences, the socioeconomic status differences and parents' educatio differences on the overall MVI and on the 4 MVI dimensions. For testing differences o the overall MVI and the above factors we used t-test and ANOVA. For testin differences on the 4 MVI dimensions we used MANOVA. Thirdly, we have explore the effect of each factor on the MVI score by using a multiple regression model.

RESULTS

To test for construct validity in comparison to the original MVI, a Principal Component Analysis (PCA) was performed on the 28 Greek items with Varimax (orthogonal) rotation. The factor analysis was supported by Bartlett's test of sphericity, χ^2 *(378) =3993.94, p< .001* and Kaiser-Meyer Olkin measure of sampling adequacy of *.9* above the recommended value of *.6.* Only factors with eigenvalues greater than 1 were retained. The criteria led to a four factor solution. The first component included the 7 items related to Interest, accounted for the *26.10%* of the variance, the second component contained the 7 General Utility items, accounted for the *11.44%* of the variance, the third component included 7 items of the Need for High Achievement, accounted for the *7.22%* of the variance, and the forth contained the 7 items related to the Personal Cost, accounted for the *5.78%.* The analysis revealed that *50.54%* of the variation in the 28 items can be explained using 4 factors (see Table 1). Overall, these results showed that the translated MVI is functioning in a similar way to the original MVI in a Greek Cypriot sample.

Table 1

Principal Component Analysis for the Greek MVI

Item	FACTOR			
I. Interest	I	II	III	IV
Mathematics fascinates me (27)	**0.80**	0.13	0.14	0.15
I am interested in doing math problem (20)	**0.78**	0.16	0.22	0.02
Solving math problems is interesting for me (24)	**0.74**	0.04	0.22	0.22
It is fun to do math (16)	**0.73**	-0.03	0.17	0.13
I find many topics in mathematics to be interesting (12)	**0.68**	0.29	0.13	-0.05
Learning new topics in mathematics is interesting (2)	**0.65**	0.23	0.12	0.03
I find math intellectually stimulating (9)	**0.35**	0.34	0.28	-0.07
II. General Utility				
There is almost no benefit from knowing mathematics (3r)	0.11	**0.68**	0.04	0.07
Having a solid background in mathematics is worthless (13r)	0.06	**0.67**	0.16	0.03
I see no point in being able to do math (17r)	0.18	**0.64**	0.09	0.23

After I graduate, an understanding of math will be useless to me (10r)	0.08	**0.63**	-0.03	0.06
I have little to gain by learning how to do math (6r)	0.07	**0.59**	-0.04	0.23
I do not need math in my everyday life (23r)	0.20	**0.56**	0.15	0.12
Understanding math has many benefits for me (21)	0.48	**0.40**	0.40	0.00
III. Need for High Achievement				
If I do not receive an "A" on an math exam, I am disappointed (4)	0.09	0.06	**0.70**	-0.03
Earning high grades in math is important to me (19)	0.23	0.32	**0.67**	-0.10
It is important to me to get top grades in my math classes (8)	0.25	-0.09	**0.67**	-0.01
Only a course grade of "A" in math is acceptable to me (25)	0.11	-0.20	**0.64**	0.36
I would be upset to be an "average student" in math (11)	0.22	0.11	**0.57**	0.07
I must do well in my math classes (28)	0.27	0.32	**0.53**	-0.22
Doing well in math courses is important to me (14)	0.32	0.40	**0.50**	-0.12
IV. Personal Cost				
Math exams scares me (26r)	0.10	0.00	-0.12	**0.67**
Solving math problems is too difficult for me (18r)	0.05	0.37	0.23	**0.63**
Trying to do math causes me a lot of anxiety (22r)	0.10	0.26	0.02	**0.63**
Taking math classes scare me (5r)	0.25	0.34	-0.14	**0.60**
I have to study much harder for math than for other courses (1r)	-0.10	-0.04	0.00	**0.60**
Mathematical symbols confuse me (15r)	0.12	0.37	0.14	**0.53**
I worry about getting low grades in my math courses (7r)	0.05	-0.07	-0.45	**0.48**

Note: The r index indicates items with reversed scores. Factor loadings in bold in the same column load on the same factor.

Descriptive statistics

The sum score for each subscale component (7 items each) ranges from 7 to 35 while the total score for the overall MVI (28 items) ranges from 28 to 140. All statements referring to the Personal Cost subscale and 6 statements of the General Utility subscales were reverse-scored in order to be aligned with the scores of the other statements. Higher scores are indicating higher value perception for the overall MVI score (highest 140) and for each dimension (highest 35). Table 2 shows the internal consistency within the current sample, as well as the means and standard deviations for overall MVI and for the 4 MVI dimensions.

Table 2

Coefficient Alpha, and Descriptive Statistics for MVI and for the 4 MVI Dimensions

	Interest	General Utility	Need for High Achievement	Personal Cost	Overall MVI
Coefficient alpha	.86	.78	.79	.76	.88
Mean	23.71	27.37	24.92	20.79	96.79
Standard Deviation	6.00	5.64	5.59	5.48	16.29

Gender Differences in Overall MVI and in the 4 MVI Dimensions

Hypothesis 1 is partly supported. Independent-samples t-tests and Cohen's d showed that there are no statistically significant gender differences for the overall MVI score *($t(405) = -.18$, $p = .861$, $d = -.02$)*. Concerning the 4 MVI dimensions results showed that there are no statistically significant gender differences for the Interest *($t(405)= .93$, $p =.351$, $d=.09$)* and for the Need for High Achievement *($t(405)= -1.69$, $p =.092$, $d= -.18$)*. However there are significant gender differences for General Utility *($t(405)= -2.00$, $p =.046$, $d= -.19$)* and Personal Cost *($t(405)= 2.24$, $p =.026$, $d=.22$)*. Girls have a higher perceived value for the General Utility of mathematics than boys; while boys have a higher value of Personal Cost of mathematics than girls (see Table 3).

Table 3

Descriptive Statistics for MVI and for the 4 MVI Dimensions for Boys and Girls

Gender	Girls (N=205)		Boys (N=202)	
	M	SD	M	SD
MVI	96.95	16.76	96.67	15.87
Interest	23.44	6.19	24.00	5.83
General Utility*	27.94	5.67	26.82	5.58
Need for High Achievement	25.39	5.66	24.45	5.51
Personal Cost*	20.19	5.51	21.40	5.42

*Note: *p<.05*

Grade Differences in Overall MVI and in the 4 MVI Dimensions

Hypothesis 2 is supported. Firstly, a one-way ANOVA was conducted to explore th relationship between grades (year 1, year 2, year 3) and the overall MVI score. The one way ANOVA was significant ($F(2,402)=4.46$, $p=.012$, $\eta^2=.02$), indicating that student in different grades do not have the same overall value for mathematics. Tukey's post hoc test showed statistically significant differences between the students in year 1 an year 2 ($t(402)=2.37$, $p=.048$, $d=.28$) and between the students in year 1 and year ($t(402)=2.56$, $p=.029$, $d=.32$). No significant differences were found between th students in year 2 and year 3 ($t(402)=.03$, $p=.951$, $d=.038$) (see Table 4). Students i higher grades (year 2 and year 3) value mathematics lower than students in the lowes grade (year 1).

To explore the relationship between the 3 grades and the 4 MVI dimensions multivariate analysis of variance was conducted (MANOVA). The results showed significant effect of grades (*Wilks' Λ= .914*, $F(8,798)=4.61$, $p<.001, \eta_p^2=.044$ Separate univariate ANOVAs were conducted for each MVI dimension. There was significant main effect of the grades on Interest ($F(2,402)=11.58$, $p<.001$, $\eta_p^2=.054$ and Need for High Achievement ($F(2,402)=7.79$, $p<.001$, $\eta_p^2=.037$). Results of pos hoc analysis with a Bonferroni correction showed that for the Need for Higl Achievement dimension, as well as for Interest, significant differences were observe between students in year 1 and year 2 ($t(402)=3.89$, $p<.001$, $d=.46$). Need for Higl Achievement scores are higher for year 1 students than for year 2 students while Interes scores are higher for students in year 1 than for students in year 2 ($t(402)=3.45$, $p=.002$ $d=.41$) and year 3 ($t(402)=4.39$, $p<.001$, $d=.54$). There were no significant difference in Personal Costs ($F(2,402)=1.26$, $p=.284$, $\eta_p^2=.006$) and General Utilit ($F(2,402)=.53$, $p=.592$, $\eta_p^2=.003$) (see Table 4).

Table 4

Descriptive Statistics for MVI and for the 4 MVI Dimensions for the 3 Grades

Variable	Year 1 (N=186)		Year 2 (N=120)		Year 3 (N=99)	
	M	SD	M	SD	M	SD
MVI*	99.25	14.22	94.78	17.96	94.12	16.99
Interest*	25.17	5.51	22.8	5.97	21.97	6.33
General Utility	27.65	5.53	27.08	6.16	27.06	5.19
Need for High Achievement*	25.97	4.88	23.48	5.90	24.57	6.01
Personal Cost	20.47	5.42	21.43	5.45	20.53	5.62

Note: *p<.05

Marks Differences in Overall MVI and in the 4 MVI Dimensions

Hypothesis 3 is supported. One-way ANOVA showed that the overall MVI score varies statistically according to the marks in Mathematics that students achieved in the first semester *($F(4,391)=32.76$, $p<.00$, $\eta^2=.25$)*. Tukey's post-hoc showed statistical significant differences between nearly all the groups of students with different marks. Only students with marks B and C *($t(391)= -2.69$, $p=.057$, $d=-.38$)* and students with marks D and E *($t(391)= -1.33$, $p=.67$, $d=-.62$)* have similar MVI scores. Students with higher marks have a higher perceived value in mathematical literacy. The highest overall MVI score was achieved by the students with the highest school performance in mathematics while the lowest MVI score *($t(391)=-4.25$, $p<.001$)* was achieved by students with the lowest marks. The effect estimate for this is large *-2.23* (Cohen's d). Descriptive statistics are reported in Table 5.

Table 5

Descriptive Statistics for MVI and for the 4 MVI Dimensions in Relation to Marks

	Marks									
	Mark E		Mark D		Mark C		Mark B		Mark A	
	M	SD	M	SD	M	SD	M	SD	M	SD
MVI*	77.00	11.71	85.03	13.82	93.55	13.91	98.86	13.5	107.47	15.35

Note: *p<.05

SES Differences in Overall MVI and in the 4 MVI Dimensions

Hypothesis 4 is supported. Firstly, a one-way ANOVA confirms that students wit different SES have statistically significant differences in relation to their overall MV score, $F(2,398)=9.20$, $p<.001$, $\eta^2=.04$. Tukey's post hoc test showed a statisticall significant difference between students from high SES and lower SES ($t(398)=4.2$ $p<.001$, $d=.53$) and between students from high and middle SES ($t(398)=2.8$ $p=.012$, $d=.36$). The effects estimates are small.

Secondly, socioeconomic status differences in the four MVI dimensions were teste using MANOVA. The multivariate effect was statistically significant (*Wilks'Λ*= .93 $F(8,790)=3.40$, $p=.001$, $\eta_p^2=.033$). Follow up univariate ANOVAs and post ho Bonferroni-corrected pairwise comparisons were conducted. There was a significan association between SES and General Utility ($F(2, 398)=5.46$, $p=.005$, $\eta_p^2=.027$ students from high SES have higher General Utility scores than students from low SE ($t(398)=3.26$, $p=0.003$, $d=.41$). There is a significant association between SES an Need for High Achievement ($F(2,398)=4.99$, $p=.007$, $\eta_p^2=.024$); students from hig SES have higher Need for High Achievement than students from low SES ($t(398)=3.14$ $p=.005$, $d=.39$). Finally, there is a statistically significant association between SES an Personal Cost ($F(2,398)=7.56$, $p=.001$. $\eta_p^2=.037$). Students from high SES have highe Personal Cost than students from middle SES ($t(398)=2.80$, $p=.016$, $d=.36$) an students from low SES ($t(398)=3.78$, $p=.001$, $d=.46$). Descriptive statistics are reporte in Table 6.

Table 6

Descriptive Statistics for MVI and for the 4 MVI Dimensions in Relation to SES

	Socioeconomic Status					
Variable	Low SES (N=145)		Middle SES (N=135)		High SES (N=121)	
	M	SD	M	SD	M	SD
MVI*	93.57	15.89	96.13	16.56	101.90	15.54
Interest	23.29	6.14	23.36	6.22	24.73	5.58
General Utility*	26.52	5.36	27.22	5.90	28.75	5.41
Need for High Achievement*	23.90	5.61	25.07	5.52	26.04	5.48
Personal Cost*	19.86	5.68	20.48	5.22	22.38	5.29

Note: * p<.05

Parents' Education Differences in Overall MVI and in the 4 MVI Dimensions

Hypothesis 5 is supported. One-way ANOVA showed that students with different PE, do not attribute the same overall value to mathematics ($F(2,405)=7.50$, $p=.01$, $\eta^2=.04$). Post hoc analysis showed that students with low educated parents have a lower overall MVI score in comparison to students with high educated parents ($t(405)=3.87$, $p=.009$, $d= .63$). Statistically significant differences were also observed between students with low educated parents, in comparison to students with medium educated parents ($t(405)= 2.96$, $p=.009$, $d= .5$). Table 7 shows the descriptive statistics for the overall MVI in relation to parents' education.

Table 7

Descriptive Statistics for MVI in Relation to Parent's Education

	Parents' Education					
Variable	Low PE(N=49)		Medium PE (N=187)		High PE (N=172)	
	M	SD	M	SD	M	SD
MVI*	89.06	14.74	96.68	15.43	99.11	17.00

Note: * p<.05

Relationship between Individual's Characteristics and the Overall MVI Score

Multiple regression was used to explore the relative relationship between the overall MVI score and the characteristics of the individuals (gender, grades, marks, SES and PE). Given that all of these characteristics are categorical variables, dummy variables had to be created. The overall model was significant $R^2=.303$, *Adjusted* $R^2=.282$, $F(11,373)=14.72$, $p<.001$(see Table 7). Overall, only grade level and marks in mathematics was related to the overall MVI score. Students in year 3 value mathematics lower than students in year 1. Students with the higher mark (A) value mathematics higher than students with lower marks (B, C, D, E). Altogether, the performance (marks in mathematics) is the most important in relation to all other variables to explain student's mathematical values.

