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Profession of Teaching

I remember my high school years talking about my history teacher - that he knew a lot about history. However, it did not mean he could teach it efficiently. I was pretty sure that *knowing* did not readily result in *being able to teach* because teaching was an ability to be earned. Even though I was a high school student, I was recognizing and pointing out in my conversations that teaching required not only knowing the subject matter but also knowing how to teach it. It was Pedagogical Content Knowledge (PCK) by Lee Shulman that I referred to without knowing him or his model yet.

Think of a surgeon who learned every bit of information about his specialization in the books. Would you trust him to do your child's operation, or would you investigate his operation skills before you decide to lay your child under his/her scalpel? Nevertheless, it is your most loved one who is under risk. We can easily adapt this question for teachers: Would you prefer a teacher who is the best mathematician for your beloved child or would you still investigate how good s/he is at teaching mathematics? It is, now, your child's educational life you need to consider. I will leave the answers to you and proceed with my argument to present my point of view.

Teaching is a profession that requires serious and dedicated education on pedagogy and content and time to practice the knowledge gained. Teaching is also a multi-dimensional profession that requires specialization. Understanding and knowing how students think, learn, and react is one dimension while knowing how to teach, interact with, and guide learners is another important one. Furthermore, having the best subject matter knowledge is an indispensable part of teaching profession. Each piece is vitally important to being a qualified teacher.

Mastering content only does not necessarily make one a good teacher. A teacher should also be good at communicating that knowledge with learners. The teacher and learner relationship could be improved as the connection between subject matter knowledge and pedagogical knowledge gets stronger. To satisfy the current trends in education, technology skills could be added as another connection point to content knowledge and pedagogical knowledge.

The idea that *knowing the content well is enough to teach* is a desperate wish. That might connotate trying to fly with one wing missing. *Every rule has its exceptions.* However, if you are with your resources, then you are much likely to be successful at what you are doing. A teacher needs at least two strong supports: content knowledge and pedagogical knowledge. As Lee Shulman emphasizes, the pedagogical content knowledge is inevitable for a teacher.

My recent research showed that even though teachers know how to divide fractions, they were lacking the pedagogical knowledge on the topic. They showed no difficulty in solving problems; however, when they were asked to explain their reasoning on the algorithm, they either gave no answer to the specific parts of questions or provided conceptually very limited answers. Similar literature shows the importance of pedagogical content knowledge for teaching and how teaching could be more subtle when teachers blend subject matter knowledge (content knowledge) with how to teach specific knowledge (pedagogical knowledge).

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Investigating the Gap between Real World and School Word Problems: A Comparative Analysis of the Authenticity of Word Problems in the Old and the Current Mathematics Textbooks for the 5th Grade of Elementary School in Greece

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ABSTRACT: *Researchers have identified the tendency of students to think superficially and abstain from sense-making when confronted with arithmetic word problems (Verschaffel, Greer & De Corte, 2000). Research has found that the nature of word problems used in mathematics classrooms is important for developing students' realistic reasoning. Exposure to situationally rich problems (which resemble authentic quantitative problems people may encounter in real life) is considered essential for triggering students' sense-making in problem solving. Our analysis focused on the new 5th-grade elementary mathematics textbook published by the Pedagogic Institute of Greece following a major mathematics educational reform that took place in 2003. This reform meant to promote inter alia critical reasoning in problem solving (Pedagogic Institute, 2003). In particular, we examined whether the word problems in the new textbook are close to authentic out-of-school quantitative problems and how they compare with the ones contained in the old textbook. For our analysis we used a classification framework developed to measure the degree of simulation at which a number of aspects of reality are represented in word problems (Palm & Burman, 2004; Depaepe, De Corte, & Verschaffel, 2009). Our comparative analysis showed some changes in the representation of different aspects, as well as persistent shortcomings.*

Key words: *Word problems, Realistic mathematical modeling, Sense-making, Curricular reform, Level of authenticity.*

INTRODUCTION

The present article begins with a definition of word problems and a discussion about their use in traditional mathematics education, followed by a brief presentation of realistic mathematical modeling. Against this backdrop, attention is then given to the

2003 curricular reform in mathematics education in Greece, which resulted, *inter alia*, in a new mathematics textbook for the 5th grade of elementary school Our decision to focus our analysis on the 5th grade was based on the fact that problem solving with emphasis on realistic modeling takes a more central place in the mathematics curriculum for this grade (Kakadiaris, Belitsou, Stefanidis, & Chronopoulou, 2006b). In the next section, we proceed to an analysis of the word problems contained in this textbook, in an effort to measure whether their level of authenticity is increased in comparison with the old textbook, which was the stated aim of its authors. Finally, the article presents an analysis of the results followed by a discussion concerning the level of authenticity in the word problems of the two textbooks. Theoretical implications and suggestions for future research are also discussed.

BACKGROUND AND RESEARCH GOAL

Theoretical and Empirical Background

Verschaffel, Greer, and De Corte (2000) define an arithmetic word problem as a short text which describes a situation, giving some quantities and purposefully hiding others. The solver is required to identify the mathematical relationships between these quantities and give a numerical answer by manipulating them. For Staub and Reusser (1995), understanding and solving word problems is “a language and knowledge intensive undertaking and should be seen as a skillful interaction of text comprehension (linguistic knowledge), situation comprehension (world-knowledge), and mathematical comprehension (mathematical knowledge)” (p. 301). Students’ engagement with word problems allows them “to explicitly apply world knowledge, discourse and language knowledge, as well as arithmetic knowledge” (Staub & Reusser, 1995, p. 302).

Research into the way students solve word problems in mathematics classrooms has shown their distinct tendency to adopt superficial and non-realistic strategies. Freudenthal (1991) attributed this tendency to the way traditional mathematics textbooks formulate word problems. According to him, word problems in traditional textbooks often require a unique numerical response leaving little or no room for realistic considerations into the solution process.Further investigating this trend, Gravemeijer (1997) explains students’ persistence in adopting a nonsensical approach to solving word problems by their need to develop routine patterns of action attributed to a “strive for efficiency” (p. 390), which characterizes all human behavior. Other researchers link these routine problem-solving techniques to the effect that years of formal mathematics education – and in particular the way teachers conceive and handle word problems – can have on students (Verschaffel, Greer, & De Corte, 2007). For Verschaffel et al. (2007), the more students learn to play the word problem game, the less they use their sense-making.

This abstention of students from sense-making in problem solving was reported by Verschaffel, De Corte, and Lasure (1994). In their study, 75 fifth-graders were given a paper-and-pencil test with 20 paired word problems. Each pair contained a standard task, asking for a straightforward application of one or more arithmetic operations with given numbers, as well as a problematic task, in which the situation model could not be translated directly into a mathematical model, if one seriously considered the restrictions of the situation described in the task. Their findings were revealing, as most children solved the standard problems correctly, while only very few pupils tackled the problematic items successfully. For these researchers, the majority of children who solved the problematic items superficially go from "text to solution without passing through a thinking brain" (Verschaffel et al., 2000, p. 181). This study has been replicated in various other countries, yielding similar results (see for example Caldwell, 1995; Csíkos, 2003; Hidalgo, 1997; Renkl, 1999; Reusser & Stebler, 1997; Xin, 2009; Yoshida, Verschaffel, & De Corte, 1997).

Researching the connection between superficial solving tactics and the instructional design of word problems, Verschaffel et al. (2000) have identified a number of characteristics of standard word problems:

1. their solution requires no more than a small number of steps taking only a few minutes,
2. they are "grouped and formulated" in explicitly titled sections,
3. they lack superfluous data, leading to the assumption that the numbers required to solve them should be clearly stated,
4. they lack lively and interesting contextual information, leading students to view such information as immaterial,
5. they are solved through an exact computation,
6. they never allow students to qualify their answer depending on the situation described, and
7. they are formulated by others and thus an important aspect of real world problem solving, namely, the identification, definition and formulation of the problem by the solvers themselves, is ignored.

Verschaffel et al. (2000) discuss several recommendations regarding how word problems could be designed to counteract the harmful effects generated by standard word problems. First of all, students should be exposed to problems the presentational structure of which would dismantle their expectation that every word problem has a unique numerical solution. Second, hints pointing at the required solution, such as key words, should be minimized. Third, creating problems enriched with information from which students critically select what is absolutely essential for solving them could dissolve the assumption that all given data are relevant. Fourth, tasks should pose realistic questions and contain numerical data likely to be encountered in real life. Fifth, improved versions of standard word problems should give students the opportunity to critically reflect on a variety of answers. Sixth, it is important that word problems are designed in a way that makes different situation models seem plausible and directs students towards examining these models critically. Seventh, teachers should not protect students from errors but instead encourage them to discover mistakes and identify their cause.

Verschaffel et al. (2000) insist on the importance of using improved word problems to promote realistic mathematical modeling. In particular, they introduce the idea that word problems should be reconceptualized as exercises in mathematical modeling which is a cyclic multi-phased process. Taking into consideration a number of essential characteristics of the modeling process (e.g., usefulness of models, goals of users of models, resources available to model users, simplification of reality by models, possibility of several alternative competing models, iterative process of modeling), Verschaffel et al. (2000) propose a mathematical model (see Figure 1), where understanding and solving a word problem resemble more the way an authentic out-of-school world problem is solved. We emphasize that this model, which is described in more detail in Figure 1, does not usually follow a strictly linear pattern.

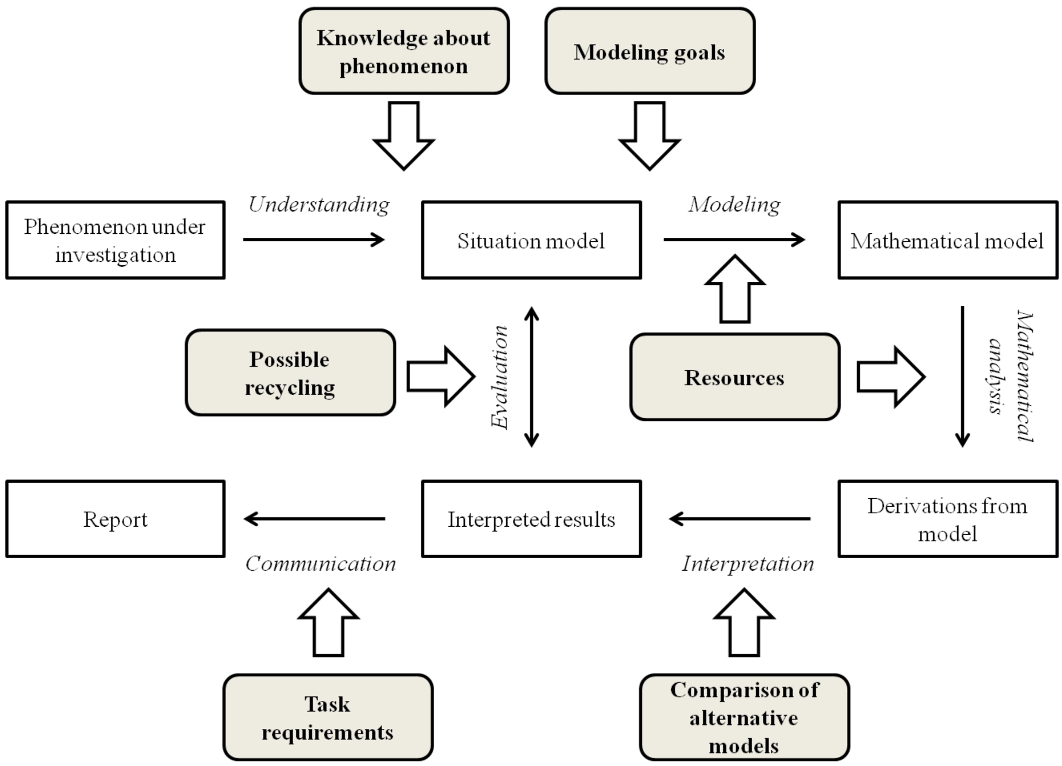

Figure 1. The modeling process (Verschaffel et al., 2000, p.168).

According to this model, solvers need to understand the given situation model often presented in different formats such as texts, drawings, tables, pictures, and videos. For this phase of the modeling process to take place, Verschaffel et al. (2000) argue that a well-organized and flexibly accessible knowledge base is required. Solvers should be able to extend this knowledge base by asking others, accessing sources of information, and carrying out mental and physical experiments. In the following phases, the solvers are required to further tap into this open knowledge base in order to generate a mathematical model. They have to be able to mathematize, i.e. "translating the situation model into mathematical form by identifying key quantities and relationships between them and expressing these by mathematical equations” (p. 169). Additionally, the use of heuristics is vital to successfully applying solution approaches that are not directly accessible. By heuristics we refer to solution strategies (e.g., making a drawing or a scheme) which help students to analyze, interpret, and understand the connections among quantitative and qualitative data in the problem. Finally, availability of mathematical resources such as techniques, symbols, graphs, manipulatives and software modeling tools are, according to Cobb (1999, p. 28), “integral to both the mathematical practices and the reasoning of the students who participate in them” and influence the derivation of the mathematical model. The next phase is the interpretation of results in relation to the situation model, in which solvers critically argue for or against a particular model and weigh up the pros and cons of competing models before they communicate their results, which is the final phase of the process.

Verschaffel et al. (2000) describe the superficial solution of word problems as the result of a truncation of this process, with several of its phases bypassed. Immediately after identifying the phenomenon under investigation, students move to the mathematical model without taking into consideration the situation model. As soon as they execute the numerical computation, they report its result without further evaluation.

The 2003 Mathematics Curriculum Reform in Greece

In 2003, primary mathematics education in Greece underwent a curriculum reform, which had implications on mathematics teaching, including word problems. The general aims of the new curriculum were the following:

1. strengthening the link between mathematics and real-life problem solving,
2. transforming the mathematics classroom into a laboratory where students work in collaboration, and
3. turning the teacher from a transmitter of knowledge to a mediator of learning (Chionidou-Moskofoglou, 2002).

Furthermore, the cross-thematic curriculum framework for elementary mathematics that was introduced in 2003 puts the emphasis on “pupils’ personal development”, “smooth social integration”, “critical thinking” and “reasoning abilities of analysis, abstraction and generalization” (Pedagogic Institute, p. 1). Based on these reform principles, which fit into a broader international reform movement towards more authenticity in the teaching and learning of mathematics (see NCTM, 1995), new mathematics textbooks for elementary school, including the 5th grade, were designed and implemented in 2006 by the Pedagogic Institute, which is exclusively responsible for designing and publishing school textbooks in Greece. So, all state and private schools are forced by law to use these textbooks.

In the teacher’s manual for the 5th grade (Kakadiaris et al., 2006b), the authors begin by critically reviewing the methodology propagated by the old textbook, stressing that the new one constitutes a radical break from it. They describe the traditional mathematics classroom environment as one based on the deep-rooted belief that doing mathematics required the application of a number of rules to reach a unique correct solution. In this environment, knowledge was thought to be transmitted linearly from the teacher to the students through written and verbal communication. In the old textbook, they argue, word problem solving gave emphasis to processes and formulas that students had to follow blindly, often at the expense of sense-making, showing no concern about the unreasonable results of these exercises. Moreover, they point out that teaching word problem solving in the old textbook consisted in the teacher solving a given problem in front of the whole class and then asking students to solve similar problems in the demonstrated fashion. As a result, pupils could not realize the usefulness of mathematical knowledge taught at school and therefore failed to transfer it to real everyday situations (Kakadiaris et al., 2006b).

According to Kakadiaris et al. (2006b), the new textbook promotes sense-making in mathematics education and marks a shift from mechanical learning to developing an expertise in problem solving through the use of alternative heuristics, in an effort to reflect everyday life, challenge students' critical faculty and be meaningful to them.

RESEARCH GOAL

Taking into consideration this curricular shift in Greece, we decided to study the extent to which the formulation of word problems in the new textbook increased their level of authenticity when compared with word problems from the old textbook.

METHOD

Data Collection

For this study, we analyzed all the word problems in the old and the new mathematics textbooks for the 5th grade of Greek elementary schools: 308 came from the old textbook (consisting of two volumes) and 264 from the new textbook (consisting of a one-volume course book and a four-volume exercise book). The new textbook we analyzed was the edition used for the school year 2012-2013 in Greece. The criterion for identifying word problems among the various tasks contained in the textbooks was based on Verschaffel et al.'s (2000) definition of word problems discussed in the introduction to this article[1].

Analytical Framework

In response to the increasing demand on teachers, textbook writers and assessment developers to design authentic school tasks, Palm (2002) developed a framework for measuring the level of their authenticity. He introduced the concept of

[1] Non-selected tasks are those presented in formats such as 4+5=?, or "How much is 40 divided by 5?" (Verschaffel, Depaepe &Van Dooren, in press)

"representativeness", which he defined as the combination of comprehensiveness and fidelity with which a number of aspects (see discussion later in this section) are represented in a school word problem. In doing so, he was able to establish the degree of resemblance between a school task and a real-life situation. Comprehensiveness refers to the "range of different aspects of the real-life situation that is simulated", while fidelity to "the degree of approximation of that aspect to a corresponding real-life situation" (Palm, 2006, p. 43). An authentic task should clearly establish the link between school mathematics and the real world, by describing a situation that is genuinely realistic outside the school setting. Various researchers (Palm, 2002; Palm & Burman, 2004; Depaepe, De Corte, & Verschaffel, 2009) have used the framework as a tool to highlight the stereotypical nature of school word problems and point out ways in which these problems could be reformulated in a more authentic manner.

In line with Palm and Burman (2004), who used the classification system to analyze task-reality concordance in Finnish and Swedish national assessments, word problems received a two level coding[2] (0, 1) for all the aspects with the exception of *specificity of data*, for which a three level classification (2, 1, 0) is applied. When a task received code 0 for any of the aspects of *event*, *question*, *existence of data* or *external tools*, the classification exercise was terminated, since such a task demanded fundamentally different mathematical solution approaches when compared to corresponding real-life situations. To analyze the nature of word problems in Flemish mathematics classrooms, Depaepe et al. (2009) also employed Palm and Burman's (2004) scheme, which they modified by removing *external tools* from the aspects where code 0 triggered the termination of the classification.

The way the aspects of the classification system are operationalized in the present study is similar to the way Depaepe et al. (2009) have done in their study, albeit with some additions. First, a three-level coding has been introduced for the aspect of *event* following Depaepe et al.'s (2009) suggestions to refine the operationalization of the framework, so that it would yield more fine-grained results. Second, discussing the limitations of the methodological tool used for analyzing word problems, Depaepe et al. (2009) observed that an important element was missing. Having identified as the root of students' modeling problems the difficulty to translate the situation model into a mathematical model, many researchers (Verschaffel et al., 2000) stress the importance of developing more challenging word problems. These "problematic" items should simulate better the complex relation between the situation and mathematical model by rendering the translation from one model to the other less straightforward and simple. In light of the above, the aspect of problematicity is included in the coding scheme to detect the extent to which word problems in the new textbook problematize this relationship. Coding all problems with respect to *problematicity*, irrespective of their codes for *event*, *question* and *existence of data*, could offer a better picture of the degree to which the problems in the two textbooks challenge superficial mathematical

[2] We should mention here that in our analysis of the word problems the terms "classify" and "code" are used interchangeably.

modeling. Moreover, when a word problem contains more than one question, separate codes are given for each.

A detailed description of the different aspects of the framework employed for the analysis of word problems from the old and the new textbooks follows. An interobserver analysis involving two independent coders scoring problems from the two textbooks has yielded a good interrater reliability rate (κ= .79).

Event: Code 2 for this aspect suggests that the "event described in the problem is closely related to students' experiential world" (Depaepe et al., 2009, p. 248). Code 1, means that the event described could be encountered in real life by people in general, whereas code 0 denotes an event unlikely to occur in real life whatsoever.

Question: Code 1 means that the question asked in the task would be posed in a corresponding real-life situation, while code 0 means that such a question would not realistically be asked.

Purpose in the figurative context: Code 1 means that the goal of problem solving in the school task coincides with the goal of solving a similar problem in real life; code 0 means that there is no such coincidence, the goal being dissimilar, unclear or absent.

Existence of data: Code 1 means that the relevant data present in the real-life event coincide with the data accessible in the school task; code 0 denotes a lack of such coincidence.

Realism of data: Code 1 is awarded when the values, figures and numbers presented are realistic, whereas code 0 is assigned when such data are fictitious and unrealistic.

Specificity of data: Code 2 indicates that the subjects, objects and places described in the task are specific. For example, if graphs are used, their source is mentioned. Code 1 is assigned when the subjects, objects and places described in the task are specific only at a minimum level and code 0 when they are non-specific.

Language use: Code 1 indicates linguistic similarity between school task and real-life situation. However, if the task terminology, sentence structure or length of text makes students use different mathematics from those that would be used in a corresponding real-life situation, code 0 is assigned.

Availability of solution strategies: Code 1 signifies that the available solution strategies in the task allow students to solve the task in the same way someone would solve it in real life; code 0 indicates a lack of correspondence between school task and real-life available solution strategies. For example, if the textbook task stipulates that students use written computation, when in a similar problem in real life mental arithmetic would be the most likely option, the textbook task would receive code 0.

External tools: If both school and real life solvers have or do not have access to concrete tools, such as a calculator, a ruler, paper and pencil, a map, a computer, a software program, etc., then the task receives code 1; code 0 is assigned when there is a discrepancy between the two situations regarding access to available external tools. For example, if, in real life, a solver could have access to a calculator and in the corresponding school task the textbook forbids such use, the task would receive code 0.

Guidance: Code 1 means that there is correspondence between the guidance provided in the school task and that normally provided in the similar real-life situation; code 0 signifies that there is no such correspondence. More concretely, code 0 is assigned when, for example, the problem's statement in the textbook is accompanied with an explicit instruction or cue or an implicit hint concerning how to approach the task and what heuristic (e.g. 'Start by making a drawing') or metacognitive strategy to use (e.g. 'Don't forget to check your answers afterwards').

Solution requirements: When a school task and a corresponding real-life situation involve similar implicit or explicit requirements for solving this task, it receives code 1; code 0 is assigned to a school task whose solution requirements are different from those in the real-world event. For instance, if the textbook problem asks for a precise numerical answer when normally the real world situation would require an approximate answer, then it should receive code 0.

Problematicity: Code 1 for this aspect means that the task is constructed in a way that calls for a non-direct translation of the situational model into a mathematical one, while code 0 is awarded when this relationship is obvious and unproblematic for the student.

What follows is an illustration of the analysis, using as an example two tasks from each textbook (Figure 2), followed by their classification (Table 1).

Old textbook

Problem 1

A grandmother divided 5 oranges equally among her 3 grandchildren. How many oranges did each child take?

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……………………………………………………………………
……………………………………………………………………
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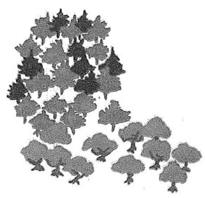
Problem 2

Keti's father has two gardens. The first garden measures 3 ares and 750 square meters and the second 3.570 ares. Which of the two gardens is the biggest?



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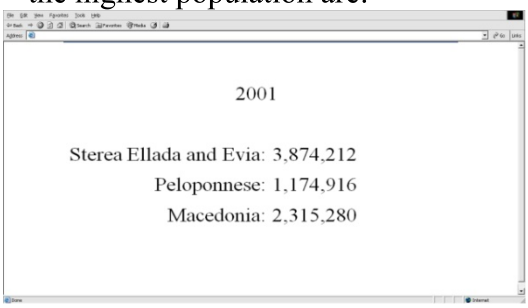
New textbook

Problem 3
The children visited the webpage of the Hellenic Statistical Authority and found information about the legal population in Greece by geographical regions.
On the basis of the data they found, they noticed that in the 4 censuses that have taken place in the last 30 years (1971, 1981, 1991, 2001), the three geographical regions with the highest population are:



- Sterea Ellada and Evia
- the Peloponnese and
- Macedonia.