Table 8

Multiple Regression Analysis Predicting the Overall MVI

Variable	B	SE	β	t
Constant	107.25	2.10		51.02
Gender	-1.62	1.42	-0.05	-1.13
Year 1 vs. 3*	6.55	1.78	0.20	3.68
Year 2 vs. 3	0.58	1.93	0.02	0.30
Marks E vs. A*	-29.58	6.43	-0.21	-4.60
Marks D vs. A*	-21.89	2.22	-0.53	-9.86
Marks C vs. A*	-14.12	1.98	-0.40	-7.13
Marks B vs. A*	-9.13	2.04	-0.24	-4.47
Low vs. High SES	-3.01	2.00	-0.09	-1.50
Middle vs. High SES	-2.88	1.86	-0.08	-1.55
Low vs. High educated parents	-2.41	2.53	-0.05	-0.95
Middle vs. High educated parents	0.71	1.63	0.02	0.44

Note: *p<.05. *In addition the second item in the vs. comparison indicates the baseline category for creating the dummy variables.*

DISCUSSION

In this article, we explored the value that Greek Cypriot students (12 to 15 years old give to mathematical literacy. Utilizing Luttrell's et al. (2010) Mathematics Valu Inventory (MVI), we examined students' valuing of mathematics with the fou dimensions: Interest, General Utility, Need for High Achievement, and Personal Cos We also offer a translated MVI to researchers who want to investigate the value o mathematics in Greek speaking samples. The Greek MVI shows four factors with hig internal consistency. Further, we identified mean level differences in the value student attributed to mathematics with regard to gender, grade, marks in mathematics, SES an parent education. In particular, we identified the groups of students who are at "risk" o low mathematical literacy. In the following, we discuss these results and we propos further research, as well as policy implications.

Our results on gender differences confirm some and contradict other results in the fiel of gender differences in mathematical literacy. We found no gender differences for th overall MVI score, which is consistent with claims that mathematics is becoming mor gender neutral (e.g., Jacobs et al., 2002; Luttrell et al., 2010; Watt et al., 2006). Th interesting finding from our Cyprus sample is that girls value the General Utility of mathematics higher than boys. Also boys are more likely than girls to devalue mathematics considering the personal cost of it in their lives. These findings contradict gender differences found in Western samples that showed boys to exhibit higher interest in mathematics (Fredricks & Eccles, 2002; Updegraff et al., 1996; Watt et al., 2006), to have higher mathematical self-concept (Eccles et al., 1993; Jacobs et al., 2002; Meece, Wigfield & Eccles, 1990; Updegraff et al., 1996), to have higher expectation for success in mathematics (Eccles & Wigfield, 2002) and to have higher value for mathematics (Chow, Eccles & Salmela-Aro, 2012) than girls. These results from Western countries are normally used to explain why girls and women get out of the track of mathematical studies and scientific high-status careers (Watt et al., 2006). In our Cyprus sample, girls recognize the General Utility that mathematics has in their lives and in their future career from early adolescence. Our finding, of girls higher valuing of mathematics, provides a good reason why girls in Cyprus do better than boys in the TIMSS test (Eurycide, 2010). The finding that boys devalue mathematics, while girls understand their usefulness, can also be interpreted within some important results from a study in Greek secondary education (Psalti et al., 2007). This study showed that boys interpret secondary school as a useless period, which works apart from society's employment demands; on the other hand, girls think that secondary education enables them to obtain the ticket for their occupational independence (Psalti et al., 2007).

As Cyprus Gymnasium students proceed from the first to the second year of secondary schooling, they value mathematics lower. This finding is in accordance to Western results showing that as students get older, they tend to devaluate mathematics (Chow, Eccles & Salmela-Aro, 2012; Eccles et al., 1983; Eccles et al., 1993; Hyde et al., 1990; Jacobs et al., 2002). The crucial point in our findings is that we identified the exact timing for that, which is the transition from the first to the second year of Gymnasium. This lower value for mathematics remains the same in the third year of Gymnasium. Moreover, we also identified that this decrease in overall valuing mathematics is due to the decrease in interest for mathematics and in the decrease of valuing mathematical achievement. It seems that second year Gymnasium students are less interested in mathematics and also have lower Need for High Achievement in mathematics than year 1 students. This is an alarming finding for Greek Gymnasiums in Cyprus, which seems not to help students to maintain Interest and Need for High Achievement for mathematics. However, a further investigation is needed in order to define the reasons that make second year Gymnasium students less motivated for mathematics. Paraphrasing Watt et al. (2006) "it is necessary to identify the multiple points at which *females (in our case both boys and girls)* opt out of maths pipeline, and to understand the reasons for their decisions to discontinue *(in our case, their interest to)* maths... can restrict or exclude (them) from certain kinds of university degrees, or other forms of education and training, which in turn lead to many high-status high-income careers (Watt et al., 2006, p. 643)."

Concerning our third hypothesis, results showed that students with higher marks in mathematics value higher mathematics in comparison to students with lower marks in mathematics. As expected students with high performance in mathematics, valu mathematics higher. As Jacobs et al. (2005) points out students' values and expectancie are dropping faster than their marks, which results in their unwillingness to pursu careers in mathematics. The finding that students with low marks in mathematics als value mathematics lower put them in higher risk groups. Linver and Davis-Kean (2005 showed in a longitudinal study that both, marks in mathematics and interest i mathematics, decline by years of schooling. However, the decline of marks differed b gender and ability groups. Girls in the high-ability group and with high appreciation fc mathematics had a less steep decline in marks. Therefore, the value that students give t mathematics can work as a protective factor against mark decline and against all th related consequences for their future career.

To measure socioeconomic status of the students we used two indicators: father an mother's occupation and education. Both indicators showed that the higher the parents occupation and education, the higher the value students attributed to mathematics. Ou findings are in accordance with other findings (e.g., Davis-Kean, 2005; Jacobs et. al 2005; Linver et al., 2005; Simpkins et al., 2006), which showed that parents' incom and education are related to higher interest in and engagement for mathematics Therefore, Cypriot students from lower SES families have more chances to be at "risk, which can lead them on the long run to lower-status jobs.

We also investigated which individual factor (gender, grades, marks, SES and PE) ca predict the overall MVI score. The results demonstrated the relative importance of th grade level and specifically the change between year 1 and year 3 students of th Gymnasium. In addition, the marks were even more important to predict the valuing o mathematics. All other factors, apart from the grade level and the performance i mathematics, were irrelevant to predict the overall MVI. These might be the mos important results for Cypriot policy makers and educational reformers. Both grades an marks are factors on which the Educational system can intervene. Specific reformatio on curriculum, on methods and on materials can help students to maintain both thei marks in mathematics and the value attributed to mathematics.

Limitations

Our study is an important contribution to the literature as it investigates the value o mathematics in the new context of Gymnasiums' students in Cyprus. Nevertheless, th study suffers from some limitations. Firstly, due to the cross-sectional design of th study we do not know the direction of influence (causality) between the variables in th study. For instance, we do not know if achieving better marks will result in a highe valuing of mathematics, or if a higher valuing of mathematics will results in bette marks. So, future studies, should use a longitudinal design to better explore the directio of effects. Secondly, the study is based on student's self-report data. Although futur studies can also focus on teachers' and parents' reports, many studies showed tha students' subjective experiences and opinions are very important to be considerec Thirdly, the construct validity of the Greek MVI is very good; expect some doubl loading of some items. Future research should focus on a representative sample of students from Gymnasiums to further validate the scale in this context.

Implications, Future Research

Despite the limitations, policy makers can get important information to improve the educational reform in Cyprus. Mathematics curriculum, methods of teaching and teaching materials, should take into account the values of Interest, General Utility, and Need for High Achievement of mathematics of students. The materials should be designed and offered in such a way that each component can be increased through the educational system. E.g. real world applied problems (increase General Utility and Interest); presenting the use of mathematics in different occupations (increase General Utility value); retain and support high expectations in students' mathematics performance (increase Need for High Achievement). Further, educators and parents should not just only devote time to develop skills and build up knowledge in mathematics, but they should also stress the values of mathematics (e.g. stress the usefulness, increase the interest, praise achievement). Moreover, further investigation should focus on the identification of the "high risk" groups of students. In the present study, high risk groups are boys with low marks in mathematics, students from low SES and also all students in the second year of Gymnasium are at risk for decreasing the interest in mathematics.

Future research is needed to investigate questions such as: (a) Does the high General Utility value that the Cypriot girls place on mathematics can predict their future-career? (b) Do boys maintain the high value that place on Personal Cost? (c) What are the reasons of the decline of the values of Interest and Need for High Achievement in the second year of Gymnasium? (d) Do any features from parents' socialization process explain girls' high General Utility value and boys' high rate of Personal Cost? Such investigations will be valuable both for Cyprus educational reformation attempts and assist Eccles et al. (1983) model researchers to delineate the role of cultural variation.

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Construction of Basic Geometrical Concepts over Spatial Information Obtained by Visual Perception

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ABSTRACT: *The epistemological question of how humans acquire the fundamental concepts of Euclidean geometry has recently drawn considerable attention in cognitive science. In this paper, we argue that fundamental Euclidean concepts such as point, line, straight line, are constructions that derive from the implementation of domain-general cognitive mechanisms over visually perceptually obtained spatial information. In particular, we suggest two such cognitive mechanisms: a) a cognitive mechanism named structural analysis mechanism that applies to perceived objects or perceptually-based abstract objects and is connected with the process of attention, and, b) a cognitive mechanism named transformation mechanism by means of which object-centered conceptual representations of perceived entities or of perceptually-based abstract entities are constructed. Our claim is that the transformation mechanism originates in the linguistic system.*

Key words: *Spatial representations, Geometrical representations, Euclidean concepts, Structural analysis of shapes.*

INTRODUCTION

Over the centuries, traditional philosophy has addressed the issue of how humans acquire the fundamental concepts of Euclidean geometry, i.e. *point, line, straight line, surface, plane*. However, modern cognitive science only recently has taken up research on the origins of the fundamental geometrical concepts. Looking at the problem from the viewpoint of cognitive science, we investigate the role of visual perception in the acquisition of the building–block concepts of Euclidean geometry, as well as the cognitive mechanisms that underlie the acquisition of these concepts. In this article we argue that the fundamental Euclidean concepts can be constructed by the implementation of two cognitive mechanisms that operate over perceptual representations, although we leave open the possibility that are more mechanisms than these two.

Because *perception* is an ambiguous notion by virtue of the lack of consensus on it relation with *cognition*, we will first sketch out the thesis we endorse with regard t perception. This is necessary for the clarification of how we consider a *perceptuc representation.* We will then focus on the nature of geometrical conceptua representations and on the ways geometrical representations transcend perceptua representations. The idea is to shed light on the kind of representations delivered durin perceptual processing and on the transformations that perceptual representations mus undergo for the construction of a conceptual representation. The third step is t introduce two cognitive mechanisms and show that they underlie the transition from th perceptual representations to the conceptual representations of points, lines, straigh lines, surfaces and planes. Finally, we briefly discuss the implications of our proposa for learning and instruction.

DICHOTOMY BETWEEN PERCEPTION AND COGNITION

Many findings coming from the fields of neuropsychology (Goodale & Milner, 1992 Milner & Goodale, 1995; Ungerleider & Mishkin, 1984), neurophysiology (Cavanagh 1988; Livingstone & Hubel, 1987; Marr, 1982; Zeki, 1993), attention-research (Drive et al., 2001; Duncan & Humphreys, 1992; Lamme, 2003, 2004; Rensink, 2000a, b Treisman, 1993; Vecera, 2000) and developmental psychology (Spelke, 1990; Spelke & Van de Walle, 1993) lay a proper ground for the development of the thesis tha perception and cognition are functionally distinct (Raftopoulos 2001a, 2001b, 2006 2007; Raftopoulos & Müller 2006). Accordingly, we endorse the thesis that *perceptio* is an organizing function sub-served by modality–specific, domain-specific an cognitively impenetrable mechanisms that together organize the piecemeal informatio processed in various sensori-motor brain areas so that an integrated representation of world scene is produced. To be precise, perception consists of two stages, namely earl vision and late vision. The former is cognitively impenetrable while the latter i cognitively penetrated and, thus, conceptually modulated. Henceforth, we refer to earl vision as perception. The mechanisms in question are modality–specific becaus disparate mechanisms sub-serve each of the senses. They are domain-specific becaus even mechanisms of the same modality are committed to different processes (e.g. i visual perception there are mechanisms that process color and other that process deptl or motion). Finally, they are cognitively impenetrable because a perceiver's knowledg cannot modify the informational content that perceptual mechanisms provide witl regard to the locations and the qualities of the entities that are found in a visual scene On account of the nature of the perceptual mechanisms, the content of a perceptua representation is determined by the received data and the correct functioning of th mechanisms in question, and not by the perceiver's knowledge. According to this view being an organizing function, perception is inherently synthetic.

Visual perception in particular, provides, as an output, a representation of individuate entities spatially organized mainly in a body-centered frame of reference. That mean that a perceptual representation provides information about the entities that occup some space in the visual field of a perceiver, about their shape and size from the viewpoint of the perceiver (also about their color and texture but these play no role in our research), and about their location with regard to the perceiver (e.g. if an entity is *in front of* him, or *on* his shoulder, or *next to* his left leg, or *very far from* him, etc). In other words, a perceptual representation is a spatial representation, where space is defined by the limitations of a perceiver's visual field. Furthermore, in perception (early vision) a perceiver is aware only of the phenomenal character of the world scene displayed in his visual field (i.e. how the entities in his visual field appear and where they are with regard to his body), and not of what kind the presented entities are. According to this view, to perceive visually is synonymous with being aware of how the space in sight is filled and arranged.

Cognition mainly serves the categorization and recognition of the perceived individuated entities, the exploration of the causal and rational relations instantiated by these entities, and accordingly, it provides the explanatory tools that are necessary for the comprehension of what is perceived. To achieve these, cognitive mechanisms construct object-centered representations, wherein "objects" are perceptually represented entities1. Cognitive representations, apart from the perceptually obtained information (shape, size, position, distance, orientation from the viewpoint of the perceiver), include information about the properties of the perceived entities per se (e.g. if an entity is transparent, movable etc), about how they are related to other perceived entities (e.g. if the appearance of one entity cause the appearance of another), and additionally, cognitive representations are filled in with stored information about the represented objects (e.g. what an object is, or what is the name of an object). Thus, cognitive representations are elaborate and their content depends on the perceiver's knowledge. The mechanisms underlying cognition are neither modality-specific nor domain-specific, but are amodal and domain-general mechanisms, i.e. they are implemented in various domains that are not connected with or governed by any particular sense-modality. According to this view, cognition is a synthetic as well as an analytic function.

Following this line of thought, to represent perceptually a straight sloping line drawn in a computer screen is to see a thing that extends from position p_1 to position p_2 of the screen, and which is inclined a little with regard to the perpendicular position of the perceiver's body, whereas to cognitively represent this line is to comprehend that the thing that is in front of the perceiver's eyes is an *oblique straight line*.

In spite of the fact that perceptual representations (the reader should recall that these are the representations constructed in early vision) and cognitive representations are inherently distinct, cognitive representations are constructed on the basis of perceptual representations, given the priority of perception in our contact with the world. What is

[1]Of course in a higher-order level "objects" are abstract entities not necessarily perceptually representable, but this is a different story.

then under consideration is how the transition from perceptual to cognitiv representations is achieved.

THE CASE OF GEOMETRICAL (EUCLIDEAN) REPRESENTATIONS

Visual perception plays a crucial role in Euclidean geometry owing to the fact that i objects are depictable. Although the objects of Euclidean geometry are regarded a ideal, they have pictorial simulations. Hence, a triangle, a line, or a cylinder, are object that can be represented by means of drawings, pictures, or material construction: Consequently, geometrical representations are not distinguished from the perceptua representations of geometrical objects. However, this is incorrect. Geometrica representations, either internal (when we think of, imagine, or make judgments abou geometrical objects), or external (drawings, graphs), are conceptual representations tha include and also convey information about the category of the objects involved, abou their properties and about the relations that hold either within or between geometrica objects. When, for example, one has the concept *triangle* and thinks of triangle (internal representation), they know that any triangle comprises three straight sides an three angles, that triangles take many forms (e.g. isosceles, right angled), or that tw triangles could be similar according to specific criteria. Also, any drawing figure of tw triangles (external representation) like the following,

Figure 1. Geometrical representations are semiotic/semantic representations.

Note: In the context of Euclidean geometry it is a common practice to draw figures and mark the parts of the figures that are equal. In this drawing the marked side and angle of the triangle on the left are equa to the corresponding parts of the triangle on the right. In that way the representations of the geometrical figures are not just drawings but contain information.

conveys information about the depicted geometrical objects, i.e. the two triangles. B drawing attention to the marked parts of the two triangles, each figure indirectl indicates that a triangle is also a configuration of units apart from an overall shape. I addition, the figure conveys information about the relation between the markedl specified parts of the two triangles - the marked side and angle of the one triangle ar equal to the marked side and angle of the other triangle, respectively. In other words geometrical representations show relations and organization of relations betwee representational units, and that make them semiotic/semantic representations accordin to Duval's definition ("a semiotic representation shows relations or, better, organizatio of relations between representational units") (Duval, 2000, p. 13).

By being conceptual and by showing organization of relations between representationa units, geometrical representations are analytic as well as synthetic. That contrast geometrical conceptual representations to perceptual representations, which are onl synthetic according to the definition of perception presented earlier. The latter indicates that when we perceive an object physical or geometrical (e.g. a drawn triangle) we perceive it only as a whole and not as a composition or a configuration of units. This is a crucial difference between geometrical and perceptual representations in spite of the fact that perception plays a significant role when the geometrical objects are presented by means of external representations such as drawings, pictures and various material constructions. The differences do not end here.

Although geometries concern the objects of a "space" and the relations that hold within and also between the objects of that "space", they are not spatial representations in the way perceptual representations are for a number of reasons. First, on account of the fact that perceptual representations serve navigation, they are strongly and unavoidably connected with reference systems (Landau,2002). Specifically, perceptual representations are framed in a reference system that has at its center the perceiver or parts of the body of a perceiver, i.e. they are body-centered representations. That allows a perceiver to see what is *in front of him*, *in his right side*, *under his feet*, or *next to his left hand*. But it is also possible the center of the reference system that frames perceptual representations to be a physical object, as when one perceives that his car is parked *far from his house*, or finds his way to a place from the positions of the stars. It is also possible having a mixed use of a body-centered frame of reference and an object-centered frame of reference, as when one perceives that something is *in front of him* and *on the top of a box*. On the contrary, geometrical representations are framed only in object-centered reference systems, wherein the objects are the geometrical "objects". This is so because a perceiver is not a part of the geometrical "space" and thus, although one may be able to perceive the included "objects" (e.g. *points*, *lines*, *triangles*, *cones* etc.) when he is presented with external representations of them, he never physically and bodily interacts with them. Therefore, when we consider geometrical "space" the only thing that matters are the relations that hold within and between the geometrical "objects".

Secondly, the strong connection of perceptual representations with reference systems, and in particular with the body-centered reference system, results in the spatial relations as encoded in perceptual representations to be strongly quantitative (e.g. "far from", "next to", "beside") and orientation-based (e.g. "above", "behind", "to the left" or "to the right"). On the contrary, the spatial relations involved in geometrical representations are substantially qualitative. When we consider, for example, the relations that hold between the angles that are formed from two parallel lines which are intersected by a third line, the distance between the parallel lines as well as the exact points of intersection are absolutely irrelevant. Or, when we compare two triangles to find out whether they are congruent or not, the relative position of the triangles, namely the distance between them and the orientation of one with regard to the other, are not involved in the criteria of congruence. The latter paradigm also indicates that apart from being qualitative, geometrical spatial relations are not orientation-based in the sense that in most cases it does not really matter if a geometrical object is "above", or "to the left" of another geometrical object.

Finally, because perceivers can move in physical space and most of the physical object are also movable (either inherently or due to physical forces), perceptual representation have a spatio-temporal dimension related to the fulfillment of the core function c perception, which is to provide a continuous flow of events that maintain the identity c objects. Thus, when we lay still and at time t_1 we perceive object A in place p_1, wherea at the moment t_2 we perceive the same object in place p_2, what we eventually perceive i object A moving from position p_1 to position p_2. On the contrary, geometrica representations are substantially deprived of the spatio-temporal dimension. Being a artificial space, geometrical space contains static objects and no internal perceiver Therefore, when two identical geometrical figures are presented in positions p_1 and p_2 c the geometrical space, the acquired conception is of two individuated figures located i different positions, and not of one object that has moved from position p_1 to position p_2.

To summarize, geometrical representations differ from perceptual representations i being analytic (i.e. they represent organization of relations between representationa units) and exclusively object-centered representations, in representing substantiall qualitative and not orientation-based spatial relations, and in lacking the spatio-tempora dimension of perceptual representations. The clear differences between perceptua representations and geometrical representations indicate that perception, and i particular visual perception, does not suffice for the acquisition of geometrical concepts although geometrical conceptual representations make use of perceptual representation (i.e. of what is seen in drawings and pictures), they pertain to a distinct and mor advanced representational level that lacks many of the perceptual characteristics. Th conclusion is, then, that a transition from perceptual representations to geometrica conceptual representations ought to be made and, according to what we have argued s far, this transition requires a cognitive process during which the perceptual spatia representations are transformed to object-centered spatial representations that encod qualitative and not orientation-based spatial relations, also deprived from the spatio temporal dimension. The question that arises is which cognitive mechanisms could sub serve such a cognitive process.

We suggest two such cognitive mechanisms, although we recognize that the assume cognitive processes may require more mechanisms. To frame the discussion, we chos the case of the fundamental geometrical objects i.e. *point*, *line*, *straight line*, *surface plane*.

THE CONSTRUCTION OF THE FUNDAMENTAL EUCLIDEAN CONCEPTS OVER PERCEPTUAL REPRESENTATIONS

In the *Elements*, Euclid offered the following definitions for the most basic geometrica concepts, those that constitute the building-blocks of the theoretical system2 know since then as *Euclidean geometry*:

[2]The primary definitions of *point*, *line*, *straight line*, *surface*, *plane* and *shape* are given in Book I of th *Elements*, whereas the primary definition of *solid* is given in Book XI of the *Elements*.

- *Point* is anything that has no parts (dimensions)
- *Line* is the thing that has length but no width
- *The boundaries3 of a line are points*
- *Straight line* is the line that lies equally to its points
- *Surface* is anything that has only length and width
- *The boundaries of a surface are lines*
- *Plane* is the surface that lies equally to its lines
- *Boundary* is anything that constitute an outer limit for some object
- *Shape* is what is delimited by one or more boundaries
- *Solid* is the thing that has length, width and height
- *The boundaries of a solid are surfaces*

Looking at these primary definitions, we can observe that a) they are strongly influenced by perceptual experience; this is evident in the definition of "shapes" as the things that are delimited by one or more boundaries (perceptually obtained information), as well as, in the conception of 3D shapes – solids- as the major shapes with regard to spatial dimensions (also perceptually obtained information), b) the notion of "boundaries" is at the core of the Euclidean definitions; the fundamental geometrical objects *point*, *line* and *surface*, are conceived of as "boundaries"; the boundaries of *solids* are the *surfaces*, *lines* are the boundaries of *surfaces*, and *points* are the boundaries of *lines*. The conception of points, lines and surfaces as boundaries has an impact on their dimensional status; being the boundaries of the 3-dimensional solids surfaces are 2-dimensional, lines are 1-dimensional being the boundaries of the 2-dimensional surfaces, and points are compulsorily zero-dimensional by being the boundaries of the 1-dimensional lines. "Straight lines" and "planes" are considered particular types of "lines" and "surfaces" respectively, namely, the uniform types of them. Taking into account these observations, a reasonable inference is that through the definitions of the fundamental geometrical objects Euclid develops a structural analysis of the shapes and their parts attempting to define the building-blocks of all shapes, at least the perceptible ones.

The structural analysis of shapes and their parts that emerge from the Euclidean definitions lays the ground for the introduction of the one of the suggested cognitive mechanisms, which we call *structural analysis mechanism*. The structural analysis mechanism is conceived as a cognitive mechanism by means of which a perceived physical object or a perceptually-based abstract object (e.g. a shape) is analyzed into its structural parts. Recall that perception is a synthetic function that provides representations of entities as wholes; hence, perceptual representations do not make

[3]The word used in Euclid's *Elements* is *πέρατα* (perata) which denotes *where something ends*.

distinctions among the constituent parts of a perceived object (e.g. do not separate th shape from the texture or the size), neither among the constituent parts of a shap regarded as a perceptually-based abstract object (e.g. do not separate its upper side fron its left side). On the contrary, cognitive mechanisms are analytic and, thus, able t provide elaborate representations of objects, either perceptible or abstract. Our claim i that when the structural analysis mechanism applies to perceptible shapes, the result i the decomposition of the shapes into their constituent structural parts. Among th constituent structural parts of a shape are those that define its outer limits, i.e. it boundaries. Thus, the implementation of the structural analysis mechanism o perceptible shapes brings on earth as constituent parts the outer limits of shapes. Whe the structural analysis mechanism is implemented in a 3D shape, it causes surfaces t come to the fore as the structural parts of the outer limits of the shape. When th structural analysis mechanism is used on these surfaces, or on any 2D shape, it elicit the constituent part of the outer limits, namely lines. And finally its application on line causes points to come out as the constituent parts of their outer limits. This way, a firs rough representation of surfaces, lines, and points as boundaries of shapes is produced The claim concerning the existence of a structural analysis mechanism gains suppor from research on attention. Attention is a selective process through which detaile representations of objects and of their parts can be built (Driver et al., 2001; Duncan & Humphreys, 1992; Lamme, 2003, 2004; Rensink, 2000a, b; Treisman, 1993; Vecera 2000). Our assumption is that the structural analysis mechanism is related to attention i the sense that its implementation requires the selective processing of attention (Griva & Raftopoulos, in preparation).

The first rough representations of surfaces, lines, and points as boundaries of shape yielded by the implementation of the structural analysis mechanism are not sufficien for the acquisition of the full-blown Euclidean concepts of "points", "lines", "straigh lines", "surfaces" and "planes". This is so because in Euclidean geometry "points" "lines", "straight lines", "surfaces" and "planes" are not only conceived as boundarie but also as individuated abstract objects. This means, on the one hand, that they ar regarded as distinct geometrical objects each characterized by its own properties (e.g "line" is anything that has only length), and, hence, as individuated objects. On th other hand, it means that they are regarded as objects that can stand, be thought of, b used and examined independently of their status as boundaries of perceptible shape (e.g. we can think of a sole line and suppose that it is extended infinitely). That is, the can be treated as abstract objects.

To consider the structural parts of shapes as distinct geometrical objects requires th construction of an object-centered representation for each, wherein the place of "object" will occupy the points, the lines, the straight lines, the surfaces, and the plane respectively. However, this is a process that cannot be carried out by the structura analysis mechanism. To clarify this point, let us suppose that one has structurall analyzed the contour of a given surface and has observed that the contour comprise parts that are spatially extended only in length, and thus they are not surfaces. T discover that surfaces consist of such parts is not identical to acquiring the concep "line". For the acquisition of the concept "line" requires detaching the structural parts of a surface from the surface itself and thinking of them not as parts of a surface but as things that can have a single separate and independent existence (i.e. as distinct entities, or better, as individuated abstract objects). This could be achieved by thinking about surfaces and their structural parts in general and not about the surface that is currently found in the visual field and is being perceived under the limitations that the perceiver's presence imposes (i.e. being perceived in a body-centered frame of reference confined by the here and now). To think about surfaces and their structural parts in general may also demand one asking themselves "how other surfaces not similar to this (the currently perceived one) are structured?", or, "do other surfaces have structural parts similar to those that this surface has?" and so on. It also requires finding out what the common characteristics of the structural parts of surfaces are (e.g. they are 1D shapes, namely, they are spatially extended only in length), and what forms they can take (e.g. straight, curved). In other words, the acquisition of the concept "line" requires conceiving the structural parts of surfaces as objects per se, and constructing a representation that refers to them separately.

How is it possible for one to think of a perceived or a perceptually-based object not as part of their perceptual experience (e.g. to think of a line not as a visible boundary of an object that is in sight) but as something that may have an existence outside the current perceptual experience (as a distinct entity, or, as an individuated abstract object, e.g. the "line")? To achieve this, one must think *about* the perceived or the perceptually-based object, and that means mainly the following:

a) to "clear out" the fact that the object is at the current time in space defined by his visual field and think of it as being "space-less" and "time-less",

b) to "clear out" the fact that the object has spatial properties by virtue of its relative position with regard to the perceiver and think of it without taking into account the spatial conditions imposed by the body-centered reference system.

In other words, to think of a perceived or a perceptually-based object as a distinct entity, one must override the spatio-temporal and body-centered reference systems that frame perceptual representations and anchor them to the perceiver's experience. Our claim is that the filtering out of the spatio-temporal and body-centered frames of reference that characterize perceptual representations is achieved by a cognitive mechanism that we call *transformation mechanism*. The transformation mechanism is conceived as a cognitive mechanism that participates in the construction of object-centered representations of perceived or of perceptually-based objects by filtering out the spatio-temporal and body-centered frames of reference imposed by the perceptual mechanisms. By means of the transformation mechanism the first rough representations of points, lines, and surfaces regarded as boundaries of perceptible shapes are transformed to the more elaborate object-centered conceptual representations of *points*, *lines*, and *surfaces*. Our assumption is that the suggested *transformation mechanism* originates in the linguistic system, given that language allows us to think of perceptual experience as well as to think about it taking distance from it (Griva & Raftopoulos, i preparation).

As cognitive mechanisms, both the *structural analysis mechanism* and th *transformation mechanism* are amodal and domain-general mechanisms. That mean first, that they do not apply only to visual perceptual representations; they may also b applied to perceptual representations related to other sense–modalities such as auditor olfactory or tactile representations (amodal mechanisms). For instance, the structura analysis mechanism may apply to a tactile representation of a physical object resultin in the decomposition of it or of its shape into its structural parts. It means, second, the do not only apply to shapes; they can be applied to any thing that has a structure such a a place, a melody or an event (domain-general mechanisms).