Study the numbers and find the total population of the three geographical regions in 2001:

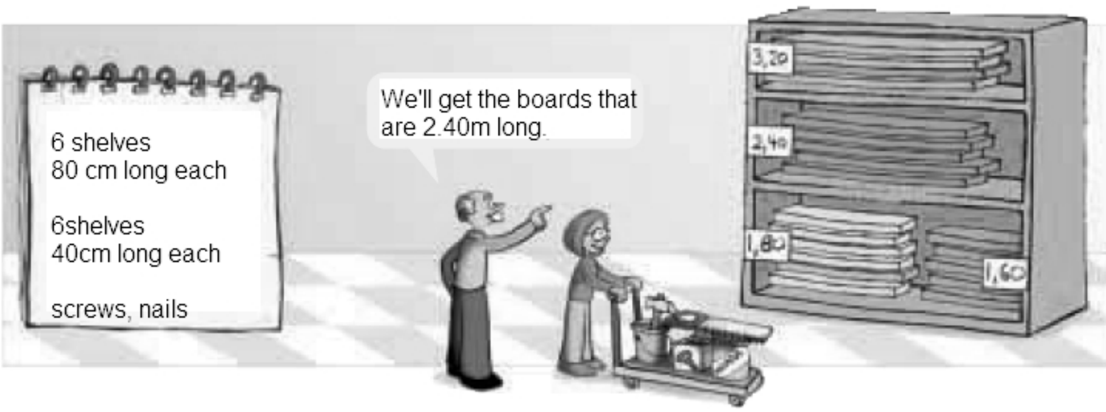
- Estimate:

  ……………..

- Calculate accurately:

- Verify by:
- Vertical addition
- Vertical subtractions

Problem 4
Nefeli with her grandfather are choosing wood that they will need to make her bookshelves.



How many wooden boards did they buy if they used every piece of them?

Figure 2. Examples of word problems from the old and new textbooks (*Alvanos, Dimou, Zervas, & Broumas, 2005a, 2005b; Kakadiaris, Belitsou, Stefanidis, & Chronopoulou, 2006a, 2006c).*

Table 1

Coding of the Word Problems from Figure 2 According to the Analytical Framework

Aspect	Problem 1	Problem 2	Problem 3	Problem 4
Event	1	1	2	2
Question	1	0	1	1
Purpose	0		0	1
Existence	1		1	1
Realism	1		1	1
Specificity	1		2	2
Language use	1		1	1
Solution strategies	0		0	1
External tools	0		0	1
Guidance	0		0	0
Solution requirements	1		0	1
Problematicity	1	0	0	0

RESULTS

Table 2 presents the results of coding the 308 problems of the old textbook and the 264 problems of the new textbook for each aspect of the classification framework.

What follows is a discussion of the results for each aspect of the analytical framework. The problems presented in Figure 2 and their classification in Table 1 will be used as illustrations.

Event

While only 9.4% of the problems in the old textbook received code 2, almost two thirds (62.5%) of the problems in the new one scored 2 in this aspect. This shift is illustrated by problems 3 and 4 from the new textbook, in which the writers attempt to enrich the context of the problems, by adding relevant details related directly to pupils' experiences. Conversely, problems 1 and 2 from the old textbook give only the bare amount of generic information concerning the described events.

Table 2

Absolute Numbers (N) and Percentages (%) Generated by the Classification of the Word Problems in the Old and the New Textbooks

Aspect	Code		Old	New	Aspect	Code		Old	New
Event	0	N	4	2	External tools	0	N	78	50
		%	1.3	0.8			%	40.4	35.7
	1	N	275	97		1	N	115	90
		%	89.3	36.7			%	59.6	64.3
	2	N	29	165	Guidance	0	N	185	106
		%	9.4	62.5			%	95.9	75.7
Question	0	N	34	37		1	N	8	34
		%	11.1	14.1			%	4.1	24.3
	1	N	271	225	Solution requirements	0	N	12	62
		%	88.9	85.9			%	6.2	44.3
Purpose	0	N	236	134		1	N	181	78
		%	87.1	59.6			%	93.8	55.7
	1	N	35	91	Problematicity	0	N	282	235
		%	12.9	40.4			%	91.6	89.0
Existence	0	N	78	84		1	N	26	29
		%	28.8	37.3			%	8.4	11.0
	1	N	193	141					
		%	71.2	62.7					
Realism	0	N	0	1					
		%	0.0	0.7					
	1	N	193	139					
		%	100.0	99.3					
Specificity	0	N	9	5					
		%	4.7	3.6					
	1	N	146	70					
		%	75.6	50.0					
	2	N	38	65					
		%	19.7	46.4					
Language use	0	N	20	2					
		%	10.4	1.4					
	1	N	173	138					
		%	89.6	98.6					
Solution strategies	0	N	15	15					
		%	7.8	10.7					
	1	N	178	125					
		%	92.2	89.3					

Question

In both textbooks, the majority of the problems based on a realistic event ask questions that are likely to be posed in real life (i.e., in 88.9% of the old textbook tasks and in 85.9% of the new textbook tasks). Only a similarly low percentage of problems pose unrealistic questions, such as that asked by problem 2 from the old textbook.

Purpose in the Figurative Context

Compared to the very low fidelity of *purpose* in the problems of the old textbook (87.1%), the new textbook shows an increased percentage (40.4%) of word problems with a realistic event and question that simulate this aspect well. Problem 1 offers an illustration of this low level of purposefulness in problem-solving activity in the old textbook. Students are asked to calculate how many oranges each child took without being given any hint as to the reason they have to solve this task. In problem 4, however, the real-life situation described in the task (Nefeli is helping her grandfather to buy the wood needed to make shelves for her room) is inextricably linked with the goal of solving this task (buying the exact amount of wood needed for the shelves).

Existence of Data

Both textbooks contain a significant percentage of problems that scored 0 for this aspect (28.8% in the old and 37.3% in the new), which indicates that their writers tend to use word problems as disguised tasks for further practice on the mathematical concept in question. For example, problems about operations with compound numbers in the old textbook (see problem 2) and problems about operations with fractions in the new one (e.g. "We bought 2 kg of oranges to make juice. The juice we made was the 7/10 of the weight of the oranges we had bought. How much grams of juice did we make?") contain data that are accessible only if the solver applies competencies that are different from those required in the real-life situation.

Realism of Data

Realistic numbers and values are provided in almost all word problems (100% in the old textbook and 99.3% in the new one) that feature a well-simulated event, realistic question and data that are accessible in real life. In the new textbook, as used in 2012, the only problem coded 0 for this aspect refers to an outdated metro ticket price (€0.80), which does not correspond with its substantially increased price (€1.40) at the time.

Specificity of Data

Only a small percentage of problems in both textbooks (4.7% in the old and 3.6% in the new) present a situation in which subjects, objects and places are non-specific. Most of the old textbook problems (75.6%) have received code 1, since the data in the tasks are specific only at a minimum level. The problems, however, in the new one are almost equally distributed between categories 1 (50%) and 2 (46.4%), signaling an increase in the number of realistic situations involving real-life data. Problem 3, for example, offers data from an official source.

Language Use

The tasks in the new textbook (98.6%) show high fidelity in simulating *language use*. On the contrary, 10.4% of the tasks in the old textbook included scientific terms, which, as Palm (2006) claims, could have an "impeding impact" on solving a task. For example, one word problem in the old textbook describes the event of an informal talk between two friends in which they use mathematical terms such "divisor" and "remainder."

Solution Strategies

Both the new and the old textbooks contain a similarly low percentage of word problems (7.8% in the old and 10.7% in the new) directing students towards a specific solution strategy, which real-life problem solvers would not necessarily use. For example, problem 3 explicitly forces solvers to apply a written computational algorithm to verify their answer, when in real life one could speed up this process by using a calculator. On the contrary, the majority of problems in both textbooks (92.2% in the old and 89.3% in the new) simulate this aspect well. Problem 4, which gives no clue as to the solution strategy the solver could use, is one such example.

External Tools

Both books simulate the aspect of *external tools* to an equal degree. Fewer than half of all the tasks (40.4% in the old and 35.7% in the new) present situations in which the solver would have no access to tools available in real life, or, conversely, would have access to tools unavailable to the real-life solver. Thus, in problem 1 from the old textbook, the student would only have access to paper and pencil, while in real life the solver would actually simply cut the oranges to divide them among the children. Similarly, problem 3 from the new textbook clearly specifies the use of paper and pencil to verify the mathematical calculation, while in real life solvers could use a calculator.

Guidance

The majority of the old textbook word problems (95.9%) do not simulate this aspect well, since they use titles or headings to inform students about the mathematical topic of each unit. Although the corresponding percentage in the new textbook is lower (75.7%), both textbooks tend to impose on students a particular way of approaching a task, when in real life no such indication would be given.

Solution Requirements

The problems in the old textbook show higher fidelity in simulating solution requirements (93.8%), whereas the new textbook places more unrealistic restrictions on students concerning the type of necessary calculation and solution. Problem 3, for example, explicitly asks for both an estimate and an accurate calculation, when in real life the type of solution would depend on the purpose of solving the task.

Problematicity

The majority of the problems in the old (91.6%) and the new (89.0%) textbooks invite students to apply the most obvious mathematical model. Problems 2, 3 and 4 fall into this category. Interestingly enough, there are even problems in which the authors exclude via explicit hints any possible realistic constraints that could complicate problem solving. In the new textbook, for example, a task asks students to calculate the number of children that ate four packets of biscuits each containing 15 pieces. The writers explicitly inform solvers not only that each child ate an equal number of biscuits, but also that the children ate all the biscuits in the four packets.

Many of the few tasks in both textbooks that scored 1 in this aspect are division with remainder problems (DWR) (see problem 1), whose solutions have to be rounded to the next number and therefore require from the solver to interpret the results against the given situation before he/she reports them.

DISCUSSION AND CONCLUSION

This section begins with a comparison and evaluation of the authenticity of word problems from the two textbooks. Then, a number of methodological issues are considered and, finally, some suggestions concerning future research are given.

Comparison and Evaluation of the Authenticity of Word Problems from the Two Textbooks

High percentages of tasks in the new textbook (see Table 3) that simulate well aspects such as *event*, *question*, *realism of data*, *specificity of data* and *language use* indicate the authors' adherence to their basic principle, i.e. to design tasks that reflect real life and make sense for the students.

The sharpest difference between the two textbooks is observed in relation to *event*. According to the writers of the new textbook (Kakadiaris et al., 2006b), word problems that are related to students' personal interests and experiences reinforce the idea that problem solving can be a meaningful activity. An interesting characteristic of the new textbook, that is absent in the old one, is the introduction of a number of children characters featuring in the stories described by the word problems both in the course book and in the exercise books. The writers claim that this will encourage students to identify themselves with these characters and as a consequence relate more to the situations described in the problems. In contrast, the majority of the problems in the old textbook provide only a minimum of contextual information and thus reinforce the idea that the real world has only a decorative function in the mathematics classrooms.

Table 3

Percentage of Word Problems in the Old and the New Textbooks which Received Code 2 or 1

Aspect	Old textbook	New textbook
Event	9.4	62.5
Question	88.9	85.9
Purpose	12.9	40.4
Existence of data	71.2	62.7
Realism of data	100.0	99.3
Specificity of data	19.7	46.4
Language use	89.6	98.6
Solution strategies	92.2	89.3
External tools	59.6	64.3
Guidance	4.1	24.3
Solution requirements	93.8	55.7
Problematicity	8.4	11.0

Moreover, the increased level of simulation of *purpose* in the new textbook, when compared with the old one (see Table 3) , indicates the authors' conscious effort to prevent students from viewing problem solving as an inauthentic activity, whose lack of a clearly delineated purpose would intensify the impermeability "between school experience and life experience" (Staub & Reusser 1995, p. 323).

Another major difference between the two textbooks, as shown in Table 3, is that the new one encourages the use of *external tools* by allowing students to use a calculator in certain tasks to verify their calculations. In the old textbook, the authors neither forbid nor consent to such use. However, the stated aim of the authors to automate students' computational competence must have implicitly ruled out the use of calculators. In light of the above, the result of 59.6% for the tasks in the old textbook simulating this aspect well appears rather misleading. Nevertheless, one could argue that it is quite common for real-life solvers to resort to paper and pencil in order to solve a problem.

A comparison of the results concerning the *specificity of data* for the two textbooks also shows clear differences (see Table 3). The authors of the new textbook attempt to develop tasks involving data from authentic sources. This could also be explained by the cross-thematic focus of the reformed elementary mathematics curriculum in Greece. Problems in the old textbook show only a limited degree of specificity, as the majority of them are phrased as “semantically impoverished vignettes” (Reusser & Stebler, 1997, p. 15), describing a situation in which the data given are not sufficiently specific.

High and moderate fidelity simulation of the *availability of solution strategies* and *solution requirements,* respectively, in the new textbook could be in line with the authors’ claim that, compared to the old textbook (see Table 3), the new one does not intend to enhance automaticity in accurate manipulation of algorithmic formulas, but to promote the use of heuristics and metacognitive strategies in problem solving (Kakadiaris et al., 2006b). Interestingly enough, the analysis indicates that, despite Kakadiaris’ et al.’s (2006b) affirmations, tasks in the old textbook simulate *availability of solution strategies* at slightly higher level. Of course, this does not mean that the way the problems were utilized by teachers allowed students the freedom to decide what kind of solution they could use, if one takes into consideration that the primary teaching aims set by its authors were students’ adherence to mathematical formulas and algorithms and the development of automaticity and computational competence (Alvanos, Dimou, Zervas, & Broumas, 2000). Even more interestingly, problems in the old textbook vastly outperform tasks in the new one in terms of simulating realistic *solution requirements*. This contradiction could be explained by the emphasis given by the new textbook to fostering students’ ability to assess the validity of their solutions by implementing alternative strategies real-life solvers would normally not use.

The authors of the new textbook also claim that through revised problem-solving activities students are encouraged to use their common sense and reasoning skills. Some findings, nonetheless, appear to contradict this claim. For example, fidelity of simulating *guidance* remains low, despite a slight improvement in comparison with the old textbook (see Table 3). In the latter, 95.9% of the tasks are part of units whose mathematical topic is signaled by a title giving explicit guidance, such as “The operation of division: Exercises and Problems” (Alvanos et al., 2005a). For Verschaffel et al. (2000), applying routine-based problem-solving strategies is attributed to the way problems are grouped and formulated in mathematics textbooks. This practice is more limited in the new textbook, which could account for the improved but still rather high percentage of such tasks. It would appear that the writers have not adequately implemented their intention to allow students to solve tasks in a less regulated environment in which problem solving does not rely on external guidance in the form of key-words, suggested rules and proposed formulas (Kakadiaris et al. 2006b, p. 9).

Similarly, as regards the *existence of data*, one could argue that the lower level simulation in the word problems of the new textbook (see Table 4), as a result of purposefully withholding or modifying information and data, reveals that the authors continue using word problems primarily for practice on certain mathematical operations. Palm and Burman (2004) explained this gap between real life and school tasks as the authors’ attempt to either simplify or make more complex the data in the school task.

Moreover, there is little improvement as regards problematicity. The persistently high percentage of unproblematic items (89%) in the new textbook means that students are hardly confronted with problems where they are specifically stimulated to use their common sense and real-life reasoning skills. Moreover, the few tasks that simulate well this aspect are mainly division with remainder problems (DWR), for which researchers (Verschaffel et al., 1994; Verschaffel, Van Dooren, Greer & Mukhopadhyay 2009) report better realistic modelling results when compared to other types of problematic tasks (such as items about union of sets with common elements or items about non-linearity). They attribute this success to the fact that DWR problems have found their way into mathematics classrooms and textbooks internationally, due to the great research attention paid to them. As a result, by excluding problematic items from the textbook, the writers, even though unintentionally, reinforce in students the belief that mathematics in the classroom is a separate domain disconnected from the real world.

In light of the above and given that only two word problems in the new textbook simulate well all twelve aspects of the analytical framework, it cannot be denied that there is still room for improvement. This observation also coincides with a finding of Palm and Burman's (2004) analysis of authenticity in assessment tasks in Swedish national assessments. They noticed that only a significantly small percentage of tasks simulated most of the aspects of the analytical framework at a reasonable degree of fidelity (p. 26). Changes to word problems, as the analysis has indicated, should focus on increasing the level of authenticity through better simulation of aspects such as *purpose in the figurative context*, *specificity of data*, *guidance* and, of course, *problematicity*. Increasing the level of task authenticity in relation to aspects involving data requires continuous and systematic revision in order to mirror changes happening in the real world.

Methodological Limitations and Suggestions for Future Research

First, some issues related to the operationalisation of the classification framework merit discussion. In line with Palm and Burman (2004), who report a discrepancy between the classifications of coders in their study, we noticed that different coding for aspects such as *event* or *question* often depended on individual perceptions, experiences and beliefs held by coders. Although such differences were not significant, it is important to mention that they have highlighted the subjective nature of the coding process. Depaepe et al. (2009), critically reviewing their implementation of the coding scheme, also argue that limited knowledge about the textbook writers' purpose of solving a problem complicates the classification of tasks in relation to the aspect of *solution requirements*. Thus, further work towards refining the meaning of the codes for the various aspects of the scheme could contribute towards improving its implementation.

Also the way the analytical framework was operationalized in this study has also shown that introducing a three-level coding scale for the aspect of *event,* as proposed by Depaepe et al. (2009), has helped highlight differences between tasks with minimum and high level of simulation, which would have been veiled by a two-level coding scale. Indeed, as our analysis has indicated, the third level in our classification scheme for the aspect of event has made clearer the difference between the word problems in the two textbooks with regard to the simulation of that aspect. Implementing a refined coding in relation to other aspects could possibly produce more helpful insights.

Another limitation of this work is that it does not take into account the way the problems are actually used by students and teachers in the context of the classroom. Thus, a research design that would focus on analyzing word problems 'in action' could help reconsider the presentational structure of word problems with a view to increasing their authenticity. Furthermore, classroom-based research into teachers' perceptions and treatment of word problems along the line of work done for example by Chapman (2009) and Depaepe, De Corte, and Verschaffel (2010) would allow us to detect the tensions that emerge in teachers' planning and executing mathematics lessons. In particular, we could identify the conflict in the teachers' strive for generality, abstraction and formality, on the one hand, and for concreteness, authenticity and realism, on the other.

Finally, one could study how the existing word problems of the new textbook and improved versions thereof affect students' realistic mathematical modeling, by carrying out a long-term intervention teaching experiment along the lines of work done, for example, in Flanders (Verschaffel & De Corte, 1997) and in Italy (Bonotto, 2009). Experimental and control groups of students would be exposed to problem solving activities involving either existing or modified word problems, in an attempt to identify the extent to which adapted instructional materials and approaches could foster adaptive modeling expertise in students. Such studies could help establish a classroom practice and culture that would encourage realistic modeling opportunities and enhance students reasoning and critical thinking.

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Pre-service Teachers' Opinions and Methods about Finding Students' Misconceptions

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ABSTRACT: *This action research study focuses exclusively on some aspects of pedagogical content knowledge, particularly addressing students' misconceptions. It mainly investigates teachers' understandings about students' misconceptions. The study was conducted with 5 pre-service teachers who participated in the study through an investigation of students' misconceptions in their field-experience schools. The results suggest that the pre-service teachers have difficulties in determining students' misconceptions and their involvement in research activities can reveal more aspects of their pedagogical content knowledge.*

Key words: *Students' misconceptions, Pre-service teachers, Pedagogical content knowledge.*

INTRODUCTION

Researches on teacher development programs indicate that pre-service teachers need opportunities to observe and reflect on students' mathematical thinking and on how effective teachers build on students' thinking (Ball, 1993; Simon, 1995, as cited in Masingila & Doerr, 2002). Teacher development programs generally include field experiences in which pre-service teachers are exposed to the complexity of the classrooms. Baki (1997) emphasizes the importance of experiencing alternative methods and approaches in this complexity. He states that without this experience, they are not able to make informed decisions about what and how their students learn, and they are not able to take responsibility for shaping the content and process underlying a task to meet the needs of their students. Thus, pre-service teachers need to engage in activities to understand more about students' thinking and build on their existing knowledge.

PEDAGOGICAL CONTENT KNOWLEDGE

Shulman (1986) discussed the need for knowledge of student thinking, knowledge of models that illustrate key concepts and understanding what contributes to the intrinsic difficulty of certain topics. In 1980's he (1986, 1987) studied the complexity of teachers' knowledge and tried to capture this complexity by identifying several categories of knowledge that have implications for teaching. He developed the construct of "pedagogical content knowledge" (PCK) which implies "an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (Shulman, 1987, p. 8). PCK includes 'the ways of representing and formulating the subject that make it comprehensible to others and 'an understanding of what makes the learning of topics easy or difficult; the concepts and preconceptions that students of different ages and backgrounds bring with them' (Shulman, 1986, p. 9). According to Shulman (1987), PCK must include the knowledge of learners and their characteristics, knowledge of educational contexts, knowledge of educational ends, purposes and values, and their philosophical and historical bases. Additionally teachers should be pedagogically powerful and yet adaptive to the variations in ability and background presented by the students.

After Shulmans' studies; An, Kulm, and Wu (2004) defined pedagogical content knowledge as the knowledge of effective teaching which includes three components, knowledge of content, knowledge of curriculum and knowledge of teaching. They point out the importance of knowledge of teaching and they accept it as the core component of pedagogical content knowledge. Knowledge of teaching consists of knowing students' thinking, preparing instruction, and mastery of modes of delivering instruction. The interrelations of these components are shown in Figure 1. Knowing students' thinking comprises the notion of teaching with understanding, which is essential to effective teaching (Carpenter & Lehrer, 1999).

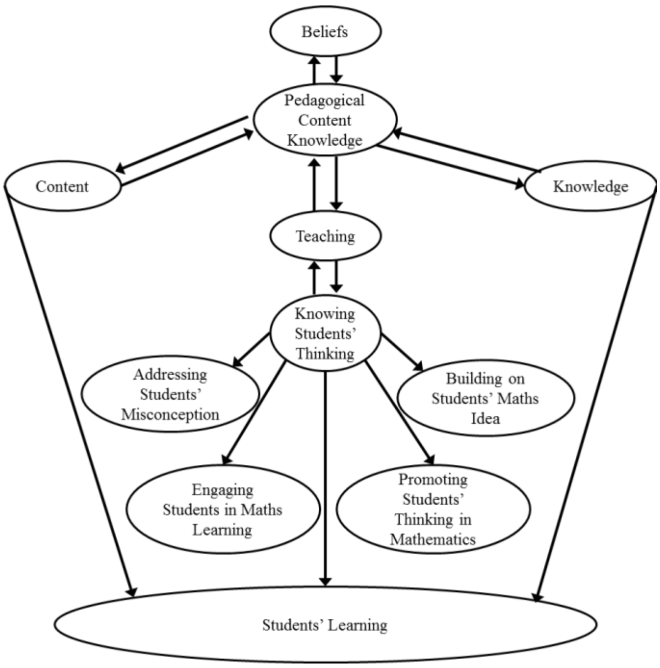

Figure 1. The network of pedagogical content knowledge (adapted from An, Kulm, & Wu, 2004, p. 147).

ADRESSING STUDENTS' MISCONCEPTIONS

"It seems that to teach in a way that avoid pupils creating any misconceptions… is not possible and that we have to accept that pupils will make some generalizations that are not correct and many of these misconceptions will remain hidden unless the teacher makes specific efforts to uncover them" (Askew & Wiliam, 1995).

Educators have widely used the "misconception" term to describe and explain students' performance in specific subject-matter domains. In the most general meaning, the misconceptions are the scientifically totally or partially wrong understandings that people create in their minds towards the situations they are in (Yagbasan et al., 2005). From a handful of investigations in a small number of science domains in the mid-1970s, misconception researches expanded to nearly every domain of science and to mathematics and computer programming as well by the mid-1980's (Smith, diSessa, & Roschelle, 1993). An understanding of common student misconceptions, and effective strategies to help students avoid them, is an important aspect of mathematical PCK (Graeber, 1999). Numerous studies (e.g., Graeber, 1999; Leinhardt, Putnam, Steinand, & Baxter, 1991) have stressed that teachers need knowledge of why confusions and misconceptions may occur (Chick, Pham, & Baker, 2006). The teachers are expected not only to recognize the misconceptions that the students have or may have, but also to turn them into advantages in teaching by fine-grained analysis of these misconceptions (Basturk & Donmez, 2011).