The claim that the process of acquiring the fundamental Euclidean concepts is constructive process, and not an abstractive or an inductive process which applies t perceptual representations, is supported by the fact that perception provide representations of whole shapes and not of their structural parts. To be more specific, i we could perceptually and separately represent the structural parts of the outer limits c shapes, then it would be possible to make generalizations on the basis of thes representations (e.g. the outer limits of this and that shape comprise parts that ar extended only in length so we may infer that all shapes comprise such parts, which wi be then named "lines") and thus acquire the Euclidean concepts in question by means c induction. But we cannot. Neither can we perceptually abstract the structural parts of th outer limits of shapes; only the whole shapes might be abstracted from perceptua representations. One could suggest that an abstractive process applied to perceptua representations of physical objects could be the source of the concepts *point*, *line*, an *surface* if one were able to "abstract" from the perceptual representation the shape of th physical object and then apply the structural analysis mechanism by means of which th whole shape is decomposed to its structural parts, "abstracting" thus, the parts from th whole. Such a suggestion is problematic because we argue that the implementation c the structural analysis mechanism is not enough for the acquisition of the fundamenta Euclidean concepts; the transformation mechanism must also be applied. Th conception of points, lines and surfaces as individuated abstract objects require intentional and systematic thinking *about* how a shape is composed, an act that is nc directly supported by perceptual representations. In fact, this act transcends what i given by perception. This conception also requires the utilization of the acquired entitie (e.g. lines as boundaries) for the formation of new constructs (e.g. the *line*) that ar cleared from the qualities that connect them with their perceptual origins (i.e. body framed and spatio-temporal specified frame of reference); therefore, it is a constructiv process.

To sum up our argument up to this point, the construction of geometrical conceptua representations of *points*, *lines*, *straight lines*, *surfaces* and *planes* as they are construe in Euclidean geometry, requires the implementation of at least two cognitiv mechanisms: a) a cognitive mechanism that decomposes perceived objects o perceptually-based abstract objects into their structural parts (*structural analysi* *mechanism*), the implementation of which over perceptible shapes provides in particular the boundaries of these shapes, and b) a cognitive mechanism that transforms the first rough representations of points, lines and surfaces as boundaries of perceptible shapes provided by the structural analysis mechanism to object-centered representations – wherein "objects" are the points, the lines and the surfaces - by filtering out the spatio-temporal and body-centered frames of reference whereon the first rough representations were built (*transformation mechanism*).

IMPLICATIONS FOR LEARNING AND INSTRUCTION

The outlined theoretical model for the construction of the fundamental Euclidean concepts on the basis of perceptual representations has some implications for learning and instruction.

It is a common practice to introduce pupils to the geometrical objects of Euclidean geometry such as straight lines, triangles, or circles, with the use of pictures and drawings, and when it is possible with the use of solid material models (in cases of 3D objects such as cubes or cones). It is assumed that the pupils will understand the relevant geometrical concepts (e.g. the concept *straight line*) through the perceptual representations provided by the presentation of these concrete objects. This assumption can be problematic for two reasons.

First, although pictures, drawings and solid models can be useful for the understanding of the concepts *square*, *triangle* or *cone* they are misleading in the case of the concepts *point*, *line* and *straight line*. The pictures and drawings of points and of lines cannot represent the *points* and the *lines* as they really are according to their dimensional status. A *point* in Euclidean geometry is a zero-dimensional geometrical object whereas a *line* is a 1-dimensional geometrical object. Therefore, a *point* is a geometrical object that lacks extent, and hence it cannot be depicted, whereas a *line* is a geometrical object that extends only in length, and hence it cannot have even the slightest width. Unfortunately, even when we draw the most delicate line there is no way for it to lack a slight width. Drawings and pictures of points and of lines can then become a source of misunderstanding because the pupils may form the notions of *points* and *lines* based on what these geometrical objects look like in their depictions. A way to overcome such potential misunderstandings is to assign to the pupils tasks like the following: after presenting to the pupils a concrete physical object, e.g. a box, we ask them to focus their attention on the outer limits (i.e. the sides) of the box and then to describe the parts that comprise these outer limits. We then ask them to focus their attention on one of the sides of the box and to describe again the parts that constitute its boundaries. Finally, we ask them to focus their attention on one of the lines that constitutes the boundaries of the side of the box and to describe again its boundaries. In other words, in order to overcome possible misrepresentations and misunderstandings of the concepts *point* and *line* we should instruct the pupils to put into practice the *structural analysis mechanism* so as to decompose the shape of a concrete object into its constituent structural parts. We assume that in this way they will form an idea of the points and the lines as boundaries of perceptible shapes - the lines as boundaries of perceptible surfaces tha are part of concrete objects (e.g. the surfaces of a box), and the points as boundaries o the thus deriving lines – that will facilitate the pupils realization of the dimensiona status of the Euclidean points and lines. We should then repeat the same process wit some other concrete object (e.g. a desk, a chair or the blackboard) and encourage th pupils to do it with their own selections of objects. During the repetition of the proces we should encourage the pupils to think about the similarities and differences betwee the various structural parts that are elicited by asking relevant questions. In this way w assume that the pupils will start to think of the elicited structural parts as bein individuated objects and will engage in the construction of object-centere representations wherein the place of objects will occupy the boundaries of shapes. Th aim is to invoke the implementation of the transformation mechanism that is necessar for the construction of the geometrical concepts *point* and *line*. Our proposal of cours requires empirical verification in the context of a well-designed experimental task.

The assumption that pictures, drawings and solid material models of geometrica objects, and thus representations of geometrical objects that are supported b perception, suffice for the understanding of the geometrical objects in question i problematic for another reason. As mentioned, visual perception provides overal representations of objects whereas a great number of concepts and processes i Euclidean geometry concern the structural parts of the geometrical objects. A grea number of definitions, theorems, and propositions of the theory refer to the relation between structural parts of the geometrical objects and concern, for example, "th distance of the *middle point* of a chord from the *center of* the circle", "the relation tha hold between *the opposite sides* of a parallelogram", or "the relation between *the side and their opposite angles* in a triangle". Therefore, in order for the pupils t comprehend the Euclidean definitions, theorems, and propositions it is necessary fo them to realize that the geometrical objects are structures consisting of parts that stan in certain relations. Our proposal is that this can be achieved by the implementation o the *structural analysis mechanism* over the geometrical objects that are perceptuall presented through pictures, drawings and solid material models. Within a proper contex the pupils should be instructed to find out which parts comprise a presented shape, wha are the qualities of these parts and what are the relations in which they stand with th other parts. In other words, the pupils should be instructed to put into practice th structural analysis mechanism in order to uncover the structure of a geometrical figure However, this should only be the first step. It is well-known to all teachers o mathematics that most of the time a great number of pupils think of the geometrica figures in a narrow way that is affected by the manner of presentation of the figures i which pupils are used to (e.g. pupils regard a triangle as isosceles provided that it doe not lie on one of its equal sides, or a quadrilateral as a square provided that it lies on on of its sides else is regarded as a rhombus). That means that the pupils largel misrepresent the geometrical figures and consider relevant perceptual elements that ar in fact irrelevant to the geometrical concepts. Therefore, the pupils should be instructe to "clear out" the constructed representations of the geometrical figures from th perceptual elements that are irrelevant to the geometrical concepts that ought to b learned. The pupils should thus be instructed to implement the transformation mechanism in order to build an object-centered representation of a geometrical object/figure that will include only the qualities that define the object and not one of the elements that simply and solely characterize a current perceptual experience.

The latest trend in modern cognitive science is to consider Euclidean geometry a mathematical theory fully characterized by the set of transformations that leave the properties of figures invariant (Klein's definition of geometry) and to examine whether the Euclidean transformations are met in spatial cognition (e.g. navigation) (Landau, 2002; Spelke, Lee, Izard, 2010). The aim is to find out whether Euclidean transformations are inherent in the human mind (nativism), or are learned by association processes (empiricism). The assumption that the Euclidean transformations are related to the mechanisms underlying spatial cognition has an impact on the teaching of geometry. It is sometimes argued that the teaching of geometry must draw on the spatial capacities of pupils and, by means of suitable tasks that require the invocation of the spatial capacities (e.g. map-tasks, navigation tasks on a computer screen) the comprehension of the geometrical concepts will be facilitated. Without disputing in principle this teaching approach, we doubt that the mere utilization of the spatial capacities is enough for the pupils to understand geometrical concepts such as "space" or "distance between a point and a line". The point we wish to make with the hypothesis of the *transformation mechanism* is that to think of an object *in* space is very different from thinking *about* space or about objects as if they were not part of the perceived space. To think *about* something requires having represented it as a distinct entity that has a single separate and independent existence from the perceiver, and thus has qualities and stands in relations that are not determined by the perceiver. For example, to think about space in the context of Euclidean geometry is to think about an artificial space infinitely extended that has nothing to do with what a perceiver experiences as "space", namely, the finite area where he lives and moves. On the contrary, to think of the things *in* space requires the invocation and simulation of the state of affairs that hold when a perceiver is in the physical space. Much of the spatial relations in the physical space are quantitative and orientation-based, and therefore irrelevant to the qualitative relations under consideration in the geometrical space.

In addition, spatial relations in the geometrical space are strictly defined, whereas in the physical space they are not. For example, to think of the distance between two lines in the geometrical space is to specify a point in one line and to draw from this point a segment that is perpendicular to the other line (the "distance between two lines" is strictly defined), whereas the metric distance between these two lines does not matter. On the contrary, to think of the distance from my computer to the opposite wall, or between two cars that are racing in a computer game, is to estimate how far or how close the two objects are (quantitative estimation of distance) without being necessary for the estimation to specify one point in the one object and draw a perpendicular segment to other (i.e. in these cases "distance" is not strictly defined). The conclusion is, then, that although the spatial capacities may help the pupils acquire a first rough understanding of the Euclidean concepts, the acquisition of the more elaborate geometrical representations that are *about* the Euclidean objects requires instruction o how to filter-out the relations that hold *in* space but are irrelevant to the geometric: space.

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Primary School Pupils' and Primary Teachers Students' Intuitions of Infinity

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ABSTRACT: *Infinity is considered a central concept of mathematics. The present study investigates young pupils' and pre-service teachers' perceptions of infinity. Intuitions of infinity of pupils in the 5th and 6th grade of primary school and the criteria they use to compare infinite sets are examined. Pupils' conceptions are compared to primary teachers students' views, who usually do not receive in-depth teaching on these concepts. The study focuses on the ability of students to apply the criterion of one to one correspondence in comparing sets before and after it is provided. A secondary aim is to explore the impact of different representations of infinity, by using verbal, algebraic and geometrical tasks. Finally, students' attitudes and beliefs towards mathematics and their relation to the intuitions of infinity are regarded.*

Key words:

INTRODUCTION

The term infinity in the English language derives from the Latin word infinitas, which means "unboundedness". Trying to understand and define infinity mathematicians, philosophers and scientists have expressed many different and contradictory theories. While the term infinity was used from ancient Greeks, many of its faces were hidden until late 18th century. The modern education systems do not deepen in the concept and refer to it in different subjects with different ways. It is not surprising to find that the concept of infinity is not clear in pupils' and students' minds.

Infinity has different meanings depended on the context it is used. As a word it is used even from young children who are impressed from it. The modern set theory and the meaning mathematicians accept for infinity could be part of school curriculum, although it contradicts the sense of our finite world.

The main objective of the present research is to examine the intuitions of infinity for the 5th and 6th grade pupils of primary school and the intuitive criteria they use to compare infinite sets. In parallel, the understanding of pre-service teachers of the same subject is investigated. In this document the word "pupils" refers to primary school pupils of 5th and 6th grade and "students" refers to pre-service teachers. The two groups are compared regarding their intuitions of infinity to reveal the differences and similaritie between different ages and educational level.

A secondary goal of the study is to determine the impact of different representations c infinity on pupils' intuitions. Moreover, different infinite sets are used to reveal th criteria pupils use to compare infinite sets and the impact of a brief description of one t one correspondence criterion. Finally, the impact of pupils' and students' attitudes an beliefs in their interpretations of infinity is examined.

THEORETICAL BACKGROUND

A Notion of Motive and Crisis

Many great mathematicians confronted infinity in different ways. The concept o infinity leads to inherent contradictions in the world of mathematics and is the reason o one of the most lasting crisis.

An important view of infinity appears in Zeno's of Elea (450 BCE) paradoxes. Zeno, i his effort to study motion, proposed the known paradoxes, which are an interestin example of the misconceptions of the use of infinity in real world. Aristotle (384-32 BCE) distinguished potential from actual infinity, accepting the first and rejecting th existence of actual infinity. For many years infinity was not formally defined, causin problems and controversies in the inner and outer world of mathematics.

John Wallis (1655) introduced the symbol ∞ for infinity, while late 17th century Isaa Newton and Gottfried Leibniz used infinity in different ways cautiously an innovatively at the same time in their formulation of infinitesimal calculus an infinitesimal quantities. When, the next century, Georg Cantor proposed his set theor and one to one correspondence, he confronted the scepticism of his contemporar scientists. For many years his theory confuted the intuitive ideas and was not accepted On the other hand, Abraham Robinson expressed the non-standard analysis acceptin the existence of infinitesimals, where all operations of arithmetic with infinity, eve division, are permitted.

The Contradictory Nature of Infinity

As Fischbein (2001) noted, infinity is opposed to our finite world and our senses. T understand the concept we have to overleap to another level of understanding an liberate ourselves from the physical world. Infinity does not exist in nature but on th other hand it is a meaningful mathematical concept.

Infinity has many different faces, which make even more difficult the understanding o its nature (Fischbein, 2001; Dubinsky, Weller, McDonald & Brown, 2005; Monaghan 2001; Tirosh, 1991; Kolar & Čadež, 2011). Potential and actual infinity, infinity as process and as an object, infinity large, infinitesimal and infinity many, discrete an continuous infinity, non-standard infinity of non-standard analysis.

The differentiation of potential and actual infinity is used in philosophy and mathematics. Potential or dynamic infinity is related to processes which are, at every moment, finite, but continue endlessly (Fischbein, 2001). Actual infinity is considered as more difficult to grasp. Thinking infinity as a totality contradicts our finite world. Most students are familiar and accept the idea of natural numbers going on forever, but cannot conceive the entire set of natural numbers.

The duality of mathematical concepts as processes and objects (Gray & Tall, 1994) can be mentioned in many concepts but is intense in infinity. While potential infinity implies a process, actual is considered as the consequent object.

The meaning of infinity depends on the context. Infinity large is reported (Kolar & Čadež, 2011) as easier to handle and students had the most difficulties in tasks of the type infinitely close. Discrete and continuous infinity and the differentiation of cardinality of infinity can confuse pupils.

Understanding Infinity

Some theories regarding the way people understand the concept of infinity have been proposed. According to Basic Metaphor of Infinity (BMI) (Lakoff & Nunez, 2000), people tend to add a metaphorical completion to an ongoing process without an end so that it is seen as having a result. There is a progression of the properties of finite to infinity. Ely (2011), in his research extends this theory claiming that the mental acts when learning about infinite processes are firstly to attend and project properties of finite cases to infinity, but afterwards to experience conflict and to resolve it by revising these projections.

APOS theory (Dubinsky et al., 2005) is based on the idea that an individual uses certain mechanisms, called interiorization and encapsulation, to build cognitive structures, actions, processes, objects and schemas. By interiorizing the steps of a process one understands potential infinity, and through encapsulation the mental object of actual infinity is obtained.

Finally, Tall (1980), based on the idea that counting and measuring concepts are consistent in the finite case but not in the infinite, proposes the extension of infinity of measuring instead of counting, as a theory which is closer to pupils' intuitions of infinity.

Intuitions of Infinity

Pupils' main intuition of infinity is like a never ending process, which is easier to grasp than the concept of infinity as an object (Fischbein, Tirosh & Hess, 1979). Our intellectual schemes are adapted to finite objects and our physical world does not trigger thoughts about infinity.

Many pupils think of infinity as a very large number (Monaghan, 1986). However if they are pressed to define infinity they will conclude to something as a number but really big and with different properties. The inexhaustible capacity of infinity leads t erroneous conclusions, such as that there is one kind of infinity (Fischbein, 2001).

Some young children have a structured representation about the infinite sets an connection between algebraic and geometrical thinking enhances their reasonin (Singer & Voica, 2007).

In other contexts infinity is thought as an ideal not existing concept (Sierpinska, 1987 Another main characteristic of pupils' intuitions of infinity is their instability. Th answers depend on the problem posed, the context, the representation and the languag (Fischbein et al., 1979). Students have different and contradictory intuitions of infinit without realizing it (Sierpinska, 1987).

Regarding the geometrical problems which are related to infinity it seems that the iconi representation does not help. People tend to think of a point as a spot. Although it i known that a spot has no dimensions the truth of this statement is not easily accepte (Fischbein, 2001). As a result the equivalence of a bounded and an unbounded set or c sets of different dimensions meets great reluctance of students (Tirosh, 1991).

According to Fischbein et al. (1979) intuitions of infinity remain unchangeable to ag and school education. Education may help students in standard problems but not reall affect their concept image of infinity.

When comparing infinite sets pupils use different criteria depending on the context an the way the problem is posed (Tirosh, 1991, Tsamir, 2001, Singer & Voica, 2007):

- The whole is greater than each of its parts. A subset of a set contains fewer element than the set.
- A bounded set contains fewer elements than an unbounded set.
- A linear set contains more elements than a two dimensional set.
- One to one correspondence.
- All infinity sets have the same number of elements.

The inconsistencies of the answers in questions about comparison of infinite sets are nc easily realized. The formal theory of one to one correspondence leads to equivalence o sets, which contradicts our finite intelligence and is not accepted.

METHODOLOGY

Participants

The sample of the research consists of 98 pupils, 61 of the 5th grade and 37 of the 6 grade, from three elementary schools in Rhodes. Moreover, 61 pre-service teachers students of Department of Education of University of Aegean, participated in th research, with the same questionnaire.

The sample was selected with the convenience sampling method. The pupils of 5^{th} grade are not acquainted with infinity and the definition of natural numbers. 6^{th} graders have met infinity in natural numbers and straight lines. Finally, the students have not received formal education about infinity or set theory but they have used it in various forms in lessons of mathematics, such as in algebra, calculus and geometry.

Procedure

The questionnaire, aimed at examining pupils' and students' intuitions about infinity, had mostly closed questions on infinity and comparison of infinite sets. Different types of representations, contexts and infinite sets were used. Natural numbers were used in many questions, by asking either the greater element of a sequence of numbers or shapes that correspond to natural numbers or the cardinality of sets equivalent to natural numbers.

After task 10 a short example of one to one correspondence criterion was presented in the questionnaire. For the pupils that had met this method only in exercises of their school books, it was further but not in depth explained. Students had been taught the method in their first semester. Pupils and students were encouraged to use this method to compare the sets of the following tasks.

The questionnaires were distributed to the pupils and students in usual classroom conditions. The participants were asked to solve the problems without any other clarification or help in 30 minutes.

Method of Analysis

The analysis of the collected data was performed using the computer statistical software S.P.S.S. (Statistical Package for Social Sciences) and Gras's Implicative Analysis with the computer software C.H.I.C. (Classification Hiérarchique, Implicative et Cohésitive). C.H.I.C. is a software for data analysis, originally designed by Regis Gras regarding the algorithms and the statistical model (Gras & Kunzt, 2008). Gras's implicative analysis enables the distribution and the classification of variables as well as the implicative identification among variables or variables' categories.

Variables of the Study

The tasks of the questionnaire correspond to the following variables. The coding of the variables refers to the subject and the number of task. A description of the type of task and the value which is categorised as 1 in C.H.I.C. are noted.

Table 1

Variables of the Questionnaire

Group	Variable	Description	C.H.I.C. Value 1
Categories	Uni	Pupil/student	Student
	Grade6	5th/6th grade	6th grade
Intuition of infinity (True/False questions)	InfInt3a	Infinity is a very large number.	Not checked
	InfInt3b	The numbers are infinite.	Checked
	InfInt3c	The grains of sand are infinite.	Not checked
	InfInt3d	A straight line has infinite points.	Checked
Natural numbers	VerNat1a	Verbal problem with symbolic representation. Infinity as a process. Greater natural number.	No
	VerNat1b	Comment.	
	VerNat2	Open, verbal. Infinity as an object. Cardinality of natural numbers.	Infinite
	InfGame4	Verbal description, game context. Infinity as a process. Greater natural number.	None of them
	Task 4. Maria and Akis are playing a game. They are saying numbers one after the other and the one who says the largest number wins. Maria begins. Who will win?		
	InfGame4b	(For students only) Verbal, game context.	
	GeoSqr5a	Iconic, geometrical. Finite.	Side 100
	GeoSqr5b	Iconical, geometrical. Sequence of squares with side 1, 2, 3, ... Infinity as a process. Greater natural number.	No
	GeoSqr5c	Comment.	
Actions with infinity	InfSum6a	Addition of 0, 5, 100	Infinity
	InfSum6b		
	InfSum6c		
	InfSum6d	Subtraction of 5	

Cardinality of sets	NoElem7a	Finite set.	3
	NoElem7b		10
	NoElem7c	Infinite set, numerable. Natural numbers.	Infinite
	NoElem7d	Infinite set, non-numerable. Points of line.	
Comparison of infinite	CAlgNat8	Numerable infinite sets. Vertical representation.	Equivalent sets
Sets	Task 8. The first row is the set of natural numbers, the second row is the set of squares of natural numbers (for each natural number we write the square). Which set has more elements? 1 2 3 4 5 6 ... 1 4 9 16 25 36 ...		
	CAlgNat11	Numerable infinite sets, after 1-1 criterion. Algebraic representation.	
	CVerSeg9	Non-numerable infinite sets. Verbal representation. Segment, line.	
	CGeoCub10	Non-numerable infinite sets. Geometrical representation with different dimensions. Segment, square, cube.	
	CGeoCir12	Non-numerable infinite sets, after 1-1 criterion. Geometrical representation. Circles.	

To avoid tasks that would cause conflict or misunderstandings, many of the tasks used, were selected from the research bibliography and could be characterized as typical questions about infinity. Squares of different sides, as in task 5, have been used in different ways as a geometrical representation of natural numbers (Tirosh & Tsamir, 1996, Tsamir, 2001). Fischbein et al. (1979) used a segment, a square and a cube to compare the numbers of points in shapes of different dimensions. Comparison of sets of natural numbers and the squares or the double of them has been used from many researchers (Tsamir, 2001; Singer & Voica, 2007; Monaghan, 1986). Actions with infinity were used in informal way to examine the intuition of whether infinity could b used in actions.

The questions about beliefs and attitudes towards mathematics correspond to th variables of Table 2.

Table 2

Variables about Beliefs and Attitudes

Variable	**Description**	**C.H.I.C. value 1**
PerfMath13	Performance	
LikeMath14	Like mathematics	Agree /strongly agree
AttImp15a	Important	
AttIntr15b	Interesting	
AttUse15c	Useful	Disagree/str. disagree
AttBor15d	Boring	
AttDiff15e	Difficult	Agree/str. agree
AttNec15f	Necessary	

RESULTS

Students' and Pupils' Performance on Problem Tasks

The highest percentage of correct answers for pupils is observed in questions 1 and 3b which are standard questions related to the greatest natural number and the cardinalit of numbers in general. For students the highest percentage appears in questions 3d, and 3b, which are related to natural numbers and points of a straight line. The lowes percentage for both pupils and students is observed in questions that correspond to tw conflicts, the minus action in task 6d and the number of points of multiple dimensiona shapes in task 10, where the number of points of shapes of different dimensions wa compared.

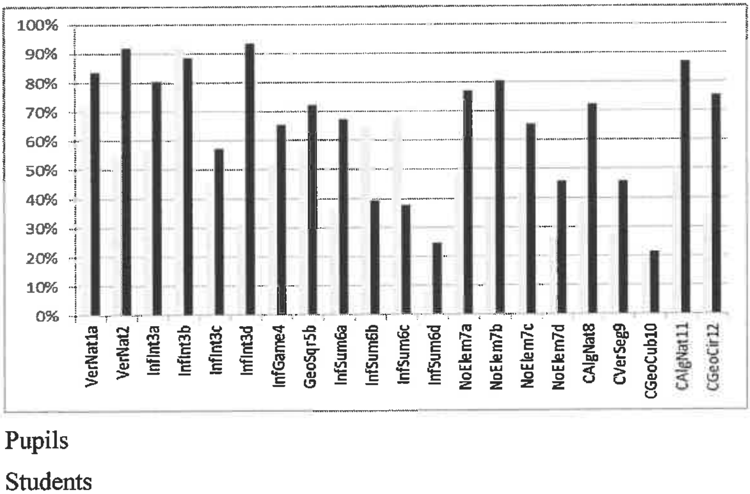

Figure 1. Students' and pupils' performance in the questionnaire.

According to Figure 1 and the comments in question 1, most of the pupils and students seem to believe that natural numbers go on forever or never end and understand infinity as a process.

Similarities and Implications between the Tasks

Statistical implicative analysis is performed, to examine the relationships between the answers. The similarity diagrams (Fig. 2 and 3) show how tasks are grouped according to the similarity of the answers.

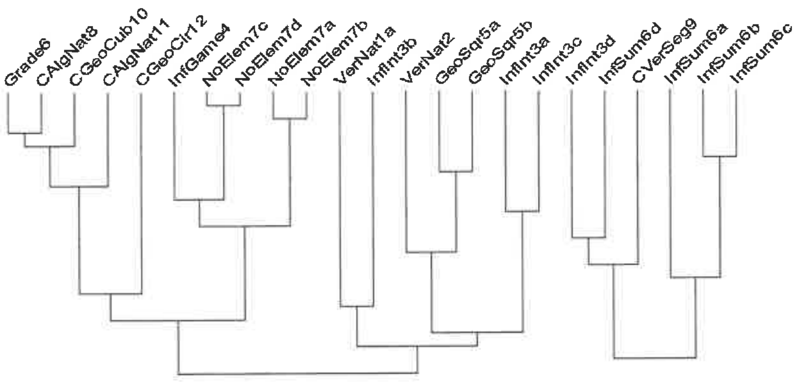

Figure 2. Similarity diagram of pupils' responses.

In the similarity diagram of pupils' responses (Fig. 2) two groups are distinguished. Th first consists of tasks 6, 9 and 3d and implies a group of pupils that are infinitists, accep actions with infinity and infinity of points of lines. The second group is divided into tw subgroups. Pupils answered in a quite similar way tasks 8, 10, 11, 12 which are relate to comparison of infinite sets and task 7 and 4. Another noticeable similarit relationship exists between the variables 2 and 5, which may handle infinity in differen ways, but are both open questions.

According to the similarity diagram (Fig. 3) of students, four groups of tasks appear The first group consists of the variables 1 and 3d, which refer to tasks set as taught, in typical way, that students can recall. The second group is divided into two subgroups The variables 2, 3c, 4, 5 are included in the first similarity group and concern th cardinality of natural numbers with different representations. The variables of th second subgroup are 3b, 8, 6a, 7c, 7d.

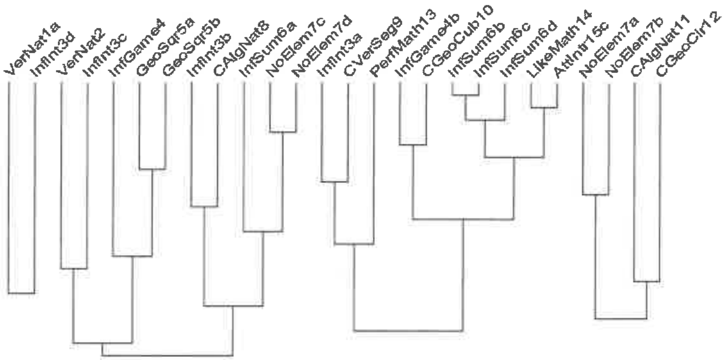

Figure 3. Similarity diagram of students' responses.

The third group consists of 3a, the more complex tasks, 9, 4b, 10 and 6, and a slightl more positive attitude towards mathematics. In the last group the variables 7a, 7b, 11 12 appear. It should be noticed that the grouping of tasks for the students is related t the complexity of the tasks and not the subject. The tasks related to the cardinality of th set of natural numbers are grouped together but not the tasks of sets' comparison.

The Implicative Graph (Fig. 4) carried out with the software CHIC, shows, with statisti percentages of 99%, the implications between the tasks of pupils' responses.

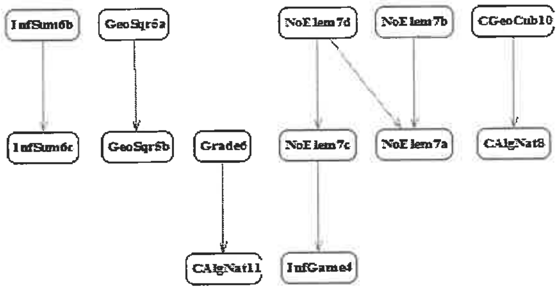

Figure 4. Implicative graph of pupils' responses.

From Figure 4 can be concluded that pupils who solved correctly the task about the comparison of points of segment, square and cube could also solve a task of comparing denumerable sets. The implications of 6b and 6c, as also of the subtasks of 7 and 5 are related to the degree of difficulty of the tasks.