Misconception studies provide valuable insights about subject matter learning which focus the attention on what students actually say and do in a wide variety of mathematical and scientific domains. Building on Piaget's studies of cognitive development, misconceptions research freed investigators' descriptive capabilities and legitimized their efforts to uncover structure and meaning in students' responses. These investigations have produced careful characterizations of students' conceptions in a variety of conceptual domains and their changes (or not) in response to experience. (Smith**,** diSessa, & Roschelle, 1993).

Although there are numerous studies (e.g., Basturk & Donmez, 2011; Jordaan, 2005; Luneta & Makonye, 2010; Orhun, 2001; Sadi, 2007) identify students' and teachers' misconceptions in various concepts, there are few studies about teachers' PCK that focus on identifying students' misconceptions. For example, McDonough and Clarke (2002) looked at 6 teachers on a single topic and tried to describe the practice of effective math teachers in the early years. In another study, Chick and Baker (2005) investigated the primary/elementary teachers' pedagogical content knowledge that was asked to describe how they would respond to some hypothetical situations involving student misconceptions and errors. They examined the self-reported practices of year 5 and 6 teachers in addressing certain typical misconceptions. They found that re-explaining, probing student thinking and evoking cognitive conflict were the common strategies that teachers suggested for misconceptions. Furthermore, they concluded that the nature of the task in which the error arose influenced teachers' emphasis on procedural or conceptual aspects, and teachers' responses revealed aspects of their pedagogical content knowledge.

According to Chick and Baker (2005), studies about finding student and teacher misconceptions generally (e.g., van der Valk & Broekman, 1999) focused on teachers' preparations on a particular topic. This study focuses exclusively on some aspects of pedagogical content knowledge, particularly addressing students' misconceptions. It investigates teachers' understandings of student misconceptions. It is significant because it addresses the preferences made by teachers while they are finding students' misconceptions in various concepts. The researchers tried to answer the question "how do pre-service math teachers identify students' misconceptions?"

METHOD

Action Research with Pre-Service Teachers

In the past decade, action research has been advocated for both pre-service and in-service teacher education as a means of improving teaching and learning through critical reflection on problems in practice (Gore & Zeichner, 1991, Lieberman, 1986, as cited in Lundeberg et al., 2003). Action research is generally defined as a form of educational research wherein a professional, actively involved in practice, engages in systematic, intentional inquiry into some aspect of that practice for the purpose of understanding and improvement (Cochran-Smith & Lytle, 1990; Kemmis & McTaggart, 1988; McNiff, 2002; Sagor, 1992, as cited in Kitchen & Stevens, 2008). Involving pre-service teachers in action research projects along with professors and/or teachers may provide opportunities to develop pedagogical content knowledge, examine beliefs about teaching, and gain confidence (Lundeberg et al., 2003).

Context of the Study

An action research study was conducted by researchers to analyze the pre-service teachers' opinions about misconceptions that engaged in action research with a teacher educator and a high school mathematics teacher. At the university level, it was conducted primarily within a teacher education course over a 4-month period with 5 pre-service teachers who were preparing to become secondary math teachers. These pre-service teachers had finished their first field experience course before the study and they attended the second field experience course during the study. They were participated in the study through this course and their task was to investigate students' misconceptions in their assigned schools. The participants were one male and four female pre-service teachers.

In conducting action research, researchers followed some routines. These routines are guided by progression through five phases of inquiry (Ferrance, 2000): (a) Identification of problem area, (b) Collection and organization of data, (c) Interpretation of data, (d) Action based on data, and (e) Reflection.

Procedure

At the beginning of the study, the instructor gathered information about areas of their interest and what expectations they had for this study. During the first three weeks, pre-service teachers reviewed the existing literature about misconceptions in mathematics education and they were required to evaluate sample research studies conducted by teachers. The purpose here was to have pre-service teachers informed about misconceptions and research methods in mathematics education.

After reviewing the literature, under the direction of the instructor, pre-service teachers made presentations about research methods; obtaining, interpreting and analyzing data. They discussed information and examples on the kinds of research methods from various topics in these presentations. At the end of this two-week period, the pre-service teachers prepared their research plans with the instructor and determined their topics that they would study. The pre-service teachers were allowed to determine a research question based on their interests. They also decided on their research sample themselves. It took again two weeks for them to review the existing literature about misconceptions related to their subject. They were then involved in finding misconceptions by various research methods. First of all, they prepared questionnaires or tests to determine students' misconceptions. Under the supervision of the instructor, they prepared questions according to the existing literature and examined prepared tests from the relevant literature. Subjects chosen by pre-service teachers and some elements of their researches are described in Table 1.

Table 1

Pre-Service Teachers' Research to Find Students' Misconceptions

Participant	Subject	Sample (Number)	Questions used for data gathering
1.	The infinity concept in limit	11th grade students (30)	5 open ended questions
2.	The induction principle subject	11th grade students (26)	5 open ended questions
3.	Understanding graphs and tables	10th grade students (20)	4 multiple choice questions
4.	The limit concept	Math teachers (10)	6 open ended questions.
5.	Algebraic expressions (equations)	9th grade students (20)	10 routine problems ask to solve equations

During the inquiry process the instructor made weekly mini interviews with the participants about research procedures. The instructor asked some questions like “What kind of tools are you going to use in your study, how did you collect your data, and what do you think about the organization of the data.” They got grades from these individual or group interviews.

All of them finished this preparation within three weeks. Then the questionnaires and tests were applied in the schools that organize training courses for pre-service teachers. They finished their applications in a week with teachers in these schools. After the application, they analyzed their data with the help of the instructor and tried to determine the students’ misconceptions. Finally, pre-service teacher-researchers reflected on what they found. Each pre-service teacher prepared a report that included what were students’ misconceptions about their subject, the challenges and difficulties students faced, and which research methods they used to find students’ misconceptions. At the end of the study the researcher asked the participants to answer a questionnaire about misconceptions and to explain their opinions about finding misconceptions in a mini interview. They were asked to define the misconception concept and to explain its differences from students’ errors. Also they explained their preferences for finding misconceptions.

Data Analysis

Analysis of the data from the pre-service teachers is based upon observations of the instructor, the data obtained from interviews with the participants, answers of the short questionnaire and an examination of their findings that emerged as a research paper with the following sections; research statement, research question, rationale (importance of study), methodology and review of related literature and results.

Content analysis on the participants’ reports was performed to illustrate how they found students’ misconceptions. Their comments were analyzed for emerging themes and charts were formed to analyze research reports more systematically. Categories of analysis for pre-service teachers’ written answers were constructed by following Strauss and Corbin’s (1990) approach to grounded theory generation. As grounded theory is known for building theories inductively from data in under-researched areas, it is suitable for the exploratory nature of this study (Grbich, 1999). Categories were sorted and compared until no categories can be produced.

FINDINGS

This study attempts to investigate the opinions of pre-service teachers about students’ misconceptions and finding them. At the beginning of the interviews the researchers investigated the understanding of pre-service teachers about the misconception concept and they evaluated the pre-service teachers’ definitions about the concept. Subsequently, the researchers focused on the question “How do the pre-service teachers determine and find students' misconceptions?" and made some comparisons between their research methods and findings.

How Do Pre-Service Teachers Define Misconception?

Generally pre-service teachers define misconceptions as different and wrong understanding or thinking. Except one, all of the pre-service students stated that the students' misconceptions are usually due to lack of information about the concepts.

They emphasized that the misconceptions are constructed by mental activities. Only one of them emphasized that misconception is accepted as a true idea by the students even if it is constructed in a wrong way. After analyzing their answers three categories emerged about the pre-service teachers' definition of misconception: (a) Thinking and naming concepts in a different way, (b) Misunderstanding and schemas, and (c) Wrongly constructed concepts that are accepted as true.

How Do Pre-service Teachers Determine Misconceptions?

In this part the pre-service teachers' reports are analyzed and discussed together with the notes taken by the researchers during the study. The researchers also made some comparisons among the research studies focusing on students' misconceptions. All of the participants gave some definitions about the misconception from the literature at the beginning of their reports. So they had some ideas about the misconception literature and tried to benefit from this literature. The following is the summary of the pre-service teachers' research reports.

First Participant

The first participant studied the infinity idea for the limit concept. At the beginning of the study, she said that she had read an article about the limit concept in an undergraduate course. After reading this article, she had seen that there have been some difficulties about the applications of this concept. So, she wanted to study this subject. After she investigated the literature, she presented three or four research issues. The instructor asked her "About which subject the literature is weak?" and "In which subject do you see more errors?" Then she stated her research problems about the infinity concept and chose the infinity in the limit concept to investigate through the lack of literature in this area.

Methodology the First Participant Adopted

She studied with 30 grade-11 students. She used qualitative research methods (content analysis, interviews and observation) to determine students' understandings. She prepared a test which was composed of 5 open ended questions to gather data about the subject. Then she interviewed four students from the working group who gave various answers to the test. She analyzed students' answers and obtained categories by content analysis.

Conclusion the First Participant Reached

She identified some mental models held by students, like "writing a very big number with many nines". But she could not notice misconceptions about the infinity concept. Although she often mentioned about the concept images of students, there was not any idea about the students' misconceptions. The researchers thought that the reason for why she could not have emphasized the misconceptions about this concept much may be the indefiniteness and difficulty of the concept for her.

Second Participant

The second participant was a silent pre-service teacher. The dialogue between the second participant and the instructor was the weakest of the class in the lessons. Although he was not very active in the presentations, his findings and analysis part of the report was better than others. He made a detailed definition of the misconception and emphasized the systematic error view of it in his report. While they were reviewing the existing literature about misconceptions at the beginning of the study, he thought that "mathematical proofs" was very important subject for his students. He stated that he wanted to choose this subject because of his experience in the lessons. Then he investigated "mathematical proofs" in more detail. Under the orientation of the instructor, he decided to study "the induction principle" concept, because they thought that "mathematical proofs" was covering a wide range of topics for him.

Methodology the Second Participant Adopted

He studied with 26 grade-11 students. He used qualitative research methods to determine students' understandings. He prepared a test to determine the students' concepts about the subject. It includes 5 open ended questions. While the 5th question is about the application of the concept, others are related to the understanding of the concept.

Conclusion the Second Participant Reached

During the analysis process, he frequently communicated with the instructor to obtain a thorough data analysis. The second participant analyzed the wrong or inadequate answers to find misconceptions about this subject. He determined some misconceptions like the ones in the literature. Some students found unnecessary to prove that the first statement in the infinite sequence of statements is true. Some students checked the first statement only for n=1. Such behavior was sine qua non for some students. Also making a generalization for every condition was difficult for students, so induction was only about `trying every condition` for some students.

Third Participant

At the end of the literature review, the third participant recognized that students have trouble with visualizing data and reading graphs. Then instructor directed her to "reading graphs" subject, because she thought that there were not many studies about this popular subject which is frequently asked in the national examinations. As a result she decided to investigate the students' ability to draw and understand graphs and tables. She tried to explain how students interpret graphs and tables.

Methodology the Third Participant Adopted

At the beginning, she tried to prepare some questions to investigate students' misconceptions. After a while, she denoted that she did not want to write a question for her data gathering tool. After her communication with her instructor, she decided to use some questions from the national examinations for which their reliabilities and validities were checked. She prepared a test consisting of four multiple choice questions. It was administered to 20 grade-10 students. Thus, the students were asked to show some abilities like finding relationships between variables, graphing verbal data, guessing by using graphs etc.

Conclusion the Third Participant Reached

She emphasized that students had understandings like "all graphs must begin at the origin and must pass through the origin". She stated that students had difficulties with constructing relationships between variables. But she could not define their misconceptions clearly and in detail. She mentioned about students' errors but there was not any clear expression about students' misconceptions in her report. The researchers observed that her understanding of misconception was inadequate and they thought that using very few questions and the limited data might cause an obstacle to determining students' misconceptions.

Fourth Participant

Since the fourth participant was influenced by one of the undergraduate courses, she wanted to study the limit concept like the first participant. She was very interested in the concept images and at the end of the literature review she concluded that the teacher presentations of a concept might cause misconceptions. She was very much interested in two examples in the literature that define the limit by a limit of a credit card and the velocity limit. She emphasized that students may be influenced by such examples to develop a misconception. So she stated that investigating teachers' concept images and examples used by them was very important. In order to study this issue, she decided to investigate the teachers' misconceptions related to the limit concept.

Methodology the Fourth Participant Adopted

To gather data, she administered a test that composed of six open-ended questions to 10 math teachers. Questions were about teachers' concept images, connections between their presentations and real life situations and examples that they give in the lessons. She tried to determine the misconceptions of teachers and analyzed obtained data by using qualitative analysis methods.

Conclusion the Fourth Participant Reached

The results showed that the teachers had some misconceptions concerning the limit concept - e.g., they defined the limit of a function with a boundary that cannot be passed, they assumed calculating limit as putting number into the changing variable. Some teachers had difficulties in distinguishing the notion of mathematical limit and the notion of limit in the real life. In conclusion, she found some misconceptions like the ones in the literature and defined these misconceptions clearly. In the literature the following types of misconceptions related to limits are described; confusion over whether a limit is actually reached, confusion regarding the static characteristic of a limit and uncertainty about whether limits are dynamic processes or static objects (Jordan, 2005).

Fifth Participant

The fifth participant was very interested in algebraic expressions and equations subjects while reviewing the existing literature about misconceptions. Then she chose the concepts related to equations to investigate for misconceptions. She emphasized that students' misconceptions related with this subject was one of the important reasons why the students were encountering problems in math lessons. Since the instructor looked for originality in their studies, he asked her that what the difference of her research was from others. Then she decided to investigate the solutions of equations and studied the operational point of equations.

Methodology the Fifth Participant Adopted

To gather data, she administered a test that asks 20 grade-9 students to solve 10 routine equations. She prepared similar questions like the ones in the existing literature for her test. She analyzed students' answers and evaluated their errors. During the analysis process, she also frequently communicated with the instructor to deepen her analysis.

Conclusion the Fifth Participant Reached

She stated that students had arithmetic misconceptions. She mentioned about students' calculation errors. Students had difficulties about arithmetic with negative numbers and using parenthesis. For example in her research report she stated that "My students made some faults like $9 + 3(2x + 5) = 12(2x + 5)$." She catched students' errors but she could not emphasize the misconceptions as a systematic error and she could not define them clearly.

A Brief Comparison

The researchers observed that the participant pre-service teachers were generally influenced by their past experiences and chose their research subject according to their past experiences about the subject. Furthermore their past experiences affect their foresights about determining students' misconceptions and related concepts. The pre-service teachers were also influenced by the past studies in the literature and they tried to make their own studies similar to the ones in the literature. The researcher observed that the pre-service teachers who benefit from the literature got better results and could define students' misconceptions more clearly. Table 2 summarizes the pre-service teachers' findings at the end of their study.

Table 2

Pre-Service Teachers' Findings about Students' Misconceptions

1. Participant	2. Participant	3. Participant	4. Participant	5. Participant
She defined no misconception.	Some misconceptions like the ones in the literature.	There are some but no clear definitions.	Some misconceptions like the ones in the literature.	There are some but no clear definitions.

Except one, the pre-service teachers generally preferred open-ended questions and under the orientation of the instructor they generally used qualitative methods. Since the development of a data-gathering tool is much more than just writing a set of questions, test development process was not easy for the pre-service teachers. The third participant gave up writing questions for the test and decided to use the questions that were used in other examinations. Except one, they did not use the benefits of interviewing much. So the reports were lacking in detailed analysis obtained from interview data.

Particularly the first participant had a difficulty in determining students' misconceptions. The researchers did not encounter enough clear explanations about students' misconceptions in the conclusion parts of the most reports. The researchers concluded that teachers' mathematical content knowledge should be adequate for the concept that they will look for misconceptions. It is a big problem for teachers to determine students' misconceptions if they have lack conceptual background in key topics. Also, it is very important for a teacher to understand the difference between the students' errors and misconceptions. Errors are visible in learners' artifacts such as written text or speech. However, misconceptions are often hidden from the undiscerning observer. Sometimes misconceptions can even be hidden in correct answers (Smith, diSessa, & Roschelle, 1993).

Teachers' Participation in Misconception Research

Teachers learn how to prevent students from misconceptions by investigating them in various subjects. All of the pre-service teachers participated in the study stated that they recommend other teachers to do research on students' misconceptions. They have similar reasons to determine students' misconceptions.

"Of course these studies must be done by teachers. Actually they are the group who communicate with students in the schools. They should determine the problems that arose from the misconceptions."

They understood that they should be careful while they are teaching subjects and giving examples about these subjects. Searching misconceptions get teachers to see students' perspectives and understandings. Teachers learn more about new student profiles.

"I see that there cannot be any teacher or student without misconception. So misconceptions should be dealt with at the beginning..."

Their Opinions about Research Methods Used to Find Misconceptions

Generally pre-service teachers asserted that qualitative methods are more suitable to obtain data about students' misconceptions. Some of them emphasized that the research method should be chosen according to the subjects for which misconceptions are investigated. The following table shows their selections for investigating misconceptions.

Table 3

The Categories Related with Research Tool and its Reason

Participants	Research Method or Tool	Reason
1.	Case study	to analyze students in problem situations
2.	Open-ended questions	to analyses the stages of students' solutions
3.	It changes according to the subject	-
4.	Individual interviews and tests	investigating points that cause misconceptions
5.	Questionnaire	economic and standard written way to obtain data

DISCUSSION

Some misconceptions were revealed from the pre-service teachers' studies. The careful articulation of misconceptions seemed to reflect deep understanding of concepts and strategies for teachers. When two teachers asserted that they would adopt a particular approach to find student misconceptions, the differences in the details offered by those teachers often revealed evidence of differences in PCK, or at least in the application of their PCK.

It should be noted that their reports about finding misconceptions alone were informative, but the follow up mini interviews and the questionnaire gave additional insights and allowed teachers to reveal more of their understanding and repertoire of strategies. It certainly seems, perhaps not surprisingly, that investigating misconceptions is intrinsically labor-intensive. Nevertheless, this approach allows investigation of a wide range of topics.

Characteristically, misconceptions are intuitively sensible to learners and can be resilient to instruction designed to correct them (Smith, DiSessa, & Roschelle, 1993). First of all, educators need to learn to determine and analyze learners' mental models and concept images carefully so that they can correctly follow learners' reasoning. This action research study encouraged pre-service teachers to elicit and interpret students' mathematical thinking. It suggests that the action research courses that are aimed to give a research notion to pre-service teachers, reveals more aspects of the PCK of the teacher.

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Teachers Learning About Student Reasoning Through Video Study

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ABSTRACT: *This paper presents research on teachers learning about critical thinking and reasoning in students through studying videos of student problem solving, by first working on the same problem tasks and studying related reading assignments in an online, graduate-level course. The teacher education study is part of a larger research and development project that makes publically available via the Internet a unique collection of videos from over 25 years of research on the development of mathematical thinking and reasoning in students. The design of the online course intervention is described. Results showing teacher growth in recognizing details of student arguments of two forms of students' mathematical reasoning are reported from a video-based assessment that was administered as a pre and post-test for the intervention.*

Key words: *Children's learning, Classroom research, Critical thinking, Online course, Rational numbers, Reasoning, Teacher education, Video data.*

INTRODUCTION

The study of teacher learning that we present here is part of a larger body of current research[1] that builds on a long-term research agenda. An extensive program of longitudinal and cross-sectional research has been ongoing for a quarter century at Rutgers University to investigate how students build mathematical ideas and forms of reasoning when invited to work on cognitively challenging tasks under conditions that support and encourage student engagement (Palius & Maher, 2011). The research has produced a unique video collection with over 4,500 hours of raw video that both fuels our current work and already has sourced data for analyses resulting in numerous publications and over 30 doctoral dissertations reporting on students learning and reasoning in mathematics. Research detailing the development of student mathematical thinking generated an evolving model for video data analysis involving seven, non-linear interlacing steps (Davis, Maher & Martino, 1992; Powell, Francisco & Maher, 2003). Resulting from this analytical method, numerous studies yielded transcripts of

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video data and identified critical events in the development of mathematical thinking and reasoning. Our current project makes videos, transcripts, and, when available, student work that emanate from seminal, long-term research accessible over the World Wide Web through efforts of a multidisciplinary team on the Video Mosaic Collaborative (www.videomosaic.org).

Video Mosaic Collaborative (VMC) is a multifaceted research and development project that is building a digital repository of videos and related resources, which mathematics education researchers have been using to conduct design research studies in teacher education. Our video collection features students engaged in mathematical problem solving across multiple content strands in classroom and informal settings. The VMC collection was generated by providing students with opportunities to extend their mathematical thinking, typically by engaging them in problem solving before formal introduction of the underlying mathematical content and algorithmic procedures in the regular curricula at their schools, while capturing their activities on video and collecting their written work. Use of these data and the analytical methods described above have enabled detailed studies of how certain mathematical ideas were built and how forms of reasoning naturally emerged in the various learning environments (e.g., Maher, Powell & Uptegrove, 2010; Francisco, 2005; Maher, 2005; Steencken & Maher, 2003; Maher & Martino, 1996). Supported through multiple grants[2] from the U.S. National Science Foundation, our video-based research includes (1) a yearlong study in a fourth-grade classroom where students make sense of fraction as number and build conceptual understanding for operations with fractions; (2) a three-year study of informal mathematics learning with urban, middle-school students in an after-school program; and (3) a seminal longitudinal study that followed a cohort of students from elementary through high school and beyond to adulthood. Well-documented examples of students' mathematical reasoning, which were initially identified through research conducted by many scholars, have now been prepared by the VMC team and populate a searchable database called the Video Mosaic.

The current design research on growth in teacher knowledge has deep roots that extend back more than 30 years. An intervention aimed at helping teachers move from a traditional, direct teaching approach of transferring mathematical information to one in which the students are actively engaged in building understanding of the mathematics, was guided by a multi-part model of teacher development (Maher, 1988). Grounded in that project and subsequent teacher professional development work that made use of videos (Maher, 1988; Maher, Davis & Alston, 1992; Maher, Alston, Dann & Steencken, 2000), we have adapted models that utilize VMC resources in order to help teachers to attend students' mathematical reasoning. As reported at an international conference (Palius & Maher, 2011), one model has focused on the education of pre-service teachers, and the second has been geared towards professional development of in-

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service teachers. These models have guided the implementation of courses and workshops that engage teachers in face-to-face settings. We also have implemented a different kind of intervention model that was created specifically for use in the context of online learning with digital resources (Maher, Palius, Maher & Sigley, 2012).

An experimental online course was designed to deepen teachers' awareness of how students develop their reasoning of fraction ideas when videos and related literature are studied and discussed critically. Using a video-based assessment instrument, the study presented here examines what change, if any, occurred in teachers' recognition of students' reasoning from pre to post with the online course as an intervention. We analyze assessment data for two different forms of reasoning with respect to the details of two arguments expressed by children in the video.

To situate the study of teacher learning about student reasoning in an online course environment, we outline key theoretical perspectives that both underlie our long-term research program and guide our current research in teacher education. We then describe the online course setting and use of videos as a context for research before presenting methods used in our study.

THEORETICAL PERSPECTIVES

A multi-decade program of research has been guided by constructivist views on the learning and teaching of mathematics (Davis & Maher, 1990; Maher & Davis, 1990; Davis, 1990). When mathematics is viewed as a sense-making activity, learners cycle through a process of creating representations, retrieving or constructing relevant knowledge, mapping representations to knowledge, and using them as a basis for action toward problem solving. Teachers, or researchers functioning in their stead, assume the role of facilitator in the learning environment, with the goal of having students view mathematics as a subject where one thinks creatively and understands the ideas. This perspective comes about from active engagement in building meaning, sometimes by making mistakes along the way, in order to learn what does and does not work to solve problems (Maher, 1988). Teachers, in their role as facilitators of student learning, can empower students to discover various ways to solve problems and learn the big mathematical ideas and ways of reasoning, as they orchestrate opportunities for students to work together and learn from one another (Maher, 1998). It also is the teacher's role to encourage learners to create personally meaningful representations, as a sense-making basis for subsequent introduction of mathematical notation, and to foster students' abilities to communicate explanations in justification of strategies used and solutions found (Maher & Weber, 2010; Maher, Powell & Uptegrove, 2010). Supporting the growth of these functions is how mathematics teachers help their students develop along the continuum from reasoning, explaining, and justifying towards articulation of formal proof (Yackel & Hanna, 2003).