The implicative graph of attitudes and beliefs is completely independent from the other variables. The implicative graph which results when isolating the responses of students implies only weak relations between the variables.

Comparisons of Pupils' and Students' Responses

Generally students gave more mathematically correct answers in the questions regarding the cardinality of natural numbers, but not as many to distinguish them from 6th graders. Significant differences between pupils' and students' responses are noticed in questions 8, 11, 5, 7b, 7a, 3d, 2, 12, but they are mainly due to 5th graders' answers (Table 3). In most questions 6th graders' percentage of correct answers is higher than 5th graders' and near or even higher than the corresponding percentage of the students. In task 3d there is significant difference between 6th graders and students. This seems to be the result of education as students have been extensively taught that a straight line has infinite number of points. Significant difference of 6th graders' and students' answers is also noticed in task 12.

Tasks 1, 2, 3b, 4, 5, 7c of the questionnaire handle the same subject, related to the cardinality of natural numbers with different representations or context.

Table 3

Percentages of Correct Responses of Pupils and Students with Significant Difference between Groups

Task	5th graders	6th graders	Students
2 (Verbal)	47,5%	64,9%	90,2%
4 (Verbal, game)	42,6%	67,6%	65,6%
5 (Geometrical)	47,5%	73%	72,1%
7a	37,7%	62,2%	77%
7b	31,1%	54,1%	80,3%
3d	37,7%	21,6%	93,4%
7d	18%	40,5%	45,9%
8 (Numerable, before 1-1)	19,7%	70,3%	72,1%
11 (Numerable, after 1-1)	31,1%	81,1%	86,9%
10 (Non-numerable, before 1-1)	16,4%	35,1%	32,8%
12 (Non-Numerable, after 1-1)	36,1%	32,4%	75,4%

Nevertheless, there exists weak relation between the tasks. Pupils' and students intuitions seem to depend on the context of the tasks, while unusual tasks caus conflicts. Especially, students could easily answer the typical tasks of the cardinalit and the greatest of natural numbers, but had difficulties when the tasks were set i verbal way as a game, with geometrical shapes or when symbols were used. Thes differences were not so obvious for the pupils. From the answers in tasks 1, 4 and where the greatest natural number is questioned, it seems that task 5 with th geometrical representation was more difficult than 4 with verbal_representation and concept of game and task 1. Task 7c where symbolism was used was more difficult.

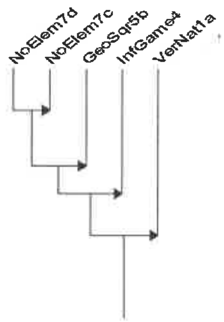

Figure 5. Part of the implicative tree of response.

The responses in tasks 3d, 7d, 9, 10, which are related to number of points in segments arse independent and there is not similarity between them. Specifically, while 31,6% o pupils and 93,4% of students believe that a straight line has infinite points, the corresponding percentage to the question related to the number of points of a segment are 26,5% and 45,9%. Consequently, the answers in tasks 9 and 10 were affected by this misconception. Students encountered difficulties in non-standard tasks (different context, different representations) as 9 and 10.

After one to one correspondence criterion pupils and students answered slightly differently in tasks related to sets' comparison. More specifically, 38,8% of the pupils and 72,1% of the students believe that the two sets of task 8 are equivalent, while after one to one correspondence criterion the percentages of correct answers in the similar task 11 was 50% of pupils and 86,9% of students. It is important to mention that the numerical-vertical representation of infinite sets, which appears in task 8, is reported to induce pupils to choose one to one correspondence criterion to compare sets (Tirosh & Tsamir, 1996). Moreover, task 11 is posed in set symbols, which as results from task 7 is not clear for a significant percentage of pupils and students.

Although tasks 9, 10, 12 refer to comparison of non-denumerable sets weak relation is observed between them. In task 9 either the part-whole criterion or the intuition of a bounded set as being smaller than an "unbounded" one, were used. Pupils were very confident for their answers in task 10 choosing the three dimensions shape as having more points. Students' answers were more uncertain but the intuition of a larger infinity in case of three dimensional shapes was intense.

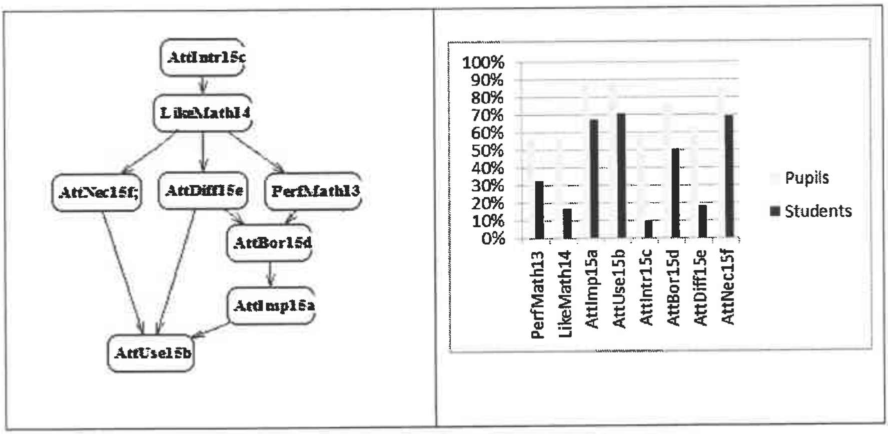

Figure 6. Beliefs and attitudes – Percentages of answers and implicative graph.

The last part of the questionnaire consisted of questions regarding pupils' and students' beliefs about mathematics. Pupils believe mathematics is important, useful and quite interesting. On the other hand students have a more negative attitude about mathematics and think that mathematics is important but also boring, difficult and not interesting.

Beliefs and attitudes do not influence intuitions of infinity. The implicative graph for th corresponding tasks is independent from other tasks of the questionnaire (Fig. 6).

DISCUSSION

Children even at the age of 11 accept that there is not a biggest natural number. Pupil seem to understand counting as a never ending process and by the same way infinity a a process with no end. This finding agrees to Monaghan's (1986) research who claime that when children talk about infinity their language repeatedly reflects infinity as process. Great differences are observed between the answers of 5th and 6th graders, wit the younger children to resist to the idea of infinity as a totality. In most questions th answers of 6th graders are alike those of the students and we can conclude that intuition of infinity are formed at the age of 12, they are related to other inner procedures o maturation and are quite stable onward. This finding is consistent with the findings o other studies which report quite stable intuitions of pupils after the age of 12 (Fischbei et al., 1979). Moreover, the teaching effect is limited to standard problems, but is no generalised to more complex questions.

A basic aim of the study is to examine whether different types of representations affec pupils' intuitions about infinity and the comparison of infinite sets. The results of th research show that success' percentages are greater when the tasks are posed in standard way similar to the teaching methods. Verbal and geometrical tasks are mor difficult for pupils and students.

An important finding is that the answers were affected by the context of the problem Pupils encounter difficulties with the concepts of sets and the non-numerable infinity o points of segments. Another issue raised from the questionnaire is the misconception related to the number of elements of sets, which was expected from pupils who had no been taught the symbolism of sets but not from students.

Students could more effectively handle problems related to number of points as the had been taught about infinite number of points in lines. Nevertheless, even fron students it is not accepted that a bounded set as a segment could be equivalent to a unbounded set or a shape of different dimensions. The iconic model is intense an overlaps the logic. As Tirosh (1991) noticed, more practice with non-numerable set could improve students' performance on these problems.

Regarding the comparison of sets most of the pupils, mainly of 6th grade, and student conclude that the compared infinite sets are equivalent but the reasons of their choic are not clear. The different criteria that pupils use when comparing infinite sets (Tirosh 1991; Tsamir, 2001; Singer & Voica, 2007) affect their answers. The idea of one kind o infinity or even the counting of the elements that appear in the problem may be som misconceptions which lead to correct answers to these problems. Consequently, th interpretation of this result is not obvious and should be further investigated. Th description of one to one correspondence was short but caused a slight change in pupil and students views, especially regarding the numerable sets, where the problems were similar and comparable.

Pupils' and students' beliefs and attitudes towards mathematics are totally independent of their intuitions of infinity. Students' more negative attitude may be a result of the specific sample, as no emphasis on mathematics is paid in order to enter a primary education department of university.

According to the findings of this research, teaching of infinity could be part of school curriculum in indirect ways and different contexts before 6th grade and with more direct reference afterwards. It may be helpful for pupils to know what infinity means for mathematicians and the ways it is used.

Research about infinity is prone to difficulties because of the abstract and contradictory subject, the different meanings and uses, the gap in language used by the researcher and the student. Moreover, our finite world does not offer examples to use in teaching or researching of infinity, there are no real referents for discourse on the infinity (Monaghan, 2001). In order to study the reasoning of pupils, interviews are required but the results could easily be affected by the researcher.

According to the result of the present study there are issues that need to be further researched. It would be useful to examine more thoroughly the difficulty of students in realising the number of points of lines and the number of elements of sets. Moreover, further investigation of the reasoning of their choices, regarding the number of elements of sets and sets comparison, would be useful to the teaching strategies.

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Sixth Grade Students' Visual Cognitive Styles and Three-Dimensional Geometrical Abilities

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ABSTRACT: *This article reports the outcomes of an empirical study undertaken to investigate the effect of students' visual cognitive styles (spatial and object imagery preferences and experiences) on their three-dimensional (3D) geometrical abilities. A group of sixth grade students (N=121) completed a Greek modified version of the Object-Spatial Imagery Questionnaire (OSIQ) (Blajenkova, Kozhevnikov, & Motes, 2006) and solved six tasks which addressed three types of reasoning of 3D geometrical thinking: measurement, spatial structuring and problem solving. The results suggest that whereas the spatial cognitive style is a statistically significant predictor of students' achievement in 3D geometrical tasks, the object cognitive style does not predict students' achievement in those tasks. In addition to this, the two types of visualizers differed statistically significant in their total mean performance in 3D geometrical test and exhibited different strategies in the solution of spatial structuring and problem solving tasks. It appears that the question is not simply whether individuals visualize or not, but whether they are spatial visualizers or object visualizers.*

Key words: *Spatial visualizer, Object visualizer, Volume measurement, Spatial structuring, Problem solving.*

INTRODUCTION

The role of visual thinking in the teaching and learning of mathematics has been examined by a number of researchers for over than thirty years (Lean & Clements, 1981; Presmeg, 1986; Eisenberg & Dreyfus, 1991; Pitta & Gray, 1999; Pitta-Pantazi & Christou, 2009a, 2009b; Pitta-Pantazi, Sophocleous, & Christou, 2013). Although substantial work has been done in this topic, the results often appear to be conflicting. A number of studies found that visualizers (those who prefer to process information visually) are amongst successful performers in mathematics (Tartre, 1990; Battista & Clements, 1998), while other studies showed the opposite (Lean & Clements, 1981; Presmeg, 1986; Eisenberg & Dreyfus, 1991). There may be several explanations for this apparent conflict in the results. One of the reasons may be that visualizers do nc constitute a homogeneous group and that they may process information differently.

Recent research studies by psychologists and neuroscientists confirmed that two distinc types of visualizers exist, spatial visualizers and object visualizers (Kozhevnikov Hegarty, & Mayer, 2002; Kozhevnikov, Kosslyn, & Shephard, 2005; Blajenkova Kozhevnikov & Motes, 2006; Blazhenkova & Kozhevnikov, 2009). According to thes studies, individuals that are characterised by spatial cognitive style prefer to represen and process schematic images and spatial relations. On the other hand, individuals wh possess object cognitive style prefer to construct and process detailed and colourfu images of objects. It appears that the question that we should be asking is not whethe individuals visualize or not but whether they are spatial visualizers or object visualizers

Despite the fact that, several researchers argued that cognitive styles may hav important implications for educational theory and practice (Dunn, Beaudry, & Klavas 1989; Sternberg & Grigorenko, 1997), the implication of the visual cognitive style i education and specifically in mathematics education is limited (e.g., Pitta-Pantazi & Christou, 2009b, 2010; Pitta-Pantazi et al. 2013). At the same time, NCTM (2000 underline that there is a need for teachers to know the strategies employed by student with different cognitive styles, so as to help them process, understand and appl mathematics in ways that make sense to them. For this, the present study tried t investigate the effect of visual cognitive styles on students' achievement in 3I geometrical tasks, and explore the ways in which students with different visua cognitive styles approached and solved 3D geometrical tasks.

The next section, offers an overview of research on 3D geometrical thinking, definition of cognitive styles and specifically of visual cognitive styles. Special attention is give in this section, on the relationship between cognitive styles and mathematical abilities In Section Three, we present the purpose and research questions of the study, a description of the participants, the procedure and setting of the study, the process i which the data was collected and the statistical analysis employed. The results ar presented in Section Four, while in Section Five we discuss the results of the study.

THEORETICAL BACKGROUND

Students' Three-Dimensional Geometrical Thinking

Three-dimensional geometry is an important part of teaching geometry in th mathematics curriculum. For example, NCTM (2000) claimed that students throug geometry lessons, they will learn about 3D geometrical solids and their characteristics will be able to perceive, build and manipulate 2D and 3D objects and they will calculat the volume and surface area of different solids. Similarly, a recent study by Pittalis and Christou (2010) suggested that 3D geometrical thinking could be described by fou distinct types of reasoning: representation of 3D objects, spatial structuring conceptualization of mathematical properties and measurement.

Many of the studies conducted until now on 3D geometry concentrated mainly on students' strategies, abilities and errors (e.g., Ben – Chaim, Lappan & Houang, 1985; Battista & Clements, 1996). Very few studies, as far as we are aware of, tried to approach these topics by considering students cognitive styles (e.g., Pitta-Pantazi & Christou, 2010). This is what we aspire to address in this article.

Visual Cognitive Styles

The construct of cognitive style has been widely researched in psychology (for a review, see Kozhevnikov, 2007). Cognitive style can be defined as an individual preferred and habitual approach to organising and representing information, which subsequently affects the way in which one perceives and responds to events and ideas (Riding & Rayner, 1998).

Cognitive neuroscience studies suggest that there is a distinction between visual and verbal processing. More recent research studies suggest that the visual areas of the brain are further divided into two functionally and anatomically independent systems: one concerned with the appearance of individual objects and the other with spatial relations among objects and components of objects (Anderson et al., 2008).

These results are consistent with research of individual differences. Kozhevnikov and her colleagues (2002; 2005; 2006) suggested that there are two types of visualizers who process visual information in different ways. Specifically, they suggested that there is one subgroup of visualizers that tend to use the spatial imagery system and a second subgroup who tend to use the object imagery system. Spatial visualizers mentally manipulate the "object location, movement, spatial relationships and transformations and other spatial attributes of processing", while object visualizers effectively manipulate the "visual appearance of objects and scenes in terms of their shape, color information and texture" (Blazhenkova & Kozhevnikov, 2009, p. 640). Spatial visualizers use imagery to represent spatial relations and process visual images analytically and sequentially, part-by-part, whereas object visualizers use imagery to construct images of objects and process visual information globally and holistically (Kozhevnikov et al., 2005).

Numerous studies attempted to relate these cognitive styles to specific abilities. Kozhevnikov et al. (2002) found that spatial imagery can be beneficial for physics, engineering tasks, technical drawing and mathematics. On the other hand, Rosenberg (1987) and Kozhevnikov et al. (2005) found that object imagery can be beneficial for visual arts.

The evidence from the field of neuroscience, cognitive psychology and mathematics education converge to the idea that any research study aiming to investigate the impact of cognitive styles should consider the double distinction between visual spatial and visual object cognitive styles.

Visual Cognitive Styles and Mathematical Abilities

In the field of mathematics education, the verbalizer/imager distinction has receive quite a lot of attention. However, it needs to be noted that mathematics educatio researchers often referred to preferred type/mode of thinking, or type of students and nc to "cognitive style" (e.g., Lean & Clements, 1981; Presmeg, 1986; Pitta & Gray, 1999) The broad idea documented by a number of researchers was that visual-spatial processe are distinct from verbal processes and that mathematics involves not only verba processes but also visual reasoning (Presmeg, 1986; Sfard, 1991).

Lean and Clements (1981) argued that research has not thrown much light on th question of whether individuals who prefer to use more visual imagery when processin mathematical information are likely to do better in certain mathematical tasks tha individuals who prefer a verbal-logical processing mode. On the one hand, a number o studies found that visual-spatial memory is an important factor which explains th mathematical performance of students and that spatial ability predicts success i mathematics (Tartre, 1990; Battista & Clements, 1998). On the other hand, there are number of studies which showed that students classified as visualizers do not tend to b among the most successful performers in mathematics (Lean & Clements, 1981 Presmeg, 1986; Eisenberg & Dreyfus, 1991). We suggest that this apparen inconsistency may be resolved in some extent by using the framework of two types o visualizers. Recently, Chrysostomou, Pitta-Pantazi, Tsingi, Cleanthous and Christo (2013) used this framework to examine the relation of the two types of visualizers, t achievement in number sense and algebraic reasoning tasks. The results of their stud indicated that spatial imagery is related to achievement in number sense and algebrai reasoning tasks. Similarly, Pitta-Pantazi et al. (2013) found that although visua cognitive styles (spatial and object imagery) predicted participants' creativity i mathematics, only spatial imagery was positively related to the characteristics o mathematical creativity: fluency, flexibility and originality. Therefore, this stud continues this line of research and examines whether this pattern of results (regardin the predictive power of spatial imagery on mathematical performance) also appear i 3D geometry.

THE STUDY

Purpose of the Study

The purpose of the study was twofold; first to investigate the effect of visual cognitiv styles on sixth grade students' 3D geometrical abilities and, second to examine the way that different types of visualizers approach and solve 3D geometrical tasks. Mor specifically, we addressed the following questions:

(a) Are spatial and object cognitive styles a predictor of their 3D geometrica abilities?

(b) In which type of 3D geometrical tasks are spatial and object visualizers more successful?

(c) What are the approaches and strategies employed by spatial visualizers and object visualizers in 3D geometrical tasks?

Participants, Procedure and Materials

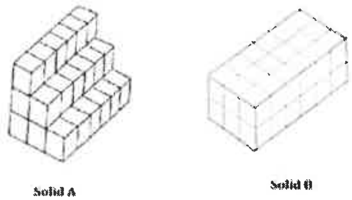
One hundred and twenty one sixth grade students participated in this study. This group consisted of 54 males and 67 females, ranging from 11 to 11.5 years of age. These students were studying in eight primary schools in Cyprus, both in rural and urban areas.

All students who participated in this study completed a Greek modified version of the Object-Spatial Imagery Questionnaire (OSIQ). After a week interval students were asked to complete a mathematical test on 3D geometry.

The Object-Spatial Imagery Questionnaire (OSIQ) is a self-report cognitive style questionnaire which is proved in previous works its validity and reliability (Blajenkova et al., 2006). In this study we translated the OSIQ into Greek and modified some of its items so that they would be accessible to 11-year-old students. Students were asked to read 30 statements and rate each item on a 5-point Likert scale with 1 indicating total disagreement and 5 total agreement. Ratings 2 to 4 indicated intermediate degrees of agreement/disagreement. Fifteen of the items measured spatial imagery preference and experiences (e.g., *My images are more like schematic representations of things and events rather than like detailed pictures*) and fifteen items measured object imagery preference and experiences (e.g., *My images are very colourful and bright*). These items addressed qualitative characteristics of the images, preferences for specific types of visual images, image maintenance and transformation processes, professional preferences, and individuals' estimations of their abilities in using spatial or object imagery. For each participant, the fifteen items ratings for each factor were averaged to create spatial imagery and object imagery scores.

The mathematics test on 3D geometry included six tasks addressing three types of reasoning of 3D geometrical thinking: measurement, spatial structuring (Pittalis & Christou, 2010) and problem solving. More specifically, there were two measurement tasks in which students were requested to calculate the area or the volume of given solids. For example, participants were asked to find which one of the solids has the maximum number of the cubes (see "Measurement task" in the Figure 1). There were two tasks labelled as spatial structuring. These spatial structuring tasks called upon students' ability to structure 3D arrays of cubes. In other words, to recognise the spatial components of an object, see the relationships between these components and construct it (Battista & Clements, 1996). For instance, in one of the items, students were asked to find the number of cubes needed to complete a rectangular box (see "Spatial structuring task" in the Figure 1). Finally, there were two problem solving tasks which required students to coordinate their knowledge in 3D geometry. For example, students were asked to put together in a three-dimensional box 24 cubes which had the sam dimensions and find the dimensions of this box (see "Problem solving task" in th Figure 1). A sample of the tasks used in this study is shown in Figure 1.

MEASUREMENT TASK
The figure below shows two different solids. Can you find which one of these solids has the maximum number of cubes? Explain your answer.



SPATIAL STRUCTURING TASK
Helen is playing with her cubes. She has put her cubes in the boxes shown in the figure below. Can you calculate the number of cubes which she needs in order to fill in completely the box?

PROBLEM SOLVING TASK
Mary has 24 cubes which have the same dimensions (The length of each side is equal with 10 cm) and she want to put them in a box. Can you find the dimensions of this box?

Figure 1. Sample of the tasks used in the 3D geometrical test.

Scoring

Students' fully correct responses in the measurement, spatial structuring and problen solving tasks were marked with 1 and the incorrect responses with 0. Students' partl correct responses were marked with 0.5.

Data Analysis

To answer the research questions of the study, both quantitative and qualitative techniques were used. It is believed, that a combination of quantitative and qualitative methods may provide a clearer picture of the data (Richardson, 2001).To examine students' visual cognitive styles and 3D geometrical abilities, descriptive analysis was used. Multiple regression analysis was also applied to examine the predictive power of spatial and object imagery scores on students' performance in 3D geometrical tasks. Then, we used multiple analysis of variance to identify the differences between spatial and object visualizers' performance in 3D geometrical tasks. Finally, to examine the ways that different types of visualizers approach and solve 3D geometrical tasks, we used coding and categorizing as described by Creswell (1998).

RESULTS

In this section, we present the results of this study, according to the research questions of this study. We first focused on providing students' spatial and object scores in OSIQ and the different groups of students that were created based on these scores. Then, we examined students' abilities in 3D geometrical tasks and investigated the predictive power of spatial and object cognitive styles on students' performance in 3D geometrical tasks. We also identified the 3D geometrical tasks in which spatial visualizers or object visualizers were more successful. Finally, we tried to explain the differences in these two groups' performance based on the various types of strategies that they employed for the solution of the 3D tasks.

We define performance, as the total score of students in the 3D geometrical test, which students completed for the purposes of this study. Along the same line, we define performance in measurement tasks, in spatial structuring tasks and in problem solving tasks as the total score of students in the tasks which were included in each one of these categories.

The internal consistency of scores measured by Cronbach's alpha was 0.72 for the 3D geometrical test, 0.72 for the spatial imagery dimension and 0.70 for the object imagery dimension. Reliability estimates of 0.80 or higher are typically regarded as moderate to high, while Cronbach's alpha of 0.70 is considered as a reasonable benchmark (Murphy & Davidshofer, 2001).

Students' Visual Cognitive Styles

First, we examined the spatial and object scores in the OSIQ of the sample of this study. Generally, the means of both scores were above the 2.5 ($\overline{X}_{spatial} = 3.16, SD = 0.60$, $\overline{X}_{object} = 3.73, SD = 0.51$, see Table 1). Although the investigation of gender differences was not one of the aims of the present study, we observed that males had higher spatial scores in the OSIQ than females ($\overline{X}_{males} = 3.31$ SD=0.58, $\overline{X}_{females} = 3.00$, SD=0.60). While females had higher object scores in th OSIQ than males ($\overline{X}_{females} = 3.81$, SD=0.54, $\overline{X}_{males} = 3.62$, SD=0.55).

Table 1
Descriptive Data for Students' Spatial and Object Scores in the OSIQ

	MEAN	SD	Minimum	Maximum
Spatial scores in the OSIQ	3.16	0.60	1.33	4.73
Object scores in the OSIQ	3.73	0.51	2.40	5.00

To identify the different types of visualizers, the sample of this study was clustered int four groups according to students' spatial and object scores in OSIQ (see Figure 2 More specifically, students who had above average spatial imagery scores in the OSI(($\overline{X}_{spatial}$>3.13) and had below average object imagery scores in the OSIQ ($\overline{X}_{object}$<3.66 were defined as spatial visualizers (see for example, Blajenkova et al., 2006 Kozhevnikov, Blazhenkova, & Becker, 2010). Twenty five students were identified a spatial visualizers (HsLo). While students who had below average spatial imager scores in the OSIQ ($\overline{X}_{spatial}$<3.07) and above average object imagery scores in the OSI(($\overline{X}_{object}$>3.71) were defined as object visualizers (see for example, Blajenkova et al. 2006; Kozhevnikov et al., 2010). There were thirty one object visualizers (LsHo). Th fact that there were more object visualizers than spatial visualizers is something tha was also found in previous research studies (e.g., Chabris et al., 2006). Additionally there were thirty five participants who had above average spatial and object imager scores in the OSIQ (HsHo) as well as twenty six students who scored below average o both spatial and imagery scales in the OSIQ (LsLo).

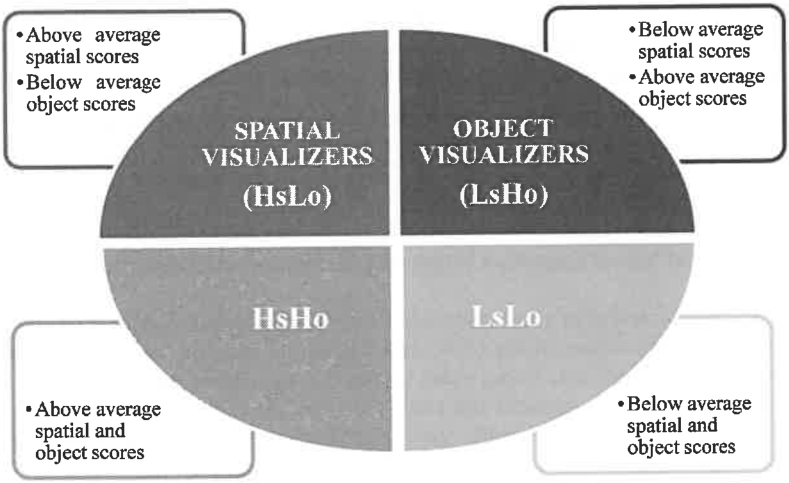


Figure 2. Diagrammatic presentation of the four groups according to students' spatial and object scores in OSIQ.

Students' Three-Dimensional Geometrical Abilities

Descriptive analysis was used to present the 3D geometrical abilities of sixth grade students that participated in this study. Table 2 provides the means and standard deviations of students' total performance in 3D geometrical test and their performance in each category of tasks. The means in Table 2 are all smaller or equal to one since students' correct answers were summed up and then divided by the total number of tasks included in each category.

Table 2
The Means and Standard Deviations of Students' Performance in 3D Geometrical Tasks

	$\overline{X}$	*SD*
Total	0.41	0.27
Measurement tasks	0.54	0.31
Spatial structuring tasks	0.33	0.31
Problem solving tasks	0.21	0.31

As can be seen in Table 2, the total mean performance in 3D geometrical test was below 0.5. However, it seems that students' abilities in 3D geometry are better in some tasks than others. More specifically, more than half of the students solved correctly the measurement tasks. This may be due to the fact that tasks similar to these often appear in their school textbooks. The mean performance of students in spatial structuring task is slightly below 0.5, while very few students solved correctly the problem solving task which required coordination of various 3D geometrical abilities

Furthermore, it is noteworthy that the standard deviations for the mean performance i spatial structuring tasks and problem solving tasks were slightly high. This indicate that the mean performance of students was not close to the mean, but a wide range of values specified it.

Relationship of Visual Cognitive Styles and 3D Geometrical Abilities

Multiple regression analysis was conducted with independent variables students' spatia and object imagery scores in the OSIQ and dependent variable their total performanc in the 3D geometrical test. Using enter method, a significant model emerged ($F_{2,114}$= 11.94, p<.00). Table 3 presents the results of the analysis. It appears that students spatial imagery can be a statistically significant predictor of students' total performanc in the 3D geometrical test (β=0.43, p=.00) and it can explain more than 17% of th variance in the total performance of the 3D geometrical test. However, object imager cannot be a statistically significant predictor of students' performance in the 3I geometrical test (β =-0.11, p=.23).

Table 3

Multiple Regression Analysis with Independent Variables Spatial and Object Cognitiv Style and Dependent Variables Students' Performance in the 3D Geometrical Test an the Subcategories of Measurement, Spatial Structuring and Problem Solving

	Total performance in the 3D geometry test		Performance in measurement tasks		Performance in spatial structuring tasks		Performance in problem solving tasks	
	B(SE)	*β*	*B(SE)*	*B*	*B(SE)*	*β*	*B(SE)*	*B*
Spatial cognitive style	0.19 (0.39)	0.43**	0.10 (0.05)	0.19*	0.324 (0.06)	0.48**	0.20 (0.05)	0.39**
Object cognitive style	-0.06 (0.05)	-0.11	-0.06 (0.06)	-0.09	-0.04 (0.07)	-0.05	-0.09 (0.06)	-0.15

R^2 = .173 for total performance in the 3D geometry test, R^2 = .034 for performance in measurement tasks
R^2 = .223 for performance in spatial structuring tasks, R^2 = .142 for performance in problem solving tasks
*p<.05, p<.01

The same pattern of results appeared when multiple regression analysis was conductec separately for students' performance on the subcategories of the tasks (measurement spatial structuring and problem solving) as dependent variable, and with students spatial and object scores in the OSIQ as independent variables. More specifically, usin enter method a significant model emerged for each group of tasks separately ($F_{2,114}$=2.01, p<.05 for measurement tasks; $F_{2,114}$= 16.36, p<.00 for spatial structuring tasks; $F_{2,114}$= 9.41, p<0.00 for problem solving tasks). From Table 3, it appears that spatial cognitive style can be a statistically significant predictor of students performance in the measurement tasks (β=0.19, p=.05), in the spatial structuring tasks (β=0.48, p=.00) and in the problem solving tasks (β=0.39, p=.00). However, object cognitive style cannot be a significant predictor of students' performance in each group of tasks separately (β=-0.09, p=.34 for measurement tasks; β=-0.05, p=.60 for spatial structuring tasks; β=-0.15, p=.10 for problem solving tasks).