Within this constructivist perspective, pedagogy is based on creating an appropriate assimilation paradigm as an experience where students build representations and map to knowledge for constructing new ideas. It thus provides a learning opportunity in which students actively discover key ideas in their experiences. As Davis (1990, p. 102) asserts, "Your mental representations must give you the power to see new possibilities and new constraints in new situations." Teachers responsible for developing students' abilities for verification, explanation, and communication in the context of doing mathematics are faced with the challenge of developing their own adaptive expertise as educator (Bransford, Derry, Berliner, & Hammerness, 2006). That is, good teaching demands the ability to spontaneously and flexibly identify, critically evaluate, and respond in appropriate ways to instances of children's learning. Of particular concern to the field of math teacher education is helping teachers to attend to emerging forms of reasoning as children express justifications using their own language. It is from this pedagogical perspective that our studies in teacher education have been designed.

Over many years and in multiple settings, researchers created conditions in the learning environments where studies were conducted that enabled student building of mathematical ideas and fostered the development of reasoning. Conditions pertain to the nature of problem-solving tasks, time for exploration and revisiting, and establishing norms for both social and mathematical behaviors (Maher, 2005). Creating those conditions has been a consistent factor across all of our studies, whether the learners be children or adults, situated in formal or informal settings, from urban or suburban communities (Palius & Maher, 2011).

In the long-term studies with children, facilitation and questioning by researchers prompted students to offer convincing arguments, which evolved into the expectation that it should be a mathematically sound and convincing argument (Maher, 2005). Importantly, the researcher was never the sole arbitrator of what was convincing, as students also had to convince one another that solutions were valid. Thus, the learning environments supported the development of students' reasoning and justification (Maher, Powell & Uptegrove, 2010). Videotaping with multiple cameras captured the developmental process and enabled it to be studied carefully. Findings from such studies inform the research community (e.g., Maher & Martino, 1996; Francisco & Maher, 2011); and videos of students reasoning and justifying in their problem solving serve as an important tool in mathematics teacher education (Maher, 2008).

Making use of video episodes with transcripts that illustrate children's reasoning in mathematical problem solving, our research investigates whether teachers are able to build the mathematical and pedagogical knowledge for recognizing components of children's reasoning. Instructional interventions with teachers are premised on the notion that they must not only know how to solve math problems, but they must also come to understand well the reasoning that justifies valid solutions to those problems (Palius & Maher, 2011). They also must move beyond their own way of approaching and solving a problem to recognize that more than one solution strategy and form of reasoning may be valid when working on a particular task, as all students do not think alike and one of our goals is development of adaptive expertise (Bransford et al., 2006). We hypothesize that our VMC collection can provide the affordances that enable teachers to develop such skills.

The design research that we have been conducting in teacher education contexts has been testing this hypothesis in several intervention iterations. Our work thus far has focused on the mathematical strands of counting/combinatorics and fractions, with pre and post assessment data to measure impact on teachers' beliefs about learning and teaching mathematics (Maher, Palius & Mueller, 2010; Maher, Landis & Palius, 2010) and on teachers' abilities to recognize students' reasoning (Maher, 2011; Maher, et al., 2012). Instructional interventions take place in the context of a university course or school-based workshop series, and generally follow one of two models (Palius & Maher, 2011). They begin by engaging teachers as learners in problem-solving tasks and require that they provide convincing arguments for solutions they find. An essential component of problem solving is the construction of representations that model solutions, and manipulative tools sometimes are made available for use. Teachers' own problem solving is an important prerequisite for subsequent study of videos. Several of our course-based interventions have included an online component, which enable threaded discussions to discuss problem solving, videos viewed, and readings that have been assigned. Online discussions also have served as a forum to talk about implications for bringing problem solving tasks into teachers' own classrooms or even sharing results from having done so.

The study presented here uses an online course as a context for design research of an intervention that targets the fractions strand. The guiding question for this study is: What change, if any, occurs in the participating teachers' recognition of students' reasoning from pre to post, as measured by a video-based assessment instrument, with the online course as an intervention?

METHODS

Online Course Design

We designed and implemented an online course in mathematics education to be taken as an elective by graduate students. The course, Critical Thinking and Reasoning, is almost entirely online, with just a single in-person meeting near the start of the term. This intervention focuses on the fractions strand with tasks that involve modeling numerical situations and relationships with Cuisenaire rods (see Figure 1). The face-to-face meeting serves to give participants an opportunity to personally connect with one another and have time to explore the rods and how they might be used to explore fraction ideas. We begin by establishing the condition that the rods have permanent color names but that their number names can vary. A few examples of tasks are given to demonstrate how the rods could be used for problem solving about fraction ideas, for instance: If we call the orange rod one, what number name would be given to the yellow rod? And what number name would we give to the red rod? Now if we call the blue rod one, what number name would be given to the light green rod? The intent of these activities is for participants to become comfortable with the idea of using rods and combinations of rods to define a unit, so that fractional relationships can be examined and fractions can be compared and examined using rod models (Palius & Maher, 2011).

An objective of this activity for the teachers is to set the stage for their subsequent viewing of children's mathematical activities that are captured on video and studied by participants in the online course. The videos selected from the VMC collection feature earlier research on how the rods and corresponding tasks were used with children before they were formally introduced to fraction operations. The videos reveal that conceptual understanding of operations with fractions can naturally emerge using the rods as tools to make models, through collaboration and sharing, and through teacher facilitation of problem solving when appropriate classroom conditions are in place (Steencken & Maher, 2003).

Figure 1. Staircase model of the set of Cuisenaire rods.

Thus, the online course was designed to make available video clips as a tool for mathematics teacher education. One purpose was to stimulate participants to consider how children in a 4th grade classroom study engaged in deep and critical mathematical thinking through problem solving on tasks in the fractions strand. A second purpose was to illustrate how the researcher serving as classroom teacher engaged the children and facilitated the learning environment. Participants in the course also had an opportunity to learn that certain tasks tend to elicit particular forms of reasoning (Yankelewitz, Mueller & Maher, 2010). Research literature connected to the video content was assigned as readings to comprise course units around which online discussions were focused. Consistent with the larger program of design research, we examined teachers' attention to children's reasoning before and after the intervention. In particular, we investigated the nature of teacher growth in identifying specific features that comprise a complete argument for two different forms of reasoning expressed by children in a video-based assessment.

Assessment

As a pre and post–assessment, the participants were shown a video of children supporting their solutions to a fraction task. The assessment prompted study participants to describe as completely as they can the reasoning that the children put forth, whether each argument offered by children is convincing, and why or why not are they convinced. The video includes footage from research conducted in an after-school enrichment program for 6th graders in an urban community, where children engaged in many of the same tasks that were explored by children in the 4th grade classroom study (Maher, Mueller, & Yankelewitz, 2009). It contains short clips of children working in groups on a task to find a Cuisenaire rod in the set (see Figure 1) that could be given the number name one-half when the Blue rod has been given the number name one. It also contains short clips of children explaining their solution ideas with rod models as justification to the whole class (Maher, Mueller, & Palius, 2010). Participants were provided with a transcript for the video and were given sufficient time to respond. The assessment prompt informed participants that their responses would be evaluated by the following criteria: recognition of children's arguments, their assessment of the validity or not of children's reasoning, evidence to support their claims, and whether the warrants they give are partial or complete.

It should be noted that the children in the assessment video offered various explanations for why they found that there is no rod in the set that can be called one half when the Blue rod is called one. Some of the explanations took the form of reasoning by cases, which included identifying rods as being even (i.e., another rod in the set could be its half) or odd (i.e., there was no rod in the set that could be its half). This study examines teacher recognition of a specific case argument expressed by the children about why the Blue rod, in particular, does not have a half because it is odd. This study also examines recognition of an argument that took the form of reasoning by upper and lower bounds, which was an idea that had emerged in a prior research setting as well as the one with urban middle-school students (Maher & Davis, 1995; Yankelewitz, Mueller & Maher, 2010). An interesting aspect of the assessment video was that more than one child's discourse contributed to the articulation of certain argument forms. We were interested in learning the extent to which teachers would recognize that children were expressing in their own language that the solution for half of Blue is bounded by the Yellow and Purple rods, with Yellow being the least upper bound and Purple being the greatest lower bound (i.e., that there is no rod in between them).

Online Course Interventions

There were two interventions done in an online, graduate course: one in the spring semester and the other during the summer session. There were 12 students in the first intervention and 10 students in the second intervention. For the 22 study participants, 50% were male and 50% were female; 50% were master's students and 50% were doctoral; 19 specialized in mathematics education, 2 specialized in educational administration, and 1 specialized in learning sciences. All but one had teaching experience at middle school, high school, or post-secondary level, and the amount of experience ranged from less than one year to more than twenty.

The two implementations of the online course were very similar in content despite varying durations. The first intervention was semester-long and contained eight course units with 15 videos and readings plus a ninth unit for a group project that made available five more videos. The second, month-long, contained seven course units with 17 videos and readings but no group project. Notably, both interventions contained a unit that focused specifically on children's mathematical reasoning about the fractions task in the video assessment. Students were assigned to study two videos, *Fractions, Grade 4, Clip 1 of 4: David's upper and lower bound argument* (http://hdl.rutgers.edu/1782.1/rucore00000001201.Video.000054465) and *Fractions, Grade 4, Clip 4 of 4: Designing a new rod set* (http://hdl.rutgers.edu/1782.1/rucore00000001201.Video.000054751).

The reading assignment for this unit was a book chapter that discussed children's mathematical exploration that leads toward proof-like reasoning, which included the example of David's upper and lower bounds argument (Maher & Davis, 1995). The prompt for group online discussions was open-ended and suggested that attention be paid to forms of children's arguments and the evidence they provide, as well as consideration of what may be evidence of obstacles encountered by the children or of their understanding of the mathematical ideas. Students were assigned to small groups for engaging in online discussions about the videos they were viewing and the related literature.

Coding and Reliability

A detailed rubric was developed by the research team in order to code the data for the presence, or absence, of the components for each of two different arguments that were articulated by the children in the assessment video. Two researchers independently scored assessment data with 90.4% inter-rater reliability. For the upper and lower bounds argument, there were four components of the children's reasoning that could combine in three different ways to be a complete argument (a, b, and c; a, b, and d; or a, b, c, and d):

a. The Yellow rod is (1/2 of one White rod) longer than half of Blue; (AND)
b. Purple is (1/2 of one White rod) shorter than half of Blue; (AND)
c. There is no rod with a length that is between Yellow and Purple; (OR)
d. The White rod is the shortest rod and the difference between the Yellow rod and the Purple rod is one White rod.

Participant responses that did not mention any of the above components or that mentioned only one or two of them were deemed to be incomplete. Although it was considered a possibility that a participant might identify the upper and lower bounds argument simply by naming as such (i.e., not describing any of its features: a, b, c or d), that did not occur with this set of participant data. For the case argument, there were three components of the children's reasoning that could combine in three different ways to make a complete argument (f and g; f and h; or f, g, and h).

f. The Blue rod is the same length as 9 White rods; (AND)
g. Nine is an odd number and cannot be divided evenly. “Halving it” would result in a decimal or remainder; (OR)
h. A rod half as long as the Blue rod would have to be the length of 4.5 White rods and there are no rods that are of “decimal” length.

Mentioning only one of these components, or g and h without f, was deemed to be incomplete, as was not mentioning any of the features of this argument. The coded data were analyzed quantitatively.

RESULTS

The data in this study indicate that from pre to post-assessment the majority of teachers as learners in the online graduate course were able to recognize and describe in complete detail two different arguments expressed by children in the assessment video. The analysis reports the significant growth rate achieved by the 22 study participants from pre to post-video assessments. We first present findings for the argument by upper and lower bounds, with details of pre-assessment and post-assessment results in Table 1, and a transition matrix in Table 2 to show how and where the growth in recognizing children’s mathematical reasoning occurred. Note that transition was only tabulated for those participants who did not express a complete argument on the pre-assessment, as that is where there was a potential for improving in recognition and description of the reasoning.

Table 1

Assessment Results for Argument by Upper and Lower Bounds

Pre-Assessment Classification Scores									
Count by Number and Percentage	Complete Detailed Description			Partial Description				No Description	Total
	ABC	ABD	ABCD	A	AB	C	D	None	
Study Participants	4 18.18	2 9.09	3 13.64	1 4.55	1 4.55	1 4.55	2 9.09	8 36.36	22

Post-Assessment Classification Scores										
Count by Number and Percentage	Complete Detailed Description			Partial Description					No Description	Total
	ABC	ABD	ABCD	B	D	AB	AD	BD	None	
Study Participants	4 18.18	3 13.64	5 22.73	2 9.09	1 4.55	4 18.18	1 4.55	1 4.55	1 4.55	22

As shown in Table 1 above, 9 of 22 (40.9%) of study participants in the online course were successful in providing a detailed complete description of the argument by upper and lower bounds on the video-based pre-assessment, and 13 of the 22 failed to do so. Table 1 also reports post-assessment results for all 22 participants. Table 2 focuses on post-assessment outcomes for only those who failed to describe a complete argument on their pre-assessment. Of these 13 graduate students, 3 provided a detailed complete description on the post assessment; 8 others provided additional component details on their post-assessment written description of children's reasoning in the video. In total, 11 of the 13 participants exhibited growth on their post-assessment. Thus, the growth rate on description of argument by upper and lower bounds is 84.6%. A 95% confidence interval for this growth rate is 57.8% to 95.7%. Only 1 student of the 13 (7.69%) failed to observe any features of this argument.

Table 2

Argument by Upper and Lower Bounds Transition Matrix: Pre vs. Post Assessment Classification for Participants Who Did Not Provide A Detailed Complete Description on the Pre-Assessment

Pre-Assessment Classification	Post-Assessment Classification: Complete Detailed Description			Partial Description					No Description	Row Totals
Count by Number and Percentage	ABC	ABD	ABCD	B	D	AB	AD	BD	None	
A							1 7.69 [b]			1
C	1 7.69 [a]									1
D					1 7.69 [c]			1 7.69 [b]		2
AB		1 7.69 [a]								1
None			1 7.69 [a]	2 15.38 [b]		4 30.77 [b]			1 7.69 [c]	8
Column Totals	1	1	1	2	1	4	1	1	1	13

[a] Signifies growth to a complete detailed description. [b] Signifies growth to a more complete description. [c] Signifies no growth.

Turning to the case argument, we report a parallel analysis of video assessment data in the following tables. The details of pre-assessment and post-assessment results for the case argument are shown in Table 3 for all 22 participants. We then present a transition matrix in Table 4 to show how and where the growth in recognizing children's mathematical reasoning occurred. Again, note that transition was only tabulated for those participants who did not offer a complete argument on the pre-assessment because that is where there was a potential for improving in recognition and description of the reasoning.

Table 3

Assessment Results for the Case Argument

Pre-Assessment Classification Scores						
	Pre-Assessment Argument Description Classification					
Count by Number and Percentage	Complete Detailed Description			Partial Description	No Description	Total
	FG			H	None	
Study Participants	16 72.73			2 9.09	4 18.18	22
Post-Assessment Classification Scores						
	Post-Assessment Aggregate Description Classification					
Count by Number and Percentage	Complete Detailed Description			Partial Description	No Description	Total
	FG	FH	FGH	F	None	
Study Participants	17 77.27	2 9.09	1 4.55	1 4.55	1 4.55	22

As shown in Table 3, 16 of 22 (72.7%) of the study participants were successful on the pre-assessment in providing a detailed complete description of the case argument and 6 of 22 failed to do so. Twenty participants described a complete argument on the post-assessment, but one only gave a partial description and one other described no component of it. Table 4 focuses on post-assessment outcomes for only those who failed to describe a complete argument on their pre-assessment. Of these 6 participants, 4 provided a detailed complete description on the post assessment and 1 other provided additional component details on the written description for their post-assessment. In total, 5 of these 6 graduate students exhibited growth on the post-assessment in describing the case argument; thus, the growth rate for this argument is 83.3%. A 95% confidence interval for participant growth rate on the case argument is 35.9% to 99.6%. Only 1 of 22 (4.5%) participants failed to mention the children's case argument on the post-assessment.

Table 4

Case Argument Transition Matrix: Pre vs. Post Assessment Classification for Participants Who Did Not Provide A Detailed Complete Description on the Pre-Assessment

Pre-Assessment Classification	Post-Assessment Description Classification					Row Totals
	Complete Detailed Description			Partial Description	No Description	
Count by Number and Percentage	FG	FH	FGH	F	None	
H		1 16.67 [a]	1 16.67 [a]			2
None	1 16.67[a]	1 16.67 [a]		1 16.67[b]	1 16.67[c]	4
Column Totals	1	2	1	1	1	6

[a] Signifies growth to a complete detailed description. [b] Signifies growth to a more complete description. [c] Signifies no growth.

DISCUSSION

The data provide evidence that by the completion of the post-assessment the majority of teachers as graduate students in the online course, specifically 12 out of 22 or 54.5%, recognized and described in complete detail the children's argument by upper and lower bounds from the video assessment. By the completion of the post-assessment, all but one student provided at least a partial description of one or more major components of this argument. The data also provide evidence that by the completion of the post-assessment over 90%, specifically 20 out of 22, were able to recognize and describe in complete detail the children's case argument as presented on the video assessment.

Situating these findings in the context of the elective, online graduate course in mathematics education, the following considerations merit explicit discussion. First, it should be noted that direct instruction in recognizing and describing forms of reasoning that constitute valid mathematical arguments was not an objective of the course. Consistent with the theoretical perspectives outlined in this paper, the researchers who also served as instructors were interested in seeing how teachers as learners would build knowledge that they did not already have about how children as students develop and express mathematical reasoning through problem solving by having opportunities to study the process in detail through videos. Much in the way that children in the videos were situated in a learning environment that supported their mathematical development, the graduate students were invited to participate in an online learning environment with tools, resources, and conditions designed to support their knowledge construction through purposeful activity.

Teachers participating in the course had the opportunity to build their own models with the Cuisenaire rods to examine more deeply the numerical relationships that the children were exploring in the videotaped research project. By studying those videos along with research literature citing evidence of children's mathematical learning from those video episodes, then reflecting and discussing observations with groups of their peers, teachers were constructing knowledge about how the children were developing mathematical reasoning about fraction ideas. Teachers had opportunities to build a richer content knowledge of fractions by attending to the conceptual underpinnings of what they already knew procedurally because they were asked to focus on how children in the videos were constructing meaning of fractional numbers, working through obstacles in their learning, and exploring relationships about fraction ideas without procedural instruction. Teachers also had opportunities to examine their pedagogical content knowledge by attending to how the researcher, acting as teacher, in the videos engaged the class of fourth graders in expressing their mathematical ideas, supporting their conjectures with Cuisenaire rod models, and evaluating one another's arguments. The videos provided teachers with opportunities to build insights into the processes of how mathematical ideas can be constructed by children in the classroom. They also reveal vividly the climate of the classroom environment and offer new ways in which a classroom teacher can manifest similar conditions. The videos and the readings reinforce one another as resources since they provide teachers with visual and research evidence of children's learning. Finally, the online discussions provide a rich forum for self-expression and evaluation of one another's ideas. In these ways, the teachers' learning process in the online course is similar to the learning environment that they were invited to witness unfolding among the children in the videos from classroom-based research.

A second consideration involves the strengths that participants may have brought to the online course in terms of their own mathematical content knowledge and prior teaching experience. For both interventions, nearly everyone had at least some teaching experience although, for most participants, it occurred at the secondary or post-secondary level, and some at the upper-elementary or middle grades. However, the intervention that took place in the summer session included two participants taking the course as an elective for their doctoral program in educational administration and supervision. Of those two, one indicated in her personal survey (collected from all students at the start of the course) that in her nine years of being a special education teacher for grades 6-8 she has taught mathematics although was not doing so at the time enrolled in the course. The other indicated that he has been "an educator for almost 20 years" but did not specifically indicate whether teaching math was included in his experiences. We can associate math content knowledge with teaching experience by noting that certification to teach secondary math requires minimally 30 credits of undergraduate mathematics, and teaching at the post-secondary level requires a master's degree in mathematics. Thus, the outcome of the 72.7% of participants who described complete case argument and 40.9% who described complete argument by upper and lower bounds on their pre-assessment was not surprising, given that the task was to make sense of children expressing the mathematical reasoning in their own words, as it was emerging as knowledge before instruction that introduced more formal mathematical language. Nor was it a surprising result that the one study participant who failed to describe any features of either argument was the school administrator who may never have taught mathematics at any grade level.

Further exploration of relationships among mathematical content, pedagogical knowledge and performance on video-based assessments of children's reasoning is an area for future study. However, the small sample size is a constraining factor for any attempt to parse out which individual characteristics of the study participants may account for variations in their assessment results. This study was limited in its analysis of assessment data as a preliminary investigation of teacher learning in the experimental online course. However, it serves as a foundation for a deeper probe into what course participants discuss online in their study of video episodes with related readings from research literature and how, if at all, these outcomes might influence their performance on the video assessments to identify mathematical reasoning (Palius, 2013).

Discourse analyses provide further insight into how the use of VMC resources in teacher education focuses attention on the development of students' mathematical reasoning. Examining the learning process by analyzing characteristics of the instructor, the individual students, the instructional resources, and how these elements combine into technology affordances for online education will reveal how essential knowledge can be constructed to inform teaching practices that build on children's emergent reasoning.

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Models and Modeling in Mathematics Education: Making Learners' Mathematical Thinking, Knowledge and Skills Visible

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ABSTRACT: *Mathematical tasks that provide a solution space rather than a prescriptive solution pathway generate opportunities for students to connect mathematical thinking and mathematical competencies with real life. Model-Eliciting Activities (MEA) go beyond creating these opportunities and provide rich information about students' mathematical knowledge and skills while also helping teachers identify strengths and weaknesses of students promptly to adjust their teaching. These affordances may not be possible with traditional simple-answer/single-solution tasks or word problems. This study presents an example of an MEA asking students to develop a mathematical model to maximize the profit in a particular context that was given to 30 mathematics and science teacher candidates. Students worked on the task in groups of three for two class periods of 60 minutes. Researchers walked around the groups, listened to their discussion, took observation notes while students worked on the task. Students' written drafts of solutions and final solutions were also collected. These qualitative data were analyzed in order to identify different and similar ways of solutions and mathematical knowledge and concepts used. The results showed that although most of the groups used a derivative concept to create a mathematical model, other mathematical concepts and tools such as graphing, tables, coordinate axes, and vertex were utilized as well. The most interesting finding was that all groups went through some kind of a test-revise cycle. This characteristic of the MEA is very meaningful as it mirrors problem-solving situations in real life.*

Key words: *Constructivism, Formative assessment, Mathematics education, Models and modeling, Model-eliciting activity, Learning and teaching.*

MATHEMATICAL TASKS FOR THE 21ST CENTURY

This study presents a complex mathematical task, a model-eliciting activity (MEA), which requires problem-solvers to develop a *model* utilizing mathematical concepts and skills as a solution rather than a single numeric answer. The importance of these types of mathematical tasks is obvious given the 21st century knowledge and skills (National Research Council [NRC], 1996). Mathematical literacy for all students has been an enormous concern of educational reform movements within the two last decades all over the world (Organisation for Economic Co-operation and Development [OECD], 2010). Moreover, recent conception of mathematical literacy has been shifted from being able to perform algorithms, memorizing prescribed mathematical rules and formulas to ability to solve complex problems and develop models that would require higher order thinking skills. As the needs of and skills used in science and mathematics related workforce, the call for implementation of inquiry-based, model-based, and/or problem-based approaches in mathematics teaching has taken on a new sense of urgency (NRC, 1996). Standards reform movement emphasizes the importance of classroom tasks that do not separate mathematical and scientific thinking from concepts and skills, that encourage multiple solutions and models to a problem, and that make connections to real-life. The use of mathematical models to represent and understand the quantitative relationship embodied in a real life problem needs to be promoted in today's education (Hoyles, Morgan, & Woodhouse, 1999; NCTM, 1991, 2000; OCED, 2010). Furthermore, prestigious organizations such as American Association for the Advancement of Science (AAAS), and the International Association for the Evaluation of Educational Achievement (IEA) underscores that systems and models are two important aspects of themes that embrace science, mathematics, and technology (AAAS, 1998; Beaton et al., 1996).