To further examine the effect of spatial and object cognitive styles on students' performance in the 3D geometrical tasks, multivariate analysis of variance was conducted. Dependent variable was students' performance in the 3D geometrical test as well as in the subcategories of the study (measurement, spatial structuring and problem solving tasks) and independent variables was the classification of students in the H_sL_o, L_sH_o, H_sH_o and L_sL_o dimensions. Table 4 presents the results of the multivariate analysis by specifying the F and p values for each of 3D geometrical tasks and the mean performance of the four groups of students based on their visual cognitive styles in the 3D geometrical tasks. The means of students' performance shown in Table 4 are all smaller or equal to one since the correct answers of students were summed up and then divided by the total number of tasks involved in each case.

Table 4

Comparison of Performance of Four Groups of Students Based on their Spatial and Object Scores in the OSIQ in the 3D Geometrical Test and the Subcategories of Measurement, Spatial Structuring and Problem Solving Tasks

	H_sL_o (N=25) $\overline{X}$ *(SD)*	L_sH_o (N=31) $\overline{X}$ *(SD)*	H_sH_o (N=35) $\overline{X}$ *(SD)*	L_sL_o (N=26) $\overline{X}$ *(SD)*	*F*
Total performance in the 3D geometry test	0.51 (0.27)	0.32 (0.22)	0.49 (0.30)	0.31 (0.17)	4.96*
Performance in measurement tasks	0.62 (0.31)	0.48 (0.28)	0.54 (0.33)	0.46 (0.28)	1.53
Performance in spatial structuring tasks	0.44 (0.39)	0.19 (0.15)	0.49 (0.40)	0.15 (0.10)	5.74*
Performance in problem solving tasks	0.28 (0.24)	0.10 (0.08)	0.31 (0.26)	0.15 (0.11)	3.46*

*Indicates statistically significant differences at p<.05.

From Table 4, it can be deduced that there were significant differences amongst the fou groups of students: H_sL_o, L_sH_o, H_sH_o and L_sL_o in the total scores in the 3D geometrica test ($F_{3,113}$=4.96, p=0.00). In addition to this, the comparison of scores in th subcategories of the 3D geometrical tasks (measurement, spatial structuring an problem solving tasks) amongst the four groups of students (H_sL_o, L_sH_o, H_sH_o and L_sL_o revealed significant differences in the spatial structuring tasks ($F_{3,113}$=5.74, p=.00) an the problem solving tasks ($F_{3,113}$=3.46, p=.02). However, no statistically significar differences were found amongst the four groups of students in the measurement task ($F_{3,113}$=1.53, p=0.21).

Furthermore, according to Table 4, spatial visualizers' mean performance in the total 3I geometrical test ($\overline{X}_{HsLo}$=0.51) was higher than that of object visualizer ($\overline{X}_{LsHo}$=0.32), students with above average spatial and object imagery scores in th OSIQ ($\overline{X}_{HsHo}$=0.49) and students with below average spatial and object imagery score in the OSIQ ($\overline{X}_{LsLo}$=0.31). It is notable that the total performance of students who ha below average spatial and object imagery scores in the OSIQ (L_sL_o) was lower than tha of object visualizers. Additionally, the total performance of students who had bot above average spatial and object imagery scores in the OSIQ (H_sH_o) was lower than tha of spatial visualizers (H_sL_o). Almost the same pattern of results appeared for eac subcategory of tasks as shown by the mean performance of students in these tasks. Mor specifically, the mean performance of spatial visualizers in measurement ($\overline{X}_{HsLo}$=0.62) spatial structuring ($\overline{X}_{HsLo}$=0.44) and problem solving tasks ($\overline{X}_{HsLo}$=0.28) was highe than the performance in those tasks by object visualizers ($\overline{X}_{LsHo}$=0.48 in measurement $\overline{X}_{LsHo}$=0.19 in spatial structuring an $\overline{X}_{LsHo}$=0.10 in problem solving) and students who had below average spatial and objec scores in the OSIQ ($\overline{X}_{LsLo}$=0.46 in measurement; $\overline{X}_{LsLo}$=0.15 in spatial structuring an $\overline{X}_{LsLo}$=0.15 in problem solving). The mean performance of spatial visualizers (H_sL_o) i the spatial structruring and the problem solving tasks was lower than the performance o students who scored above average in the spatial and object imagery scales in the OSIQ ($\overline{X}_{HsHo}$=0.49 and $\overline{X}_{HsHo}$=0.31 respectively).

However, there was a need based on the above findings, to further investigate th differences between spatial visualizers (H_sL_o) and object visualizers (L_sH_o). Thus in th analyses which follows we concentrate only on the comparison of the two group o students H_sL_o and L_sH_o. For the sake of brevity in the remaining of the paper we wil call these two groups of students spatial visualizers (H_sL_o) and object visualizers (L_sH_o) For this purpose, a multivariate analysis of variance was conducted with students achievement in the 3D geometrical test and the subcategories of tasks as dependen variables and the classification of students in spatial visualizer and object visualizer a independent variables. The analysis revealed that spatial visualizers' (H_sL_o performance in total 3D geometrical test was statistically significant higher than objec visualizers' (L_sH_o) performance in the same test ($F_{1,54}$=7.92, p=.01). Furthermore whereas there were statistically significant differences between spatial (H_sL_o) and object visualizers (L_sH_o) in the total scores, in the spatial structuring tasks ($F_{1,54}$=6.05, p=.02) and in the problem solving tasks ($F_{1,54}$=5.90, p=.02), there were no statistically significant differences in measurement tasks ($F_{1,54}$=3.36, p=.07).

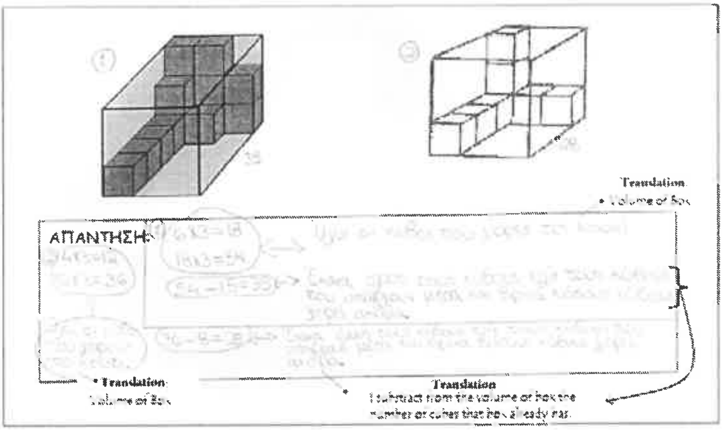


Figure 3. Sample of answer given by a spatial visualiser (HsLo) in the spatial structuring tasks.

These differences between spatial (H_sL_o) and object visualizers (L_sH_o) in the spatial structuring tasks and in the problem solving tasks can be explained by the different strategies these two types of students used in these tasks. More specifically, in the spatial structuring tasks most of spatial visualizers (H_sL_o) (62% of spatial visualizers (H_sL_o)) used an analytic and systemic way to solve these tasks. For example, we observed that spatial visualizers (H_sL_o) in order to find the number of missing cubes in the given rectangular boxes, they first calculated the volume of the boxes and then the number of cubes already inside the boxes (see Figure 3).

In contrast object visualizers (L_sH_o) did not use an analytic solution as the one used by spatial visualizers (H_sL_o). We found that most of the object visualizers (L_sH_o) (91% of object visualizers (L_sH_o)) tried to find the missing cubes by drawing on the sides of the box the missing cubes (see for example Figure 4). This approach inhibited most object visualizers (L_sH_o) from calculating the correct number of missing cubes since they forgot to calculate the numbers of cubes which were inside the box. This is in line with other researchers' findings, who suggest that object visualizers (L_sH_o) "tend to encode images globally as a single perceptual unit, which they process holistically" (Kozhevnikov et al., 2005, p. 723).

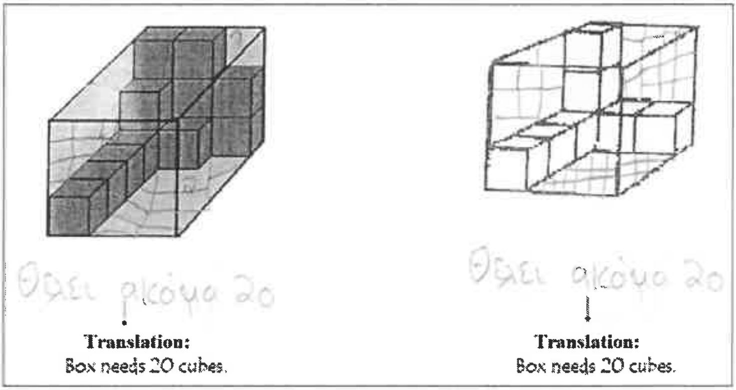
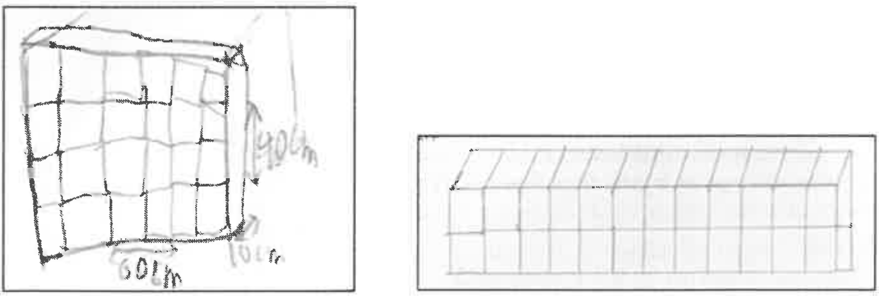


Figure 4. Sample of answer given by an object visualiser (LsHo) in the spatial structuring tasks.

In the problem solving tasks, half of the spatial visualizers (H_sL_o) use diagrammati methods. The other half based their answer on the arithmetical data of the tasks. Spatia visualizers (H_sL_o) who used a diagrammatic method, drew 24 cubes and put them i such way as to create a rectangular box and also wrote the dimensions of box (see fo example Figure 5a and Figure 5b). It appears that these spatial visualizers (H_sL_o thought of the structure of the box. Analysed its components part-by-part so as to find : way to put 24 cubes inside it and calculated its dimensions.



(a) **(b)**

Figure 5. Sample of answers given by two spatial visualisers (HsLo) in the problem solving tasks when approaching the problem diagrammatically.

Spatial visualizers (H_sL_o) who relied on the arithmetical data of the problem solvin tasks in order to provide their answers, used an analytical way to write down th possible solutions of the tasks and later on checked them (see for example, Figure 6).

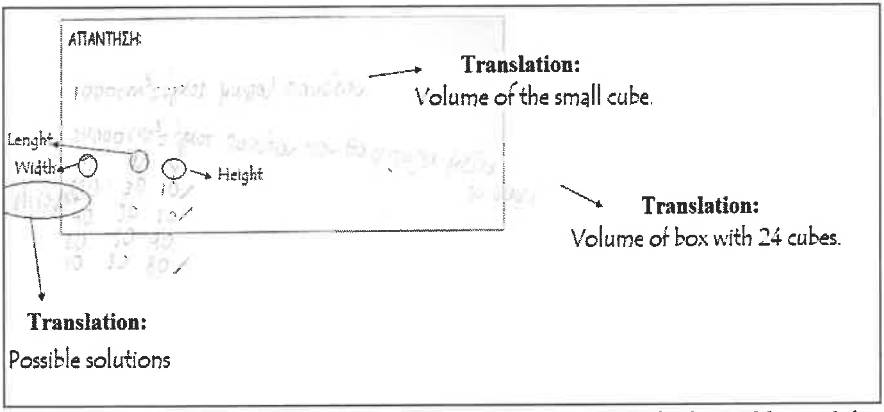


Figure 6. Sample of answer given by a spatial visualiser (HsLo) in the problem solving tasks when approaching the topic arithmetically.

On the other hand, all object visualizers (L_sH_o) who tried to solve the problem solving tasks took the numbers that they were given in the problem and either multiplied them (24x10=240 or 24x10x10x10=24000) or/and divided them (24x10=240÷3=80) without considering the meaning of these numbers.

DISCUSSION

The current paper examined the effect of students' spatial and object cognitive style on students' abilities in 3D geometry.

Overall, the results of the study indicated that spatial cognitive style can be a statistically significant predictor of students' performance in 3D geometry, while object cognitive style cannot be a significant predictor of students' performance. Furthermore, spatial visualizers (H_sL_o) performed statistically significant better than object visualizers (L_sH_o) in 3D geometrical test by employing more successful strategies. Spatial visualizers (H_sL_o) used analytical approaches, combined the components of tasks and drew the appropriate diagram, while object visualizers (L_sH_o) used the data of tasks holistically and could not see the relationship of the various components of the tasks. The above characteristics are in accord with the characteristics of spatial (H_sL_o) and object (L_sH_o) visualizers suggested by Kozhevnikov et al. (2005). These differences in their characteristics suggest that spatial visualizers (H_sL_o) are better problem solvers than object visualizers (L_sH_o).

The above results provide a possible explanation for the existence of conflicting results in regard to visualizers's abilities in mathematics. More specifically, it seems that visualizers who were successful performers in some studies (e.g., Tartre, 1990; Battista

& Clements, 1998) may have been spatial visualizers (H_sL_o) and those who did not ten to be among the most successful performers in mathematics (e.g., Lean & Clement: 1981; Presmeg, 1986; Eisenberg & Dreyfus, 1991) may have been object visualizer (L_sH_o). However, further research is needed to verify this claim in relation to othe mathematical concepts.

In summary, the results of this paper extend previous research results (Kozhevnkov e al., 2005; Blajenkova et al., 2006) by providing information about the way in whic spatial (H_sL_o) and object (L_sH_o) visualizers perform and process information in 3I geometrical tasks. This information may prove to be useful to mathematics teacher since they highlight differences in the abilities, strategies and approaches that spatia (H_sL_o) and object (L_sH_o) visualizers employ in 3D geometry.

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