The MEA implemented to a group of science and mathematics teacher candidates in this study makes an example of how such tasks, defined and characterized above, could be powerful in demonstrating what students know and think mathematically. Modeling activities could be used in many ways: introducing a new topic in a meaningful way, teaching a mathematical concept in an effective way, and assessing students' mathematical knowledge and competencies both for formative and summative purposes. The modeling activity used in this study was used as a formative assessment tool. These tasks differ substantially from traditional word problems, drill and practice problems and procedural problems that merely dominated mathematics curricula for over decades. This paper discusses the affordances of these tasks and how they are different from conventional mathematics problems.

The remaining sections of this paper explain why mathematics literacy is so crucial and what the current approach and recommendations to reaching mathematical literacy are. Then, model-eliciting activities and their extensions are introduced. The *Research on Models and Modeling* section presents a summary of studies revealing why modeling and model-based teaching and learning are important and how they could improve mathematical literacy. After presenting the study and what has been found, lastly, this paper concludes with the discussion of model-eliciting activities from a formative assessment perspective.

THE RATIONALE

There is an obvious decline in the number of American students enrolling in science and mathematics related programs (National Academy of Sciences, National Academy of Engineering, and Institute of Medicine, 2007). Students who do pursue a technical career often find themselves not very well prepared for the rigors of real-world problem solving. Many educators and policy-makers worry that this decline in science and mathematics literacy could have long-term effects detrimental to the national economies (Bybee, 2007; NRC, 1996; Sensenbrenner, 1998; Toulmin & Groome, 2007). For example, National Mathematics Advisory Panel formed upon President's request recently prepared a report based on an extensive review of the research in science, technology, engineering, and mathematics (STEM) education in 2008. This report emphasizes quantitative reasoning and mathematics education being to be very crucial to the national leadership, well being, economy, and security as well as to the improvement in technology, medicine, commerce, and overall future developments (National Mathematics Advisory Panel, 2008).

These concerns made educators, researchers, and policy makers to call for inquiry-based, model-based, project-based, and/or problem-based approached to science and mathematics teaching and learning. Not only do these approaches differ from one another in terms of their design principles, focus of content areas, implementation and so forth, they also overlap at many points regarding the theory behind them, the themes they embrace, and the objectives at which they aim. For example, inquiry-based and project-based approaches are usually used in science areas, and problem-based and model-based approaches usually take their places in mathematics classrooms while they all focus on learning by understanding and take on constructivist learning approach. So, although these non-traditional approaches may have different names and pursue different ways in going beyond the traditional views of learning, in essence they all embrace a constructivist approach that standards reform movement call for. MEAs are only one of constructivist approaches among many mentioned above that are informed by continually developing learning sciences and that are called for in almost every comprehensive reports prepared by leading educational institutions.

The rationale of studying MEAs is twofold: first is the need to improve STEM education in the U.S. Secondly, there is a growing interest in non-traditional teaching methods. MEAs are promising examples of these constructivist approaches as the cutting edge of mathematics learning, teaching, and evaluation as in other STEM disciplines. Empirical research has found that these innovative approaches to STEM education provide a deeper and interdisciplinary understanding of STEM concepts, while at the same time narrowing the achievement gap in science and mathematics related disciplines (Diefes-Dux et al., 2004).

Although, this study might seem to be concerned only about the STEM education in the U.S., this is not the case. It is true that this study took place in the U.S. However, the lessons learned from this study, results and conclusions made here could also apply to other education systems worldwide to improve mathematics education. It might be useful to remind that models and modeling is a universal concept recognized internationally as a very important aspect STEM education (English, 2003; Eric, 2008; Lesh & Doerr, 2003; Vos, 2007).

MODELS AND MODELING IN MATHEMATICS EDUCATION

Model-based approaches assume that the fundamental principle of mathematical and scientific thought is the understanding of systems: "Knowledge development in the areas of mathematics and science utilizes the construction of systems to interpret, explain, predict, and describe situations and phenomena" (Ashmann, Zawojevski, & Bowman, 2006, p. 192). Whether it is a system of rational numbers, an ecosystem, a solar system, or a circulatory system, students are taught to approach these systems - defined by the relationship of each constituent part - with the aid of a model. In this case, a model is a conceptual structure that enables students to represent such systems effectively. Learners then could apply these models to solve authentic or real-world problems (Hamilton, Lesh, Lester, & Brilleslyper, 2008). American Association for the Advancement of Science (AAAS, 1990) defines models as

> A model may be a device, a plan, a drawing, an equation, a computer program, or even just a mental image. Whether models are physical, mathematical, or conceptual, their value lies in suggesting how things either do work or might work. (p. 168)

Lesh and Doerr (2003) also defines models as:

> Models are conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behavior of other system(s)-perhaps so that the other system can be manipulated or predicted intelligently. (p. 10)

Therefore, models could be any conceptual structure that a student uses to solve a problem or to understand the structure of complex systems, and modeling refers to the internal process by which students construct and adapt these conceptual models to solve a real-world problem or understand dynamic structures (Clement & Rea-Ramirez, 2008; Lesh, Hoover, Hole, Kelly, & Post, 2000). This process is called either *"mathematizing" the problem* by some researchers, as students generate not simply a solution but a system that can be applied to solve multiple problem scenarios (Lesh & Kaput, 2007); or *mental modeling* by others, as students abstract conceptual models of systems (Clement & Rea-Ramirez, 2008). Clearly, the concept of modeling has been considered a foundational part to the mathematics education since the beginning of this century and an important agenda in mathematics education research.

Below, the three examples of most widely used modeling activities are presented. Summary descriptions for model-eliciting activities, model-exploration activities, and model-adaptation activities are given. Then, some examples of studies on models and modeling demonstrating positive effects on mathematics learning are provided.

Model-Eliciting Activities

Model-Eliciting Activities are "open-ended, real-world, client driven problems that require the creation of a mathematical model for a given solution" (Diefes-Dux, Bowman, Zawojewski, & Hjalmarson, 2006, p.52). These activities require students to engage in problem solving processes, evaluating their solution against real-world needs. These activities reflect the complexity of actual STEM challenges in which the correct solution method is not readily apparent, and client needs and experimental limitations must be considered. MEAs also enable educators the chance to evaluate not simply a student's answer, but their entire thought process. This method of evaluation has the advantage of allowing teachers to more accurately address any conceptual problems that their students may be having. Lesh et al. (2000) outlined six principles to ensure that new problems are effectively Model-Eliciting and therefore, "*thought-revealing*" (Lesh et al., 2000):

1. *The Model Construction Principle*: Is it desirable for the goal of the activity to include the development of an explicit construction, description, explanation, or justified prediction?
2. *The Reality Principle*: Can students make sense of the situation based on extension of personal knowledge and experience?
3. *The Generalizability Principle*: Does [the model] provide a way of thinking that is shareable, transportable, easily modifiable, and reusable?
4. *The Self-Assessment Principle*: Are students able to judge for themselves when their responses need to be improved, or when they need to be refined or extended for a given purpose?
5. *The Construct Documentation Principle*: Will responding to the question require students to reveal explicitly how they are thinking about the situation?
6. *The Effective Prototype Principle*: Does the solution provide a useful prototype, or metaphor, for interpreting other situations?

Model-Exploration Activities

A Model-Exploration Activity (MXA) typically follows an MEA, and introduces the students to several conventional professional engineering models that could be used to solve the problem introduced by the MEA. In the MXA, students consider their own models in comparison to traditional engineering models, evaluating the implications and assumptions associated with each (Diefes-Dux et al., 2006). After applying their own model and the traditional model to the problem set, students compare the results. This self-evaluation process forces the students to consider the practical effectiveness of their own model, and often reveals to them the benefits of viewing a problem from several different perspectives.

Model-Adaptation Activities

The last part of the model development sequence is the Model-Adaptation Activity (MAA), in which students are given a new problem or situation, and must use their developed model to generate a solution. In the process of applying their solution to a new problem, students learn to value the "generalizability" of solutions, and experience the challenge of real-world engineering in which solutions must be flexible to remain effective (Lesh, Hoover, & Kelly, 1993).

Given these three examples of modeling activities, model sequence can serve as "vehicles for teacher candidates and their eventual students to be introduced to the notion of systems and models in science and mathematics" (Ashman et al., 2006, p.192). This topic is at the frontier of STEM learning, education, and evaluation. Modeling activities could help towards significant, widespread enhancement of STEM teaching methods. MEAs have been proven useful in high school and college level classrooms. For example, Purdue University has fully adopted MEAs in its first year engineering courses, and demonstrated the effectiveness of this method on group problem solving and complex mathematical thinking (Keller, 2007; Moore, Diefes-Dux & Imbrie, 2005, 2006, 2007).

Research on Models and Modeling

Researchers found many positive outcomes as a result of teaching through modeling tasks (e.g., Eric, 2008; English, 2002, 2003). Some of the studies paid particular attention to social gains such as collaborative working, communication skills, and interactive engagement while others investigated meta-cognitive developments. Although there are many studies reported in the literature, selected ones that focused only on mathematics domain will be presented.

Eric (2008) did a classroom study involving two sixth grade classrooms comprising small groups of four or five students via a modeling perspective to determine the students' conceptual interpretations during problem solving. He found that children could develop conceptual interpretations mathematically and engage in mathematizing as the problem solving-becomes more authentic. What was more striking in the study was that students were seen working independently of the teacher most of the time, and therefore, more self-directed. Teachers were also found to find opportunities to elicit their students' thinking. He concluded that students were working in similar ways that engineers do (Ashmann et al., 2006) by means of the nature of this modeling activity.

English (2002 and 2003) studied key features of mathematical modeling activities. He found that children made significant mathematical and "social" gains from working authentic modeling problems. English (2002) also studied children's mathematical and social development over the course of modeling activities with a class of 30 ten-year-olds and their teachers from a co-educational private school. He found that children progressed from focusing on isolated subsets of information to applying mathematical operations that helped them aggregate the given data.

In a study by Ikeda, Stephens, and Matsuzaki (2007) on students' thinking about the key features and important aspects mathematical modeling, modeling problems were found to be somewhat effective even in multiple choice format when combined with small group discussion and careful teacher direction. This study shows that when modeling aspects are incorporated into the instruction in some way, students talk about problems with using important mathematical concepts such as average and ratio (for the particular problems used in this study). In the study, students were found to be able to articulate about what happens to the dependent variable as independent variables change. So, even in multiple-choice format, modeling problems are promising in development of student thinking about key mathematical concepts.

Amit and Jan (2007) studied the nature of the environment that could reinforce the development and the understanding of concepts in probability and could contribute to build probabilistic models. 12 students from grades six to eight who had no prior formal instruction in probability were asked to create a model for a probabilistic situation. The results revealed that the students were able to grasp important concepts of probability such as possible versus certain, theoretical versus actual probability, probabilistic advantage and the interface between mathematics and chance, fairness and so forth. Moreover, the nature of modeling task and the environment made the students going beyond empirically proving their knowledge and test the ideas scientifically. The researchers called this additional process "meta-argumentation" to arrive at group consensus, during which the results were generalized and a model was formed.

A study on problem solving and mathematical modeling of pupils in secondary schools (Greefrath, 2010) demonstrated that students were able to plan and discuss simplifications of reality in depth, use mathematical terms and correctly apply them to reality; associate objects from reality with the relevant mathematical actions and simplifications; and discuss planning process in depth and were generally successful.

A modeling task developed by Carmona and Greenstein (2010) was implemented with 40 elementary students and eight post baccalaureates in central Texas. Although the quality and complexity of the solution pathways were different from elementary kids to graduate students, every solution revealed conceptual systems implanted in representations and demonstrated the problem solvers' "*rich and deep understandings of the mathematical concepts*" (Carmona & Greenstein, 2010, p. 90) underlying those solutions. The study is unique in showing that a modeling task no matter the grade level it's used at could be powerful for understanding and applying mathematical concepts.

Vos (2007) analyzed pre-college students' scores on mathematics part of TIMSS-2003 test in Netherlands to see how modeling-based curricula were effective. Netherlands has adopted two innovative curricula for pre-college students: modeling-as-vehicle and modeling-as-content. TIMSS data analysis showed that especially pre-college students of Netherlands did better than their peers in five other countries: Australia, England, Flanders, Sweden, and USA. The results also showed that lesser gifted students benefited from context-based curricula of Netherlands. One might argue here that this type of approach could shrink the achievement gap between different student populations.

A MODEL-ELICITING ACTIVITY IMPLEMENTATION

As summarized above, the findings from the literature on models and modeling in mathematics education show student gains in terms of both meta-cognition and social learning. However, an in-depth analysis of how problem solvers proceed when modeling a problem is missing. That is, what kind of action do individuals take to reach a solution? In this study, pre-service teachers' solution ways to a modeling task were analyzed. This modeling task given to students represents a typical optimization problem that could be solved by a derivate method. However, the way the problem posed and the context of it are substantially different from traditional mathematics problems. The description of this MEA and its details are given in the next section.

Historic Hotel Problem

The problem of Historic Hotel is an MEA designed by Aliprantis and Carmona (2003). The MEA starts with a newspaper to make an introduction to the problem and more importantly to make the problem meaningful to students. Students then need to answer a few readiness questions to warm-up for the real task. This introductory stage makes students get engaged with the problem to be posed next. The problem asks to develop a mathematical model to maximize profit that could be calculated with a quadratic equation if presented in traditional word problem. In the problem, there is a client who inherited a historic hotel and does not have management skills. He wants to determine the rate per room where he was told by the previous owner that all of the 80 rooms are occupied when the daily rate is \$60 per room; the rate per room increases by \$1 for every vacant room; and each occupied room has a \$4 cost for service and maintenance. The problem can be solved by quadratic formulas, 1^{st} and 2^{nd} derivative method, or simply looking at the profit for each value. The final answer the problem asks is a letter that explains the mathematical model that solves the given problem and a generalization method that could be used in different situations (i.e., when number of rooms or rate per room changes.) See Appendix for the complete version of the task.

Research Questions

This study analyzed the implementation of this MEA from a formative assessment perspective. Specifically, the goal was to explore *modeling cycles* (Ekmekci & Dominguez, 2007) in which students develop a model to the problem, test or discuss it, and make modifications during the process, producing a final model. Three research questions guided this study:

1. What *mathematical knowledge* and *skills* do problem solvers utilize when they are engaged in a non-traditional problem?

2. What *symbols* and *tools* did students use in solving an optimization problem?
3. What *modeling cycles* could be identified in common in students' solutions to an optimization problem?

Methodology

Participants

The study took place in a core teacher education course in a secondary science and mathematics teacher preparation program at a large research university in Southwest U.S. *Historic Hotel MEA* was given to 30 pre-service science and mathematics teachers. All participants were enrolled full-time undergraduate students majoring mathematics, biology, chemistry, physics, or earth sciences. Although there was a variation in the coursework they had completed so far, all of them had taken at least pre-calculus level courses.

Method

The study was conducted during two sections of the course of 75 minutes each. In the first episode of the activity students read the introductory newspaper article setting the context of the problem. Then students answered readiness questions as a class. Next, students were asked to form groups of 3 by themselves. In the last 45-50 minutes of the first meeting, they solved the MEA in their groups. In the second meeting, a few more minutes were given to students to finalize their solutions. Then, each group presented their models and received feedback from peers for about 5-7 minutes. After the presentations, students went back to their groups and revised their models based on the feedback received from peers if they needed. At the end of the class groups turned in all their work along with the final solution letter in which they described their model in detail.

The data collection in this study includes field notes, observations, and students' written solutions to the problem. Filed notes were taken while students were discussing and solving the problem in their groups, presenting their solutions to the class, and discussing the problem and solutions to it as a whole class. The focus of the observations was on the mathematical ideas that emerged and evolved during the time students received the problem until each group reached their final solution.

In order to describe the mathematical ideas developed by students when solving The Historic Hotel MEA, the data collected was coded and analyzed qualitatively (Miles & Huberman, 1994; Strauss & Corbin, 1998). Students' responses were sorted by identifying the common representations or conceptual systems utilized when developing a solution.

The following coding schema were developed and used to categorize students' solutions to show common modeling functions and optimization strategies students developed.

Definition of variables

n = number of vacant rooms
r = dollar amount raise in rate per room
x = number of occupied rooms
y = daily rate per room

Modeling functions

F-I. Profit = $(80 - n)(60 - n) - 4(80 - n)$ or Profit = $(80 - r)(60 - r) - 4(80 - r)$

F-II. Profit = $xy - 4x$ and $y = 140 - x$ (function of profit with respect to x)

F-III. Profit = $xy - 4x$ and $x = 140 - y$ (function of profit with respect to y)

F-IV. Profit = $(80 - (y - 60))y - 4(80 - (y - 60))$

Optimization strategy

DER : Use of the 1st derivative test to determine a maximum point.

GRA : Use of a graphing calculator to determine the maximum point of a function.

TBL : Use of table to systematically compute the profit by decreasing the number of occupied rooms and increasing the daily rate per room.

VER : Use of the vertex formula, $x_V = \frac{-b}{2a}$, for a quadratic function of the form $f(x) = ax^2 + bx + c$.

RESULTS

This particular type of MEA seemed to make students' thinking about mathematical concepts related to optimization visible to both their peers and teachers. Here, first presented are the numerical solutions showing what variables, modeling functions, mathematical strategies, tools, and symbols students used in their work. This fist part answkers research questions 1 and 2. Then, qualitative information about students' *modeling cycles* in which testing-and-revising occurred is presented. This part answers the third research question.

Table 1 summarizes the mathematical models each group developed in numerical ways. As shown in Table 1, five groups used F-I; four groups used F-II; and 2 groups used F-III modeling functions. The optimization strategy of two groups was not clear enough to make any judgments about their solution strategy. These were labeled as N/A in Table 1. The groups that used F-I seemed more comfortable when using a single variable in their equation. Moreover, they preferred to use notations n or r that were more meaningful to them as they wanted to represent *number* and *rate* in their equations as opposed to using x and y which were more abstract to them. Two of F-I groups were able to turn their equation into quadratic form and apply derivation methods to reach to a numeric answer.

Table 1

Summary of Numerical Solutions

Group	Variables	Modeling Function	Optimization Strategy	Graph	Generalization	Answer
G1	r	F-I	GRA	Yes	No	r = 12
G2	r	F-I	N/A	Yes	No	r = 12
G3	n, r	F-I	TBL	No	Yes	n, r = 12
G4	r	F-I	DER	No	No	r = 12
G5	x	F-II	DER	No	Yes	x= 68
	n, r	F-I	DER	No		n, r = 12
G6	x	F-II	N/A	Yes	Yes	x = 68
G7	x	F-II	DER	No	Yes	x = 68
G8	x	F-II	DER	No	No	x = 68
G9	y	F-III	VER	Yes	Yes	y = 72
G10	x	F-III	DER	Yes	Yes	y = 72

In the groups that used F-II and F-III, it was obvious that they were able to handle two unknowns in a mathematical equation. These groups knew what each of these two variables really meant. The discussions took place in these groups and their use of x's and y's showed that they were open to more abstract thinking when talking about variables.

Five of the groups used 1st derivative rule to obtain a critical point in the equation. Only one group used a vertex formula to find the critical point. However, only one group among these made sure that this critical point was the maximum by graphing the equation. From calculus point of view, solution set of the 1st derivative only shows the critical points and the 2nd derivative solution is needed to determine the concavity of the curve at those points in order to identify which of them are maximums, minimums, and/or just an inflection point. This means that many of groups that used 1st derivative method have not really grasped the concept of derivative methods or vertex formula to find optimum points.

Five groups had skills to use a graphing calculator to graph a quadratic function and find maximums. G3 was the only group that did not utilize any derivative methods. From the discussions among the group members, it was clear that they did not have a thorough understanding of derivation concept. They knew that this could be solved by derivatives but they were not very comfortable in applying it.

Four groups were able to explain how to modify the mathematical model they produced for different situations. This demonstrates that students in these groups were able to transfer the mathematical knowledge and skills particular to this task for different situations.

First part of results showed what variables, modeling functions, mathematical strategies, tools, and symbols were used. Now, the results related to third research question will be presented. This second part of the results report *modeling cycles* (Ekmekci & Dominguez, 2007) students went through. Field notes from listening to students working in groups as well as their written work were coded and analyzed (Miles & Huberman, 1994). Modeling cycles given in Figure 1 emerged from these analyses.

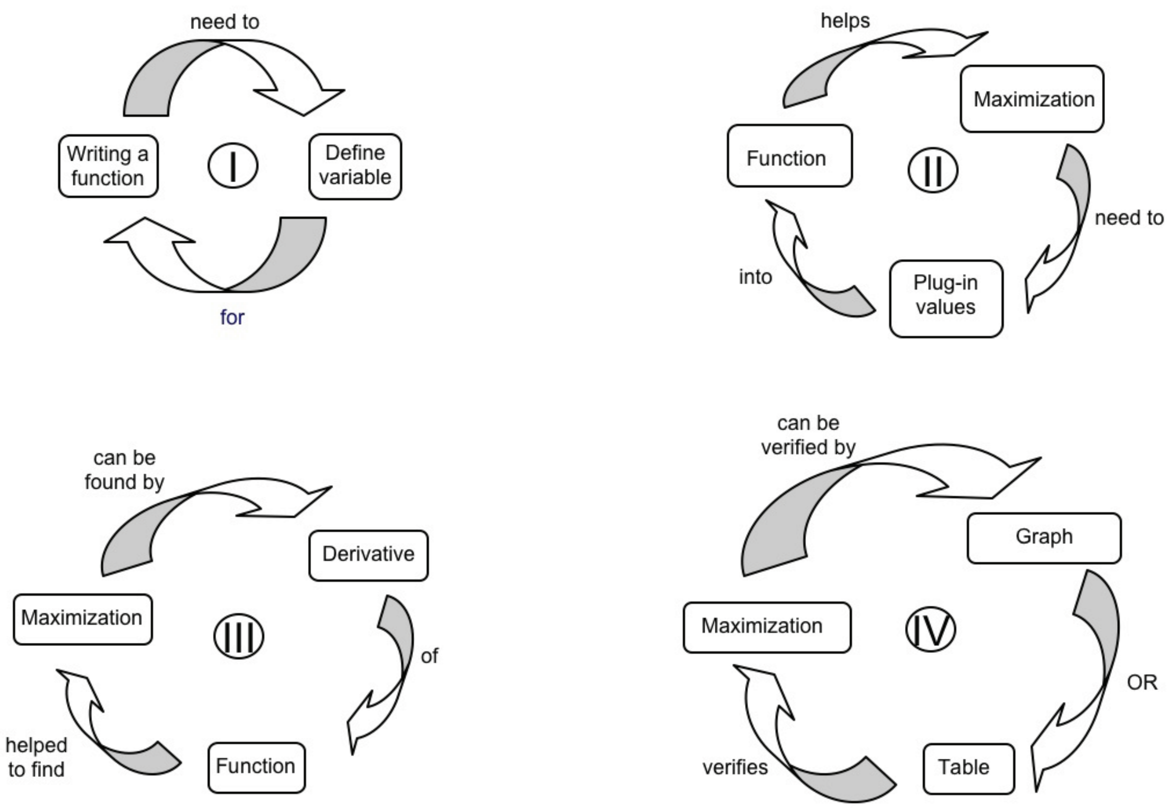

Figure 1. Modeling cycles in modeling process.

Four types of model-production have been identified in students' modeling process. Model-I represents a relatively primitive model with which the students started. Model-II could be considered as an improved version of the first model. Model-III seems the most complete model among all. Model-IV, on the other hand, also provides a complete model that could solve the problem. However, this model lacks information about how quadratic functions work and what they mean.

In five of the groups students pursued the pathway I - II - III. That is, they started out with first model, developed the second one, and lastly, they ended up with the third model as their final solution model to the problem. One of these groups, Group-10 went further and did the Model-IV, too. It could be said that the students in Group-10 were able to make connections between derivation and graph of the functions. The other five groups followed I - II - IV pathway. The students in these groups did not demonstrate a thorough understanding of the concept of derivation in calculus.

DISCUSSION AND CONCLUSION

Historic Hotel MEA is used as a formative assessment tool in this study. Through this activity, students' mathematical thinking, knowledge and skills were made visible to both their peers and their instructor. They were able to communicate well among themselves. In other words, within all the groups, students were able to talk mathematics and understand each other. There were instances where some students did not know the concepts to utilize very well or were not self-confident enough to use them. However, these obstacles that might have hindered their learning and understanding have been overcome with the help of a more knowledgeable one in the group. The role of the instructor was facilitating the activity and triggering good discussions where needed. Students learned from each other and collectively produced mathematical models they understood very well at the end.

The Historic Hotel MEA as stated at the beginning of the paper would be given in a traditional fashion (i.e., in word problem format, and at a particular time-point in a calculus course) in which a single numeric answer was requested and the concepts, formulas, and procedures that the students would use would be pre-determined. In other words, this problem could have been posed as a summative assessment. However, Historic Hotel MEA is designed in a way that it can potentially satisfy six design principles outlined by Lesh et al. (2000).

This particular MEA is still criticized by many colleagues in models and modeling in mathematics education field that it has a single numeric answer, which is contrary to MEA design principles. Indeed, this was the case in our study that students arrived in the same single numeric answer. Although, this seems a big constraint from modeling perspective, it is not. In this study, students in some of the groups were actually talking about other very interesting solutions such as profiting from vacant rooms by renting them long-term at a reduced rate and reduced maintenance frequency, or converting them into a conference room and so forth. So, this problem has shown the potential for allowing multiple solutions rather than a single answer (Dominguez, 2009). These multiple ideas were very valuable to bring possible other mathematical knowledge and skills into the solution. However, students did not pursue these ideas because they were so accustomed to traditional mathematic problems so that they ended this part of the discussion by saying something like "I do not think that's *allowed.*" Although, the instructor made it very clear that they could develop any model that could work, although this was not a calculus course at all, students' perceptions of mathematical tasks were very orthodox. These perceptions constrained their way of thinking mathematically and limited the space of different solutions.

On the other hand, this MEA activity provided the instructor with very rich information about what students knew and did not know, what their strengths and weaknesses in specific concepts. Above all, this activity seemed to elicit students' way of thinking. That is, where they started with, how their ideas evolved, and why they dropped some models and adopt others were clear enough to see. Needless to say, this rich information is not possible to obtain with classical tests, word problems, or traditional teaching methods.

Lastly, this mathematical task seemed to make all of the students participate in the activity. Each and every student were highly engaged whether they were or were not good at mathematics, female or male, and regardless of their ethnicity. This is very promising, indeed, in terms of inclusion of diverse groups of students into the instructional activities that would, in turn, hopefully increase knowing, learning, and understanding of all students. This is also consistent with the research in the literature. For example, Diefes-Dux et al. (2004) studied MEAs in undergraduate engineering courses for increasing women's interest in engineering (NSF-HRD Gender Equity Project). They found that MEAs had the potential to provide a learning environment in which a more diverse population than typical engineering course could engage because MEAs allowed students with different backgrounds and values to emerge as talented. They found that the MEA framework fostered significant change in the way engineering faculty think about their teaching and their students. They reported that engineering faculty working on the MEA development changed their notions about what it meant for a problem to be open-ended, client-driven, and realistic.

As stated above, our goal was to elicit what mathematical knowledge and skills students had and were able to apply in order to accomplish given tasks and how their thinking evolved from a modeling perspective. Although, this study served good enough to address these questions, there are other important pieces to be answered that would inform the mathematics education research field. For example, how could this rich information gathered through these kinds of formative assessment tools inform the instruction? What strategies the do instructors follow or what modeling cycles do they go through in modeling, planning, and teaching their courses? As Black and William (1998a, 1998b) suggested, assessments should be useful for instruction to improve teaching and learning in order to be formative. Therefore, next steps would be studying teachers' decision-making, planning, and modeling strategies based on the information gathered through formative assessment approaches.

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APPENDIX

Historic Hotel Problem

Newspaper Article: An Enchanting Vacation

Going on vacation is something that everyone looks forward to. But staying in a historic hotel transforms any vacation into an enchantment. Finding these charming places is a task to which the National Trust of Historic Hotels of America is committed. To be recommended by the National Trust of Historic Hotels of America, hotels have to prove that they have faithfully maintained their historic architecture and ambience. Several of these hotels hold great pride in their stories, myths, and legends. For example, the

French Lick Springs Resort and Hotel in Indiana was named after an early French outpost and its rich mineral springs that included a naturally produced salt lick (a lick is a deposit of exposed natural salt that is licked by passing animals). The hotel was originally built by Dr. William A Bowles, and it flourished during the mid-nineteenth century.

Whether for health reasons, or just for curiosity, visitors were compelled to visit the rich mineral springs, which were said to possess curative powers.

In 1897, the hotel burned down, and it was not rebuilt until 1902. The new owner, Thomas Taggart, built the new French Lick Springs Resort on the ruins of the original hotel. Mr. Taggart, mayor of Indianapolis, made the resort grow in size and reputation in the early decades of the 20th century. Surrounded by lush gardens and landscaping, the six-story hotel, with its sprawling sitting veranda, was a more than relaxing environment many wished to enjoy. Among the most interesting celebrities that frequented the resort were John Barrymore, Clark Gable, Bing Crosby, The Trumans, The Reagans, Al Capone and President Franklin Roosevelt. In fact, Roosevelt even locked up the Democratic nomination for president in the hotel's Grand Colonnade Ballroom.

Maintenance for a hotel like The French Lick Springs Resort, with all its services, is not an easy task. In 1929, Mr. Taggart died, and inherited it to his son - the only boy among six children - Thomas D. Taggart. With the Depression, however, the popular French Lick Springs began to decline. World War II brought a monetary revival, but in 1946 young Tom Taggart sold the hotel to a New York syndicate.

Today, French Lick Springs Resort rests on some 2,600 acres in the breathtaking Hoosier National Forest. Newly acquired by Boykin Lodging Company, the resort eagerly embraces a "New Beginning". It provides 470 rooms, full service spa, two golf courses, in-house bowling, a video arcade, indoor tennis center and outdoor courts, swimming, croquet, horseback riding, children's activities, skiing, boating, and fishing. Fine and casual dining are also available at a variety of restaurants. Two main meeting rooms, the Grand Colonnade Ballroom and the Exhibit Center, accommodate large-scale events.

Besides The French Lick Springs Resort, the National Trust of Historic Hotels of America has identified over 140 quality hotels located in 40 states, Canada, and Puerto Rico.

Readiness Questions

1. What do hotels have to accomplish in order to be recommended by the National Trust Historic Hotels of America?

2. What are the main features of The French Lick Springs Resort?

3. How many owners has the French Lick Springs Resort had since it opened?

4. What are some responsibilities that a hotel manager might have?

Problem Statement

Mr. Frank Graham, from Elkhart District in Indiana, has just inherited a historic hotel. He would like to keep the hotel, but he has little experience in hotel management. The hotel has 80 rooms, and Mr. Graham was told by the previous owner that all of the rooms are occupied when the daily rate is $60 per room. He was also told that for every dollar increase in the daily $60 rate, one less room is rented. So, for example, if he charged $61 dollars per room, only 79 rooms would be occupied. If he charged $62, only 78 rooms would be occupied. Each occupied room has a $4 cost for service and maintenance per day.

Mr. Graham would like to know how much he should charge per room in order to maximize his profit and what his profit would be at that rate. Also, he would like to have a procedure for finding the daily rate that would maximize his profit in the future even if the hotel prices and the maintenance costs change. Write a letter to Mr. Graham telling him what price to charge for the rooms to maximize his profit and include your procedure for him to use in the future.

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Accessing Mathematical Understanding through the Hermeneutic Circle

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ABSTRACT: *The purpose of this paper is to present a conceptual framework for accessing mathematical understandings that is based on the hermeneutic circle. Brown (2001) proposed applying the metaphor of the hermeneutic circle as a means for interpreting how mathematical understanding evolves over time through the interplay between explaining and understanding. The design of this conceptual framework is based on three inter-related elements. The first element is the concept under investigation, which in the study used to illustrate this framework was the idea of function. The second element consists of the types of mathematical activities that engage the learner in the process of explaining their understanding of the key concept. In this case, the study of patterns was believed to provide an appropriate setting for developing and communicating the idea of function (Smith, 2003). The third, and perhaps the most important element, consists of the mathematical processes one employs while engaging in the mathematical activity. In this illustrative study, algebraic thinking formed the matrix, or third element, in which the concept of function and the study of patterns were embedded. This paper describes how the conceptual framework based on these three elements guided the research activities of the author and includes suggestions for modifying the framework to access individual understandings of other important concepts in mathematics.*

Key words: *Hermeneutics, Mathematical understanding, Conceptual framework, Research in mathematics education.*

ACCESSING MATHEMATICAL UNDERSTANDING THROUGH THE HERMENEUTIC CIRCLE

The sociocultural lens has been utilized extensively in research on mathematics teaching and learning. Calling on the works of Vygotsky, sociocultural theorists adopt the stance that learning takes place within social settings as individuals make use of cultural tools such as language, gestures, and other symbols to transfer knowledge (Cobb, 2007; Forman, 2003). Under this premise, the discourses produced within a social setting would contain artifacts that could then be analyzed in terms of what learning has taken place (Lerman, 2001; Sfard, 2001). Sfard (2001) posited that evidence of individual learning is manifested in the communications between members of a group. For instance, the use of a newly introduced mathematical term or procedure would be an indication that the student is learning how to use the tool (Lerman, 2001). Under this premise, the artifacts of learning manifested in the discourse between participants should be accessible to the researcher. However, Sfard (2001) described the reconstructive efforts of the researcher as analogous to the efforts of an archaeologist attempting to recreate an ancient vessel without having all of the original pieces. The task of explicating the understandings of individuals, particularly with respect to complex concepts, poses a problem for researchers in mathematics education.

The purpose of this paper is to present a theoretical framework for accessing the mathematical understandings that are embedded in the explanations students offer while involved in problem-solving activities. This theoretical framework is based upon the hermeneutic circle, as described by Brown (2001). Drawing from the works of Gadamer and Ricoeur, Brown proposed using this model as a means for interpreting how mathematical understanding evolves over time through the interplay between explaining and understanding. When an individual engages in a mathematical activity, one must interpret the meaning of the problem within the context of one's prior experiences. However, this interpretation results in new understandings of the problem at hand which, in turn, alters the original interpretation. The hermeneutic circle is formed as the elements of interpreting and explaining a problem enhance each other, creating a cycle of developing understanding. Brown suggested that this understanding is captured, in part, by the texts produced by the individual. These texts, whether in the form of conversation or written work, provide a means for understanding mathematical learning from the perspective of the learner (Brown, 2001).

The premise of hermeneutics requires one to first examine the constituent parts before being able to make generalizations about the whole (Paterson & Higgs, 2005). From the perspective of the learner, this involves grappling with concrete examples of a concept before one can explain its meaning to self and others (Brown, 2001). For the researcher seeking to access mathematical understanding through the hermeneutic circle, he or she must first focus on the explanations offered by individuals as they engage in a mathematical activity. In this manner, the researcher enters into a circle of developing understanding while reading, describing, and interpreting the texts created by the individuals. The insight gained concerning individual understandings can then be assimilated to describe how a group of learners experience a concept. In this manner, the hermeneutic circle, when applied to qualitative analysis, opens a space for interpreting the experiences of others.

The idea that learning in mathematics is hermeneutic formed the basis for the theoretical framework that will be described in this paper. First, an explanation of why learning in mathematics can be viewed as hermeneutic and how this view can be used to develop a framework for analyzing individual understandings of mathematical concepts will be offered. The use of this framework will then be illustrated using a study conducted by the author on pre-service elementary teachers' understanding of pattern and function. In this particular case, the connections between communication and conceptualization in mathematics, the concept of function, and the supporting role patterns play in developing algebraic thinking formed the conceptual framework that guided the research activities. The conclusion of this paper will offer suggestions for modifying the framework to access student understanding of other important concepts in mathematics.

BACKGROUND INFORMATION

The purpose of this paper is to describe a theoretical framework suitable for unpacking the mathematical understandings individuals hold concerning key concepts in mathematics. However, the notion of understanding in mathematics should be explored before attempting to examine how individuals conceptualize a particular idea in mathematics. Sierpinska (1992) presented a theoretical framework of understanding based on the works of Locke (1985), Dewey (1988), and others (as cited by Sierpinska, 1992). Under this framework, acts of understanding can be broken down into four categories. The first of these acts, identification, occurs when an individual recognizes that an object is of special interest. In other words, something now stands out as different from other objects around it. The second act, discrimination, occurs when the individual distinguishes both the differences and commonalities between two objects in mathematics. Subsequently, the third act of understanding, generalization, is made possible as the individual expands the notion to other settings. In the fourth act, synthesis, a cohesive concept is formed as the individual merges the various properties and facts about objects together. Therefore, it is not possible for understanding to simply arise from reading the definition presented in the textbook. Instead, the understanding of a concept, Sierpinska (1992) explained, emerges after:

> …we have seen instances and non-instances of the object defined, when we can say what this object is and what it is not, when we have become aware of its relations with other concepts, when we have noticed that these relations are analogous to relations we are familiar with, when we have grasped the position that the object defined has inside a theory and what are its possible application.” (p. 26)

In such a manner, understanding appears to evolve over time through engagement in mathematical activities. Taking snapshots of the ways a learner responds during these activities may help ascertain the hallmarks associated with the stages of concept development.

Concept Development

Vygotsky (1934/1986) described concept development as a dynamic enterprise that arises out of the need to communicate complex ideas. He described three stages in the development of a concept, but rebuked the idea of studying concept development as a fixed course, instead viewing it as “…a live thinking process” (p. 105). In the first stage, objects are clumped into unorganized categories based primarily on trial and error. The decision to begin sorting these objects is tied to the first act of understanding described by Sierpinska (1992), as it is through the act of identification that we begin to notice the properties of an object. Through experiences with the objects, the individual formulates a set of rules for joining these objects into groups that Vygotsky referred to as complexes. The formulation of these rules is made possible by noticing the features that a set of objects have in common, as well as their differences, thus involving what Sierpinska referred to as the act of discrimination. In this second stage, the complexes formed evolve into "pseudoconcepts" that are held together by concrete facts the individual has derived from his or her experiences. This evolution from a complex to a pseudoconcept is made possible by the act of generalization and serves as a bridge between complexes and a true concept (Vygotsky). At this point, the individual may adopt the same word or expression to describe the ideas as an adult, but his or her "…framework is purely situational with the word tied to something concrete and the adult's frame is conceptual" (Vygotsky, 1934/1986, p. 133).

Vygotsky (1934/1986) asserted that to arrive at the third level of a true concept, the individual must move beyond the concrete bonds of a pseudoconcept. The generalizations used to formulate a pseudoconcept need to be analyzed or separated into their constituent parts for conceptual understanding to occur. In this manner, Vygotsky stated, "the connections between concepts are neither associative nor structural, but are based on the *principle of the relations of generality*" (p. 204). For illustrative purposes, consider the acquisition of the concept of place value. Although an individual may have a full understanding of place value within the context of a base ten system, the problem of interpreting the value of a base five numeral poses a problem that requires them to explore the structure of base five. However, in doing so, the individual recognizes the commonalities between base 10 and base 5, generalizing the properties that were at one time, generalized within the specific context of base 10. This act of synthesis enables the individual to acquire the true concept of place value. Vygotsky stated that this type of broader understanding requires "…shifting to a plane of greater generality" (p. 205). In doing so, the individual is able to transcend the notion from the concrete to the general, thus synthesizing what is understood about a concept. This final stage corresponds to the act of synthesis, as described by Sierpinska (1992).

According to Vygotsky (1934/1986), thinking about a concept is part of one's understanding and speech is used to explain this understanding to self and to others. The meaning of the word used to describe a concept evolves in the cycle between understanding and explaining that stem from experiences. Vygotsky (1934/1986) voiced the concern that "…to understand another's speech, it is not sufficient to understand his words – we must understand his thought" (p. 253). The image of the hermeneutic circle offers a means to access the thought processes of the individual engaged in mathematical activity (Brown, 2001). Under this image, the thinking process undertaken to understand a concept is partially revealed in the words used to explain these understandings while they are evolving. Brown (2001) asserted that such "…notions of hermeneutic understanding as applied to mathematics require a shift in emphasis from the learner focusing on mathematics as an externally created body of knowledge to be learnt, to this learner engaging in mathematical activity taking place over time" (p. 50).

Mathematics as Hermeneutical Understanding

Brown (2001) proposed a framework for illustrating how mathematical understanding continually evolves through the process of reconciling present experiences with prior understandings. When individuals engage in mathematical activities, they initially interpret the problem in terms of what they already understand about mathematics. Through their attempts to explain their understandings, the explanations offered changes what they initially understood about the situation. A process of reconciliation between explanation and understanding develops to form a recursive relationship referred to as the hermeneutic circle. Therefore, Brown claimed, the process of learning mathematics "…continuously evolves, oscillating between understanding and explanation" (p. 80). Drawing upon this idea of the hermeneutic circle, Brown viewed understanding as a dynamic process that can be partially captured in the statements produced by the learner.

Sierpinska (1992) also described understanding in mathematics as a hermeneutic process, but she rejected the cyclical nature implied by the metaphor of the hermeneutic circle. Sierpinska asserted that understanding in mathematics is more likely to be a discontinuous process, littered with instances of stagnation followed by giant leaps in understanding. She attributed these periods of stagnation to epistemological obstacles that occur due to misconceptions held by individuals or by certain societal groups. Conflicts between these misconceptions and new evidence that challenges them open a space for new understandings to develop. Sierpinska (1992) viewed the metaphor of the hermeneutic circle as a trap when the preexisting knowledge structures are incorrect or inadequate. She stated "…it is possible to escape the paradox if we abandon the metaphor of "circle" and bring forth the idea of spirality in describing cognitive processes" (p. 28).

Modifying the hermeneutic circle to incorporate the spiraling effect created by an evolutionary change in conceptual understanding aligns with Vygotsky's notion of "a higher plane of thought" (1934/1986, p. 202). Vygotsky spoke of how generalizations lead to new levels of understanding that bring the individual to this higher plane. In this model, concept development is not simply a matter of acquiring a fixed body of knowledge. As Vygotsky stated, a concept is "more than the sum of certain associative bonds formed by memory, more than a mere mental habit; it is a complex and genuine act of thought that cannot be taught by drilling" (p. 149). The word that represents the concept is but a generalization of the idea whose meaning evolves through the experiences that the individual has with the idea. Vygotsky asserted that these generalizations arise out of the need to communicate one's thought processes. Without words, it is impossible to think in terms of concepts.

The idea of mathematics as hermeneutics provides a framework for accessing mathematical understanding. The key characteristic of this idea lies in the view that mathematical understanding is communicated through language. Sierpinska (1992) stated that understanding in mathematics can be linked to Ricoeur's hermeneutics because of the "…relationship between the symbol side of the mathematical concept and the object side" (p. 30). This analogy can be interpreted to mean an individual uses traditional words or symbols related to a concept as well as the image he or she has of that particular concept in order to communicate with others. This image is continually shaped and refined by the experiences the individual has with the concept or idea both in and outside of the mathematics classroom (Vinner, 1992). Brown (2001) had suggested that mathematics "…can be seen as a subject of hermeneutic understanding if the emphasis is placed on interpreting mathematical activity, which itself might embrace the generation of mathematical statements" (p. 50). Thus, determining how an individual conceptualizes a mathematical idea entails taking snapshots of this image in the statements produced during mathematical activity (Brown, 2001).

DEVELOPING A THEORETICAL FRAMEWORK

The task of developing a theoretical framework for accessing mathematical understanding begins with the identification of a concept, along with the types of activities and underlying processes that contribute to the development of that concept. In this paper, the concept of function is used to illustrate the development of a suitable framework based on the idea of the hermeneutic circle. The process of developing this framework began with an investigation into how others experience the concept of function.

The historical refinement of the concept of function offers two ways to conceptualize functional relationships (Confrey & Smith, 1995; Slavit, 1997; Smith, 2003; Billings, 2008). The first of these views is based upon Euler's definition of a function as a dependent relationship in which one data set covaries with another. Viewing a function as covariation places an emphasis on how changes in one variable result in changes with another. The second view draws upon the modern definition of function as a correspondence between two data sets (Slavit, 1997). Under this perspective, emphasis is placed on stating the relationship that maps members of one set, usually referred to as the domain, to members of another set, known as the range (Confrey & Smith, 1995; Slavit, 1997; Smith, 2003). Sierpinska (1992) stated that "…the notion of function can be regarded as a result of the human endeavor to come to terms with changes observed and experienced in the surrounding world" (p. 31). She emphasized that the basis for understanding functions lies in the ability to identify such changes and the relationships between them. In mathematics, this basis of change translates to the covariation definition of function wherein changes in one variable are associated with changes in another variable (Smith, 2003). The ability to view the idea of function as covariation is believed to be a crucial step in the development of algebraic reasoning (Smith, 2003). However, the view of function as correspondence is also believed to be a key stepping stone to understanding the concept of function (Slavit, 1997). With this view, one is readily able to see the relationship between two data sets.

Smith (2003) proposed using the study of patterns as a context for experiencing function as both covariation and correspondence. He discussed two approaches to analyzing pictorial patterns which can be used to build a framework for developing connections between patterns, functions, and algebra. The first of these approaches focuses on how the pattern is constructed, which he referred to as stasis. Examining how a pattern is constructed and noting the constant pattern between the position of the figure in the sequence and how it is constructed allows one to establish a relationship or correspondence. The dual focus on stasis and change within the context of patterns potentially leads to generalization, a key feature of algebraic thinking (Smith, 2003). If teachers are going to support algebraic thinking, they need to be able to identify and generalize change (Bezuska & Kenney, 2008). In response to this problem, the design of this study focused on how the ideas of stasis and change can be used to generalize patterns. The pre-service teachers in this study were invited to experience the idea of function by engaging in conversations about how they construct, extend, and ultimately generalize patterns.

Paterson and Higgs (2005) created a model of hermeneutic understanding applied to research based on the works of Bontekoe (1996) in which examination of different aspects (the parts) of professional practice judgment artistry communicated by the participants in their study could be used to define the meaning of the term (the whole). Drawing inspiration from this model of the hermeneutic circle, I proposed that engaging in pattern-finding activities leads to an understanding of function as embedded in algebraic thinking. A diagram of the interaction between communication and conceptualization of the concept of function is presented in Figure 1. The study of patterns and function is embedded within the matrix of algebraic thinking which is represented by the rectangle. The study of patterns provides a context for developing the idea of function. Through these patterning activities, the participants have the opportunity to communicate and demonstrate their understanding of function. The circle of arrows represents the hermeneutic circle that evolves between the processes of explaining and understanding. Research based on this framework would involve capturing individual interpretations of pictorial growth patterns. In the next section, a brief description of the research design employed in this illustrative study of pre-service elementary teachers' understanding of function will be provided.

RESEARCH DESIGN

Data Collection

The particular study described in this paper sought to explain how pre-service elementary teachers conceptualize and communicate their understanding of function while experiencing the concept through pattern-finding activities. The data collected for this study came from three different sources: group conversations, written documents, and individual interviews. All three of these sources were used to create an ongoing account of the experiences these participants had while working with patterns in the mathematics classroom. This ongoing process of data collection was followed to take snapshots of the evolving mathematical understandings of the six primary participants in this study selected from a larger pool of 29 participants.

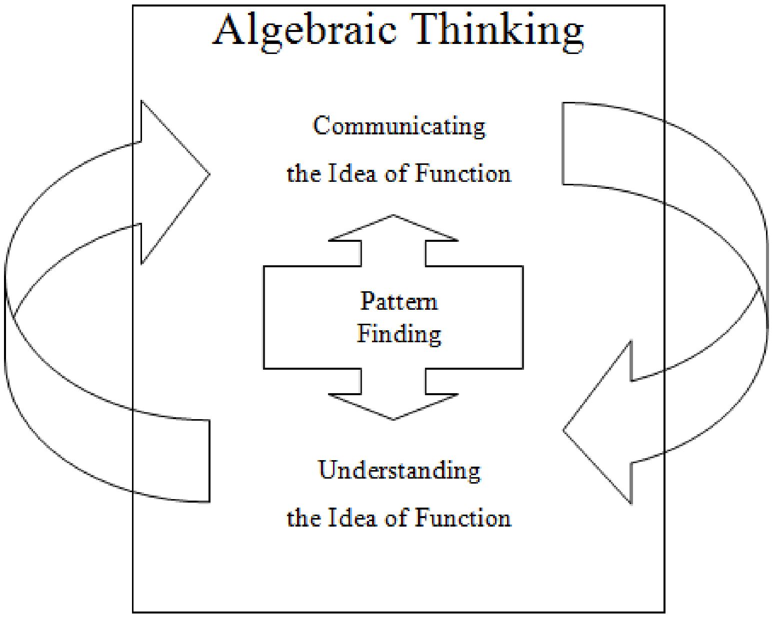

Figure 1. Communicating the idea of function as embedded in algebraic thinking. Diagram of framework for context of study based on a model used by Paterson & Higgs (2005) which was adapted from the works of Bontekoe (1996).

Data Collection Format

The basic format employed resembled the 'think-pair-share' instructional strategy advocated by McTighe and Lyman (1988). In the typical 'think-pair-share' strategy, students are given time to ponder a question or problem posited by the teacher. After sufficient time is spent working independently on the problem, the students pair up and discuss their strategies. Closure is then achieved by having the pairs come together as a group and share their understandings with the entire class (McTighe & Lyman, 1988). This strategy was modified by replacing the group sharing session with a paired problem-solving activity. The participants first worked independently on two pattern finding tasks, then paired up to share their strategies. Immediately after sharing their strategies, the participants worked together to solve four problems of a similar nature. Closure was achieved by asking the participants to write a reflection on their experiences during the patterning activities. This four-stage strategy will be referred to as 'think-pair-together-reflect'.

The 'think' session. During the think stage, the participants examined two patterning problems involving arithmetic (linear) sequences. The first problem used geometric figures (small squares) to represent an arithmetic sequence and the second problem presented a numerical sequence in tabular form. In general, the participants were asked to determine the next figure or term in each pattern, find the fifteenth term, and then write a rule for the *nth* figure or term in the sequence. After completing the two patterning tasks, the participants were asked to write a definition for the concept of function based on their experiences with the two previous pattern-finding tasks. One purpose of the 'think' session was to produce a written text; therefore, the participants were asked to write an explanation of how they completed each of these tasks. The 'think' session was also designed to provide a snapshot of the prior understandings each individual brought to the classroom. This stage occurred before any class instruction or group activities on these particular types of patterning activities had taken place.

The 'pair' session. Each pair or triad was responsible for recording their conversations while sharing the strategies employed during the previous 'think' activity. The teacher/researcher provided one audio recording devise to each pair/triad and instructed the participants to record their conversations throughout the session without interrupting the recording. The participants were asked to explain their strategies and to make note of any differences in approaches and/or answers. In general, the participants followed the same line of questioning as printed on the assignment. This stage was designed to obtain a snapshot of their understandings which were embedded within the statements offered in their explanations.

The 'together' session. After sharing their individual work, the participants were given four similar pictorial growth patterns to analyze together. These patterns were presented in both pictorial and tabular formats. The pictorial format consisted of two-dimensional drawings of figures built using cubes. Each participant was given a bag of centimeter cubes with which to build models of the figures shown in the pictorial representations. The second pattern analyzed is shown in Figure 2 for the reader's convenience. The 'together' session provided a unique opportunity to capture the participants' evolving understanding of the patterns in the explanations that they offered to one another. This snapshot was taken to preserve the process of learning, which, as Brown (2001) stated, "…continuously evolves, oscillating between understanding and explanation; that is, between an on-going learning process and statements generated within this process…" (p. 80). As in the 'pair' stage, each pair/triad was responsible for recording their conversations without interruption.

The 'reflect' session. At the conclusion of the 'together' session, the participants were asked to respond in writing to a series of prompts. The prompts served two general purposes: 1) to gain insight into how the participant experienced the pattern-finding activities and 2) to determine what connections the participant might be making between this experience and their past experiences with patterns and functions. Specifically, the participants were asked to choose one of the six patterning tasks just completed and explain how they were able to determine a general rule for describing the pattern. They were also asked to include a discussion of any difficulties they and/or members of their group encountered while analyzing the pattern. Additionally, participants were asked to reflect on the ways these patterning tasks strengthened their understanding of patterns and functions. These reflections were written in class immediately after the conclusion of the 'together' session.

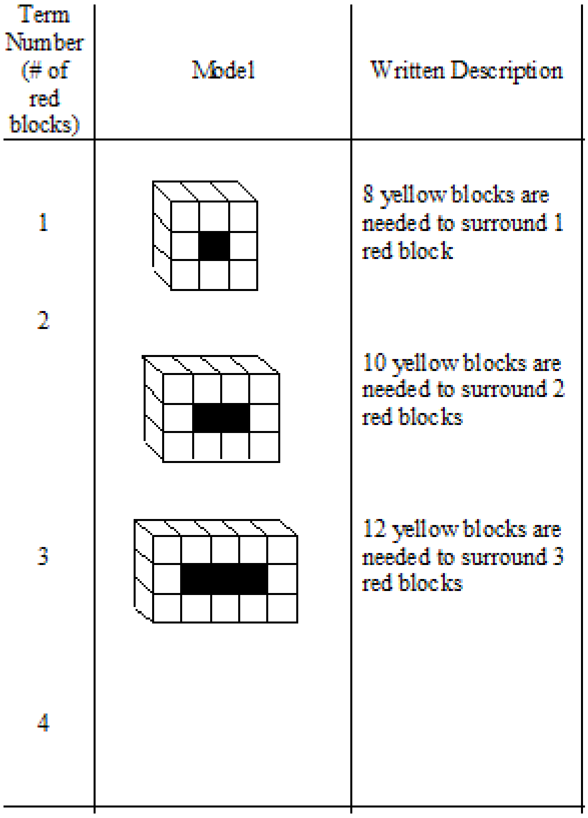

Figure 2. Second figure in 'together' session.

Selection of Primary Participants

The written texts gathered from the 'think' session were analyzed for the purpose of selecting the primary participants in this investigation. This preliminary analysis took place during the week after collecting the data. The participants (*N* = 29) had completed three tasks during the 'think' session, two tasks involving patterns, and one follow-up question concerning the definition of a function. Both patterning tasks involved finding the next term in a sequence as well as the 15th and *nth* terms. Responses to these first two tasks were sorted based on whether or not the participant abandoned the problem, extended the pattern to find the 15th term but did not write a rule for the general term, or successfully extended the pattern and wrote a verbal or symbolic rule for the general term. A summary of the end results on both patterning tasks is displayed in Table 1. A slight majority of the participants were able to write rules for both patterns and all 29 participants were able to extend at least one of the two patterns. Eighteen of the 29 participants wrote a definition of function; the remaining eleven either did not respond to this prompt or simply wrote "not sure" or "don't know" in the space provided. These eighteen responses were sorted into four categories (pattern, rule, relation, or generalization) based on the idea of function represented. An attempt was then made to select a heterogeneous sample whose responses mimicked the population as a whole. A brief description of the six primary participants selected based on these criteria is displayed in Table 2.

Table 1

Cross-Tabulation of End-Result

	Score	Think 2 Did not extend	Extended Pattern	Verbal Rule	Symbolic Rule	Total
Think 1	Did not extend	0	2	1	0	3
	Extended Pattern	4	5	0	1	10
	Verbal Rule	0	2	0	1	3
	Symbolic Rule	1	0	1	11	13
Total		5	9	2	13	29

Note. The first linear problem appears on the left, labeled as Think 1. The second linear problem appears along the top of the table, labeled as Think 2.

Table 2

Summary Description of Primary Participants

Pseudo-Name	Age	Patterning Activities (First/Second)	Image of Function
Tara	27	Symbolic/Symbolic	Pattern
Cathy	28	Symbolic/Symbolic	Rule
Shelly	24	Extended/Did not extend	Pattern/relation
Matt	34	Verbal/Extended	Relation
Ashley	22	Symbolic/Symbolic	Rule
Jill	25	Verbal/Symbolic	Generalization

Interviews with Primary Participants.

Individual interviews were conducted with the six purposively-selected participants after in-class data collection was completed. The interviews were structured around the think-pair-together-reflect activities completed in class. The participants were asked to interpret each problem and explain how they solved it. In addition, participants were asked to complete a similar problem that they had not worked previously.

Data Analysis

Data analysis was guided by the purpose of the study and the theoretical perspective that informed the research process (Patton, 2002). This research endeavor employed the interpretive lens of hermeneutic phenomenology in an attempt to understand how pre-service elementary teachers conceptualize and communicate the idea of function. The use of this paradigm includes the assumption that access to understanding is gained by reading and interpreting the texts of those who have experienced the phenomenon under investigation (Van Manen, 1990). Hermeneutic analysis provides a systematic framework for interpreting these texts in light of the purpose of the study (Patterson & Williams, 2002).

According to Brown (2001), hermeneutics views texts as an avenue to understanding the perspectives of others. The researcher navigates this avenue by engaging in a circular dialogue with the text, referred to as the hermeneutic circle (Patterson & Williams, 2002). The hermeneutic circle is a metaphor for an analytical process that entails "…understanding the whole through grasping its parts, and comprehending the meaning of parts through divining the whole" (Crotty, 2003, p. 92). In research, the whole is constituted by the phenomenon under investigation with the individual data making up the parts. Analysis of the individual parts is made possible by the researcher's understanding of the phenomenon. These new-found interpretations are then integrated with the holistic understandings such that "…by circuitously viewing a phenomenon as a whole and as a sum of individual parts, the researcher gains knowledge to build increasing understanding of the experience" (von Zweck, Paterson, & Pentland, 2008, p. 119).

Analysis begins with the understandings the researcher brings to the phenomenon under investigation. These understandings may have come from prior experiences and/or a review of the literature pertaining to the phenomenon (Patton, 2002; Patterson & Williams, 2002). Patterson and Williams described these prior understandings as the "…scaffolding upon which knowledge is built" (p. 23). They referred to Heidegger's 'forestructures of understanding' as a metaphor to describe how knowledge is shaped by prior exposure to the phenomenon.

Analysis thus begins by examining the parts or data with the illuminating lens of what the researcher already understands about the research problem (Patton, 2002). This illustrative study sought to understand how pre-service elementary teachers communicate their understanding of function through the study of patterns. The researcher's prior experiences teaching pre-service elementary teachers and the literature review conducted for this study led to the following assumptions:

1. Understanding in mathematics is a hermeneutic process.
2. We communicate our understandings through language; therefore, access to these understandings is gained by interpreting texts produced by the individual.
3. The dual characteristics of patterns in mathematics, stasis and change, support the development of algebraic thinking and functional reasoning.

4. The concept of function may be viewed as a correspondence between two data sets, e.g. by looking across the table of values or as a covariational relationship between two data sets by looking down both sides of the table simultaneously.

These assumptions formed the whole of what was understood about how to interpret the conceptualization and communication of the idea of function in pre-service elementary teachers.

The researcher starts the analytical process, Patterson and Williams (2002) stated, by reading the texts produced by the individual to understand how this relates to the phenomenon under investigation. These understandings are then used to re-examine the text in closer detail as the next stage in the part-whole analysis. The circle of understanding at this individual level is used to identify themes and create a written interpretation of the text. Subsequent analyses of the texts produced by other individuals may produce similar themes or ones that had not been recognized before; therefore the researcher should reexamine previously interpreted texts as appropriate. Ultimately, cross-case analysis may be pursued if shared meanings are uncovered. In summary, Patterson and Williams (2002) stated that:

> Hermeneutics is an empirical enterprise characterized by critical and "meaningful" thought beginning with a particular perspective (the forestructure of understanding) progressing through a rigorous and systematic cyclical analysis (the hermeneutic circle) in which interpretations are evaluated and modified on the basis of the data that is then presented as evidence of the warrants for conclusions" (p. 36).

In the next section, I will explain how I followed this systematic process of analysis.

Steps Taken in Data Analysis

The texts collected for this investigation were analyzed in three stages. The first stage was conducted to identify a pool of primary participants. This was the briefest of the three stages and took place before the follow-up interviews were conducted. The second stage of analysis focused on the experiences of the six primary participants. The data analyzed for this stage included the individual written work completed during the 'think-pair-together-reflect' sessions and transcriptions from the 'pair-together' and individual interview sessions. The analysis of data followed the same order as the patterning activities, in that texts associated with the 'think' session were analyzed before those texts revolving around the patterning tasks completed during the 'together' session. In addition, the analysis took place at the individual level and followed a somewhat circuitous route by considering each participant's responses associated with one patterning task before considering the next task. This in-depth analysis was cyclical in nature and, subsequently, was the most time-consuming.

This systematic process was repeated with the texts associated with the remaining pattern-finding activities. Each cycle produced a series of interpretive memos that were then reviewed and rewritten as understanding of the research phenomenon continued to evolve in the process of explanation. At this point in the analysis, organizing the various codes and looking for interrelationships between them became a crucial step in the interpretive process (Patterson & Williams, 2002). This third stage of analysis began with an attempt to sort these codes under the initial framework derived from the participants' descriptions of the idea of function.

Hermeneutic analysis requires the researcher to take a circuitous path while interpreting the data. Deciding when and how to break free of this circular path and come to a conclusion can be a difficult task for the researcher. A portrait of pre-service elementary teachers' understanding had evolved over time as individual records were analyzed and interpreted by the researcher and each new interpretation was examined in light of what was understood about how others in the group had described their experiences with patterning. This integration of parts and whole resulted in the identification of common themes associated with the idea of function and common approaches to generalizing patterns. However, the final analysis did not take place until the researcher re-considered what other researchers had said about how individuals conceptualize the ideas of pattern and function. This integration of the parts understood from the present study with the whole of what is understood by others made it possible to sketch a picture of pre-service elementary teachers' understanding of pattern and function.

SUMMARY OF FINDINGS

The purpose of this illustrative study was to examine pre-service elementary teachers' understanding of pattern and function so as to better understand how to prepare them for supporting the development of algebraic thinking in their own students. To address this purpose, the texts produced by six pre-service elementary teachers while they were engaged in pattern finding were collected and analyzed. The design of this study was based on the assumption that understanding in mathematics is a hermeneutical process that evolves through the recursive relationship between explanation and comprehension. This assumption implied that access to pre-service elementary teachers' understanding of pattern and function could be obtained, at least in part, by analyzing the explanations each offered while engaged in pattern-finding activities. These texts were analyzed under the same assumptions of hermeneutics, in that we communicate our understandings with others through language, these understandings emerge through the interaction between the texts we produce, and the learner/researcher's understanding evolves over time through the interplay between explaining what was understood and interpreting what was explained.

The Idea of Function

The first research question asked how pre-service elementary teachers conceptualize and communicate the idea of function while engaged in pattern-finding activities. Four overlapping ideas of function emerged through the analysis of their conversations and written texts associated with the task of pattern finding. Further examination of these four ideas revealed their interdependency, with the kernel of algebraic thinking, the idea of generalization, at the center. All four of these ideas (pattern, relation, rule, generalization) formed a model of algebraic thinking.

Pattern. The participants communicated the idea of function as “a pattern you must find” by referencing how the model changed and how the model stayed the same from one term to the next. In general, statements associated with change, “how many are added each time,” were made before the individual noted the constant part of the model that is “always going to be there.” The identification of the constant part often made it possible to generalize the relationship between the position of the term and the variable part of the model. For example, Cathy explained the advantage of identifying the constant part by stating that “…once you find out what’s not changing…it’s easier to find out what is changing.”

Relation. The idea of function as a relation was manifested in references to a function as “basically a pair” or in attempts to relate the position of a term to its corresponding construction or value. The attempt to formulate this correspondence often made it possible to describe the construction or value of the general term. For example, in Think Pattern 1 Matt described how to construct any term in the sequence by stating that “…you have a center tile and with each figure you add tiles based on the number of the figure to the top, bottom, right, and left of the center.”

Rule. Ashley described a function as “…an equation…and you have to figure out what goes into the equation and what comes out.” This idea of function as an equation or a rule was communicated through direct references to a rule or through statements that indicated the individual understood that a rule “always works.” The idea of function as a rule that always works often operated in conjunction with the idea of function as a relation. Jill applied both ideas to test her “little theories” on the relationship between position and value in a table. This potential connection between the idea of function as a rule and as a relation was also exemplified in Tara’s statement that she “…knew to make a rule, it had to relate to the *nth* term.” Recursive rules, like the one Shelly wrote to describe Think Pattern 1 when she noted that “you always draw or add an additional square to each side to complete the next figure,” are also rules that *always work* but these rules do not consider the idea of function as a relation.

Generalization. Jill described a function as a process that requires one “…to experiment with numbers to see how they related to each other just because that’s how I found it…I had to sit there…mess around with it until it came.” Jill explained how she struggled with the process of generalizing a pattern, stating, “I can see how that one changes…I see how this one changes but I don’t see how they change together.” Statements associated with *how* the pattern, rule, or relationship was forged were coded under this theme and were more likely to be stated when these participants struggled to describe the general term.

Interdependency of the ideas of function. The explanations Tara offered while analyzing the second pattern in the ‘together’ session (Figure 2) illustrate the relationship between these four ideas of pattern. Although Tara had been the first one to state a way to describe the *nth* term of the first pattern the trio analyzed together, she struggled with the second pattern they analyzed. She initiated the task by explaining how the model was changing.

> If you have two in the middle so you add those to make 10 and when you have three in the middle…so you add 1 red block and you add two yellow blocks each time…right?
>
> So one red block and two yellow blocks each time you go down…so what could be our pattern…if we have…*n* red blocks…how do we get a pattern…hang on…let's back up…maybe we should do the first one first.

Tara described the pattern of change as "one red block and two yellow blocks each time you go down." Her explanation of the pattern incorporated the idea of a constant rate of change, a fundamental characteristic of linear functions. Although she has communicated the idea of function as a pattern, she indicated that the rate of change is not the pattern she is looking for.

When one of her partners suggested that "… you just add two to the outside of it," Tara agreed, stating "…two yellow blocks. Okay so we got that but how do you get a relationship for like a rule…from your red block to your yellow block. What is the rule for that?" At this point, Tara demonstrated her understanding of function as a relation by stating "how do you get a relationship…like a rule…from your red block to your yellow block"? She communicated her understanding as a 'relationship…like a rule' that would map one set to another set, in this case, red blocks to yellow blocks.

In contrast, her partner, Anna, continued to consider how the rate of change might help the trio analyze the pattern. She pointed out the similarity between this particular pattern and the second 'think' pattern as both sequences grew by the addition of two and suggested the rule would be the same, $2n + 2$. Tara tested to see if the rule would work when the number of red blocks was two, but quickly discovered the rule did not apply. She returned to her idea that the rule had something to do with the number of sides around the red blocks in the center.

> T: I'm sorry…I know…I was thinking along those lines but I was trying to count the yellow blocks but it extends…but it's all the way around the sides plus four…I think…see if you do 1...2...3...4...5...6…7...8...9…10 plus four whereas if you do this 1… 2...3...4…5...6…7…8 plus four is twelve. That's it. It's the number of sides all the way around the red cube plus four. See what I mean?
>
> A: So how do you put that in a thing…it would be *n* plus…
>
> T: *n* is the number of red blocks so it's the number of sides around the red block…so um…how would you word it or like in a formula for the number of sides around *n*

Tara arrived at the general term in a verbal way but struggled to express her rule symbolically. Anna tried to assist Tara make use of the symbolic.

A: You'd put *r*…ur…*y*…you're saying that the number of red blocks you'd have *r*…plus the number of yellow blocks…that would be.

T: No I'm not saying the number of red blocks…I'm saying the number of sides outside the red blocks. See there's one red block it's got four sides…there's two red blocks it has six sides…oh but that's plus two…six…seven...eight plus 2…that's it! It's um…you were really close…look at that. It's your *r* times 2 plus 2 in parentheses plus 4. Try it out. Let's go. Or let's do *n*…cause it's gonna go *n* times 2 in parentheses plus 2 in brackets plus 4. Let's try it. Okay…so *n* would be 4 red blocks…right?

A: uh huh

T: Times 2 is 8 plus 2 is 10 plus 4 is 14. Yeah!

Although Tara noted how the model was changing with the addition of each red cube, she actively sought to find a pattern that would connect the number of red cubes to the number of yellow cubes in each figure. Once she arrived at a verbal way to connect the two parts of the model, she struggled to find a way to symbolically express that relationship. However, through the process of explaining how to apply her verbal rule, Tara noticed a connection between the number of red cubes and the corresponding number of sides "all the way around the red cubes." Tara was successful because she engaged in the mathematical process of finding "a pattern… a relationship…for like a rule…from your red block to your yellow block" that enabled her to describe the general term.

Examining the process of analyzing patterns revealed the interdependency of all four ideas communicated by this group of pre-service elementary teachers. The illustration of the interdependency of these four ideas of function, displayed in Figure 3, can serve as a model for understanding the nature of algebraic thinking. The process of examining patterns usually began with the analysis of change. This analysis of change is depicted by the largest of the three inner circles and represents the idea of function as a pattern. The user operating under the idea of function as a pattern considers how changes in position relate to changes in the figures used to represent the sequence. This coordination of change, or analysis of covariation, often led to the idea of function as a relation.

The two overlapping, inner circles represent the ideas of function as relation and as a rule. The attempt to forge a relation between the two data sets did not always result in a rule, such as in the misapplication of proportional reasoning sometimes used by Matt. In the same manner, not every rule described the relationship across the table, as in the recursively-defined rule Shelly wrote for Think Pattern 1. Therefore, these two ideas of function, though not disjoint, are not subsets of each other. The region where these three ideas intersect represents the kernel of algebraic thinking, the idea of generalization as stated by Kaput (2000) and Smith (2003). Radford, Bardini, and Sabena (2007) described the "…idea of generalization as a shift of attention that leads one to *see* the general *in* and *through* the particular" (p. 525 – 526). The idea of generalization incorporates the ability to describe the value or construction of any term in a sequence. With pictorial growth patterns, this would mean one is able to visualize the construction of any figure based on its position. With quantitative relationships, one would need to be able to visualize the arithmetic or, as Jill's partner described it, "see the math in it."

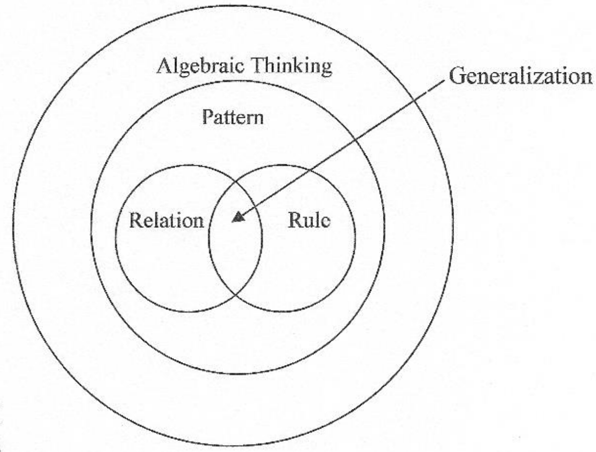

Figure 3. Model of the interdependency of the ideas of function as represented in algebraic thinking.

Demonstrating an Understanding of Function

The second research question addressed how pre-service elementary teachers demonstrated their understanding of function while engaged in the process of generalizing patterns. The approaches taken by the six primary participants in this study were identified and categorized as either pathways or roadblocks. An approach served as a pathway when coupled with the ideas of function as pattern (covariation), relation (correspondence), and rule. Roadblocks occurred when one of these ideas was missing from the explanations offered by an individual, particular the ideas of function as a pattern or a relation.

Pathway. The process of generalizing pictorial growth patterns usually began with the analysis of how the model was changing from one term to the next. These changes were often described as a unit rate of change such as in Tara's description of the pattern of change in Identifying Patterns 2 ("…so one red block and two yellow blocks each time you go down"). The impetus to switch from analyzing sequential change to an analysis of correspondence was provided by the push to find an "easier way" to locate a non-sequential term. The primary participants in this study described looking for an easier way as "what to do to this to get these." Formulating a relationship across the table was achieved using various approaches. Jill resorted to checking "little theories" for connecting the values across the table. Her approach was based on the idea of function as a relation and operated as a pathway because Jill understood that her rule had to work for all data pairs. Her partner sometimes looked for a pattern of differences across the table, as did Ashley. They were successful because they were able to visualize the connection between the changing pattern of differences and the position of the term in the table. Both of these examples illustrate how the successful application of a specific approach was dependent upon an understanding of function as a pattern, as a relation, and as a rule.

Roadblock. The label of roadblock was marked by the failure to incorporate the ideas of function as both a pattern and a relation. For example, Shelly's rule for Think Pattern 1 ("You always draw or add an additional square to each side to complete the figure.") generalized the pattern of change but not the relationship across the table. Matt's use of proportional reasoning was a roadblock in some instances, but a pathway under other circumstances. When Matt focused on the physical construction of the figures in a pattern he usually found a direct way to describe a non-sequential term. By making use of the visual, Matt was able to identify the constant and varying parts of the model. He was then able to apply proportional reasoning to forge a connection between the position of the figure and the part of the model that was changing. On the other hand, when Matt only focused on the quantitative relationship represented in a t-table, his tendency to apply proportional reasoning failed him. In these cases, he would simply work with the last known pair in the table in his attempt to find a non-sequential term, a tendency Kaput (1992) associated with a pre-algebraic understanding of function. Although Matt was working across the table to establish a relationship between position and value, he did not consider how the pattern of change did not represent a simple proportional relationship.

Understanding of function. This group of pre-service elementary teachers demonstrated their understanding of function through the process of generalizing patterns. This act of generalization was achieved through the employment of the ideas of function as a pattern, a relation, and a rule. The successful application of the idea of function as a process to generalize a pattern was dependent, primarily, on the understanding of function as a pattern and a relation. The idea of function as a rule that "always works" often acted in conjunction with function as a relation to explain the construction of the general term. Difficulties generalizing patterns arose due to the omission of one or more of these three ideas of function, most notably the ideas of function as a pattern and function as a relation. The fusion of these two ideas of function worked in tandem to flush out the image of the general term which could then be described either through the use of symbols or words.

SUMMARY

The primary intent of this investigation was to explain the nature of pre-service elementary teachers' understanding of pattern and function. This intent was accomplished, in part, through the analysis of how pre-service elementary teachers communicate and demonstrate their understanding of function while engaged in the process of pattern finding. The first level of understanding involved the identification of the pattern of change which often led to the second level of discriminating between how the model changed and how the model stayed the same. This level of discrimination pointed towards the structure of the relationship, culminating with the generalization of the pattern. These levels of understanding a pattern were described by Sierpinska (1992) as representative of the nature of understanding in mathematics.

According to Sierpinska (1992), the first act of understanding, identification, occurs when an individual recognizes that an object is of special interest. In the case of exploring linear patterns, the individuals in this study first attended to the constant rate of change. The second act, discrimination, occurs when the individual distinguishes both the differences and commonalities between two objects in mathematics. The two objects, in the case of linear patterns, represented how the figures changed and how they stayed the same. Sierpinska noted that the third act of understanding, generalization, was made possible as the individual expands these notions to other settings. This act of understanding a linear pattern was marked by the shift from recognizing the elements of stasis and change in specific terms to the perception of how these elements were represented in the general. This level of understanding resulted in either a verbal or a symbolic rule for describing the construction of any figure in the sequence. Sierpinska included a fourth level of understanding in mathematics; the level of synthesis. At this level, the individual formulates a cohesive concept by noticing the properties shared by all of the objects under study. Someone who recognized the properties shared by all of these linear patterns and applied them in their analysis of patterns, would have developed a cohesive concept of linear patterns and function.

In the pictorial models used in these patterning activities, the constant part of a linear equation was visible and could be separated from the variable part. By identifying the constant and the rate of change, the individual at the synthesis level could quickly generalize the pattern using the properties of a linear function. Cathy progressed from the level of generalization to that of synthesis after her experiences with the second pattern in the 'together' session (Figure 2). She explained how she arrived at that point in her written response to the first reflection prompt:

> "Once I realized that the 6 blocks on each end were constant in every figure, this became much easier to solve. I then realized that for every red block that was added, 2 yellow blocks were added, 2n, and the 6 constant yellow blocks on the end, making the formula, $2n + 6$.

After that point, she stopped trying to get her partners to just see the relationship between position and construction and instead asked them to identify the constant and the rate of change. She then used the constant, *b*, and the rate of change, *m*, to help them write the general rule for a linear pattern as the rate of change times the term number plus the constant.

Under Slavit's (1997) property-oriented view of functions, students assimilate properties of functions through their experiences with different classes of functions. The recognition of these properties creates a library of functions which contributes to the formation of a more complete understanding of the concept of function. The experiences this group of pre-service elementary teachers had with linear patterns initiated the assimilation of the properties of linear functions, one of the basic building blocks in the library of functions. In many ways, the assimilation of these properties emerged through the interplay between explanation and understanding as represented by the hermeneutic circle. Jill explained this phenomenon in her response to the second reflection prompt, stating how her experiences not only strengthened her ability to find patterns, "…but also to EXPLAIN how…(she)…found them." She went on to offer the example of how she discovered "what basic components the sequence should follow." Like others in the group, Jill had begun to notice the properties of a linear sequence.

The interpretations offered in this discussion paint a picture of how pre-service elementary teachers communicate their understanding of pattern and function while analyzing linear patterns. This picture includes a model of how the ideas of functions communicated by this group of pre-service elementary teachers incorporated the important features of algebraic thinking. A description of how pre-service elementary teachers demonstrated their understanding of functions while analyzing patterns was also offered, including a critique of the approaches taken to complete the task. The final part of this picture described four acts of understanding linear patterns, as experienced by this group of pre-service elementary teachers. This multi-layered picture was made possible through the employment of a theoretical framework based on the hermeneutic circle. This paper will conclude with a discussion of how this framework can be modified to investigate individual understandings of other important concepts in mathematics.

A GENERAL FRAMEWORK FOR ACCESSING MATHEMATICAL UNDERSTANDING

Developing a theoretical framework for accessing mathematical understanding through the hermeneutic circle begins with the identification of the types of activities that lead to the development of the concept under investigation. In this particular investigation, pattern-finding was believed to provide a setting for developing and communicating the idea of function (Smith, 2003). If one were interested in student understanding of the concept of limit, then perhaps the task of graphing functions might provide an appropriate venue. A thorough review of the literature on the development of a particular concept as well as personal experience as a mathematics educator should lead to the identification of suitable activities.

In this illustrative study, algebraic thinking formed the matrix of the framework used to study pre-service elementary teachers' understanding of function. This matrix represented the underlying process one employs while engaging in the mathematical activity of pattern-finding as well as in the study of functions. Functional reasoning would most likely form the matrix of a theoretical framework designed to investigate the development of the idea of limit as experienced through the task of graphing. An individual exploring the graphs of function would assumedly experience an evolving understanding of the idea of limit as they communicate the concept to self and others.

Figure 4 represents a general diagram of a theoretical framework for accessing understanding of mathematical concepts through the hermeneutic circle. The use of the hermeneutic circle as a point of access requires capturing an individual's explanations while he or she is engaged in mathematical activities. Key considerations include identifying the underlying process through which a concept is embedded as well as the types of mathematical activities that would engage an individual in the evolving cycle between explanation and understanding of a concept.

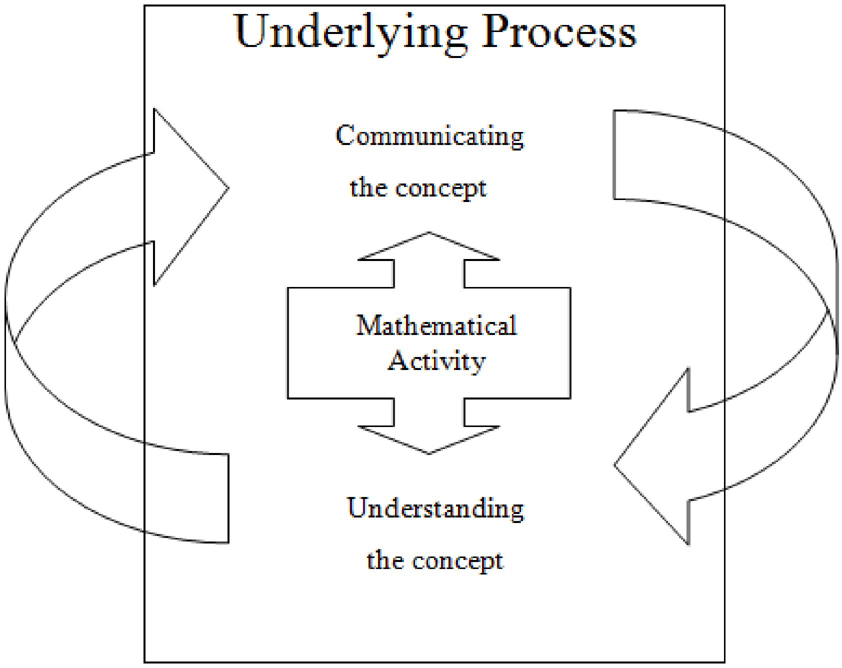

Figure 4. A General Framework for Accessing Understanding through the Hermeneutic Circle.

CONCLUDING REMARKS

The use of the hermeneutic circle as a means of accessing individual understandings of mathematical concepts holds both limitations and promises in mathematics education research. Perhaps the most important limitation is due to the very nature of this type of investigation. Like most qualitative research, the sample size is restricted due to the time required to assemble data records and analyze them for common themes and trends. Once this task is completed, the reported findings are not likely to be replicated with another group of individuals, or even with the same participants. However, as Van Manen (1990) stated, hermeneutic phenomenology is the "…*theory of the unique,* it is interested in what is essentially not replaceable" (p. 7). The results from a study built on this framework adds to the body of knowledge about how others come to know and understand a concept, but also opens the door to more questions in this regard.

The study described in this paper resulted in a rich portrait of how the participants communicated and demonstrated their understanding of function. This portrait included a model of how the ideas of function generated by this group of pre-service teachers overlap to form the basis of algebraic thinking, the idea of generalization. Similarly, the acts of understanding patterns and function were made explicit by reviewing how others had described the nature of understanding in mathematics. However, this study only considered the ways pre-service elementary teachers communicate and demonstrate their understanding of function while examining linear patterns. Other classes of function were not included in this investigation leaving the question of whether or not the same ideas of function or acts of understanding would still apply unanswered. In addition, this study did not consider how different representations of the same class of function might influence the process of generalizing patterns. However, repeating the study with other classes of function and/or with other mathematical activities would create an evolving understanding of how others conceptualize the idea of function.

The promise of using a framework based on the hermeneutic circle lies in its ability to pinpoint how others come to understand key concepts in mathematics. Repeated applications of the framework using different mathematical activities and/or representations of a concept have the potential of creating a multi-layered description of how individuals come to understand a concept in mathematics. Much like individual understandings evolve over time as one engages in mathematical activities, researchers' explanation of the concept images held by others would also be developing over time.

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Gender Differences in Spatial Ability: A Review of the Literature

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ABSTRACT: *In this study, a broad literature has been scanned on gender differences in spatial ability. Clarification of what has been meant by spatial ability and what it incorporates has been followed by illustration of arguments on the first appearance of gender related performance differences. Subsequently, a variety of researchers' attempts to explain the whys and wherefores of such discrepancy have been presented. Finally, significance of spatial ability and possible starting points to facilitate resolution of the issue has been discussed.*

Key words: *Gender differences, Spatial ability, Mathematics education, Mental rotation.*

INTRODUCTION

"If a square that has sides of 4 cm. is rotated 180 degrees around one of its sides, then what would be the volume of this shape?" is a typical question that can be seen in mathematics books or examinations and that requires mainly ability to think in 3-dimensional space rather than ability to calculate. When you ask someone for an address, in order to understand his/her explanations you normally either summarise and memorise them or visualise a map in your mind. Cognitive activities such as these, again, demands mental visualisation of the 2-dimensional space. Many examples can be given about our usage of ability known as "spatial ability" as human beings. This complicated cognitive ability is used in daily life as well as in our education particularly in mathematics. The idea that spatial ability is a requirement for mathematical achievement is widely acknowledged. Halpern (2000), for example, stated that:

> If you think for a minute about the nature of advanced topics in mathematics-geometry, topology, trigonometry, and calculus- you will realise that they all require spatial skills (p 118).

Mathematics is a subject, which comprises both spatial and verbal/numerical components. The former constitutes two- and three-dimensional shapes and images, whereas the latter is made up of letters, symbols and numbers. It is rare to be very skilled in both components (Clausen-May & Smith, 1998). Throughout the years, the interaction and interrelationship among mathematics achievement, spatial-visual skills and verbal skills has been investigated. It is acknowledged that there is a strong connection between spatial-visual abilities and mathematics achievement. Guay & McDaniel (1977) examined this association among elementary school children and concluded that high achievers in mathematics have greater spatial ability than low achievers. In a relatively more recent study, almost a replication of the previous one, stronger and much clearer evidence has been found. Seng & Chan (2000) came to the conclusion that there was a significant positive relationship between spatial ability and mathematical performance. The spatial ability score was a significant predictor of students' mathematics performance (Delgado & Prieto, 2004; Casey, Nuttall, Pezaris, & Benbow, 1995; Battista, 1990; Sherman, 1979).

In spite of the significance of spatial ability as one of the predicting parameters in students' success in mathematics, there seems to be differences in how girls and boys perform in spatial tasks (Voyer et al., 1995). This may result in gender related differences in students' mathematical performances.

In this vein, this study has three main aims. First aim is to look at the present studies regarding gender differences in spatial abilities from the eye of a teacher, because the studies in this area have generally been conducted by researchers from a variety of background ranging from psychology (Hegarty et al., 2006) to architecture, medicine (Keehner et.al, 2006; Stull et al. 2009), meteorology (Hegarty, M. et al., 2009; Hegarty, M. et al., 2010) engineering (Mohler, 2008; Nemeth & Hoffmann, 2006), and even apparel design (Workman, Caldwell, & Kallal, 1999). Secondly, it has an aim to contribute to awareness of teachers, mathematicians and those who are in charge of and responsible for teaching mathematics. Thirdly, to collect relevant resources pertinent to spatial ability in existing literature in order to help those, among mathematics educators, who will make further research on this area.

This paper will look at gender differences (GD) in spatial abilities (SA) from educational perspective in the light of the findings of several studies. In the first section definition and categorisation of spatial ability will be given. Subsequently, discussions on the magnitude and first appearance of these differences will be examined in the second and third sections. In the fourth section, possible explanations of these differences will be explored. Teachers' current approaches to spatially gifted students and some implications for the curriculum are also incorporated in the last section.

WHAT ARE SPATIAL ABILITIES?

Although it is generally accepted that spatial ability is an important component of intellectual ability (Linn & Petersen 1985), it is difficult to define spatial ability, compared to verbal ability or quantitative ability (Maccoby & Jacklin, 1974, p 91). The difficulties in defining spatial ability resulted in a variety of definitions. Some of them will be briefly introduced. McGee (1979, p 893) defines it as "the ability to mentally manipulate, rotate, twist, or invert a pictorially presented stimulus object". Petersen (1976) regards spatial ability as the ability to visually manipulate images without recourse to verbal mediation. Spatial ability is also defined as "skill in representing, transforming, generating, and recalling symbolic, non-linguistic information" (Linn & Petersen, 1985). Although among some authorities the perception of objects and their mental rotations, cognition of horizontality and/or verticality and location of simple figures within complex figures are accepted as spatial skills, there is no consensus on the definition of spatial abilities (Linn & Petersen, 1985). Finally, Halpern (2000, p 60-61) approaches the issue from spatial abilities' measurement aspect. Hundreds of tests might be used to measure the different aspects of these abilities and obviously not produce the same results. Therefore, a variety of tests should be employed to provide empirical evidence for illuminating an aspect of spatial abilities (ibid.).

Several researchers proposed a number of ways to categorise components of spatial ability (McGee,1979; Lohman, 1988; Carrol, 1993). Hegarty & Waller (2006) can be referred to for detailed information about various categorisations. Following categorisations of large scale meta-analyses (Linn & Petersen, 1986, Voyer et al., 1995), three main components of spatial ability will be considered in this review.

These are: Spatial Perception (SP), Mental Rotation (MR), Spatial Visualisation (SV).

Spatial Perception can be defined as the ability to locate the subjects horizontally and vertically, despite distracting information. This ability has mainly been measured by the Rod and Frame Test, introduced by Witkin (1949), and the Water-level task (Linn & Petersen, 1986). This skill might be the basic ability that underlies children's geometric understanding.

Mental Rotation is generally referred to as the ability to rotate two or three-dimensional objects mentally. Its nature has been thoroughly identified in Corballis (1982).

Spatial Visualisation is defined by Voyer et al. (1995) as 'the ability to manipulate complex spatial information when several stages are needed to produce the correct solution'. It is commonly measured by the paper-folding test, the hidden figures test and the spatial relations section of the Differential Aptitude Test, and requires complicated analytic and visual processing of spatially presented information (Linn & Petersen, 1986, p 71). Although Halpern (1992, p 211; 2000, p 101) gives, as a fourth component, spatiotemporal judgement which was defined as the perception of objects moving in space, in this paper, spatial ability will refer to these three components as do most of the pertinent literature.

MAGNITUDE OF GENDER DIFFERENCES IN SPATIAL ABILITIES

Throughout the years, the difference between sexes in spatial abilities has been one of the unsettled topics of research in education and psychology. This is due probably to spatial ability's complexity and/or the controversial outcomes of research and debates regarding gender differences. As far as mathematical and spatial abilities are concerned, many researchers have devoted much of their time to clarifying their nature. Some, like Macfarlane-Smith (1964), believed that spatial ability is the "key" ability underlying mathematical ability. However, many psychologists, who see the mathematical ability as “the combination of general intelligence applied to the mathematical context” (Bishop, 1980), have barely taken the classroom into account. Yet, they have made a considerable contribution to education (ibid.). Many researchers have investigated human spatial skills from the point of view of the psychology of sexes and that of gender differences. They have also examined the magnitude of gender differences in spatial ability, its first emergence in the life span and the possible reasons why there is such a difference. Because of the advances in meta-analysis technique, relatively some studies in this area, such as Hyde (1981), Linn & Petersen (1985, 1986), Baenniger & Newcombe (1989) and Voyer et al. (1995), have employed this technique, which is essentially synthesising empirical studies and approaching that synthesis statistically. These meta-analytic studies provided an excellent overview on the researches of gender differences in spatial ability for relatively early research. Eagly (1986) can be referred to for detailed information about the meta-analytic approach to gender difference research.

Despite the existence of some exceptions, majority of the studies reported that males outperform females in the first two of the three spatial abilities (Maccoby & Jacklin, 1974; Halpern, 1992; Stafford, 1961; Macfarlane Smith, 1964; Nordvik, Amponsah, 1998; Heinrich, 1998; Hines, M. (2004). Very recently Estes (2012) argued that “of all cognitive sex differences, the mental rotation of abstract figures in 3-dimensional space is the most robust”. Peters et al. (2007) is possibly the study with largest sample size on this area. They observed “clear” sex differences in mental rotation test for a sample of 255,100 people from seven ethnic groups (134,317 men and 120,783 women). However, there are arguments particularly regarding the magnitude, first emergence and basis of these sex differences for all three components of spatial ability. In their literature review, Maccoby & Jacklin (1974), for instance, conclude that GD in SA are “well-established”. On the other hand, Hyde (1981) applied meta-analysis technique to the studies reviewed by Maccoby & Jacklin, and his conclusion was that GD account for only 1%-5% of the population variance. But the deficiency of his study was omitting non-U.S. studies, which might increase sampling concerns. However, relatively more recent research such as Rilea et al. (2004) and Halat (2006) shows that the issue at hand is much more complex than it seemed in the earlier research. Having reviewed the literature, it seems safe to state that at best males hardly underperform females.

The book of Macfarlane Smith (1964) was regarded as the first well-organised source of information in the area of spatial ability for both psychologists and educators by many researchers. The studies that were conducted up to 1974 were reviewed by Maccoby & Jacklin (1974). In 1985, Linn & Petersen examined the researches from 1974 to 1982; finally Voyer et al. (1995) questioned almost all the studies made up to 1995. The question of “what is the magnitude of gender difference in spatial ability?” has promoted many of the studies. According to Linn & Petersen (1985), see Table 1, mean effect size for spatial perception tests is 0.44 ($p<0.5$). It is 0.73 ($p<0.5$) for mental rotation tests and there is a tendency to increase in this figure when the items in tests needs to be rotated a larger angle. In other words, males dominance over females in mental rotation test increase with a larger angle of rotation. Additionally, Kail et al.

(1979) noted that 30 % of all females were slower than all males. Recent studies also shown similar results (Peters, 2005; Maylor et al., 2007; Lippa et al., 2010).

Table 1

The Results of the Meta-analysis of Linn & Petersen (1985)[1]

Group	N	Weighted Estimator of ES	95% Confidence interval for ES
Spatial Perception			
All ages	62	.44	.04-.84
Under 13	26	.37	-.06-.81
13-18	23	.37	-.11-.85
Over 18	13	.64	.31-.97
Mental rotation			
All ages	29	.73	.50-.96
Spatial Visualisation			
All ages	81	.13	-.24-.54

Voyer et al. (1995), see Table 2, provided further evidence for the existence of such a difference in spatial perception and mental rotation in favour of males. On the other hand, in terms of spatial visualisation, the outcomes of the studies showed no significant difference (ibid.). Hence, a satisfactory research to clarify the gender difference in spatial visualisation ability is needed.

THE FIRST APPEARANCE OF GENDER DIFFERENCES IN SPATIAL ABILITIES

The emergence age of spatial ability was one of the dimensions of the spatial ability researches. Its changes in the life span were also investigated. As far as educators particularly mathematics teachers are concerned, it is important to notice this age in order to adjust their teaching technique according to students' understanding level. However, it is difficult to give a precise age, at which this difference emerges, instead, only some estimations have been made. There is a rising trend in gender differences parallel to subjects' age (Table 1 and Table 2). In fact, the differences in question come out in early adolescence and increase significantly by the age of 18, which is generally the first year of university. Linn & Petersen state:

> Our meta-analysis results contradict the assertion that sex differences in spatial ability are first detected in adolescence. For spatial perception,

[1] Note: Adapted from Linn, M. C. & Petersen, A. C. (1985). Emergence and characterization of sex differences in spatial ability: a meta-analysis. *Child Development*, *56*, 1479-1498. Adapted with permission.

differences are detected in individual studies at about 8 and, for grouped studies, emerged at age 18. For mental rotation, sex differences are detected whenever measurement is possible, although some versions of the test are inappropriate for those under 13. For spatial visualisation, there are no differences. Thus sex differences are detected prior to adolescence for some categories of spatial ability and not at all for others (Linn & Petersen, 1985).

Table 2

The Results of the Meta-analysis of Voyer et al. (1995)[2] *(* p<0.5)*

Category of Tests	N	Weighted Estimator of ES	Test of significance for ES (Z)
Mental rotation			
All ages	78	.56	4.63*
Under 13 years	13	.33	2.00*
13-18 years	23	.45	4.21*
Over 18 years	42	.66	5.55*
Spatial Perception			
All ages	92	.44	2.25*
Under 13 years	21	.33	1.73
13-18 years	18	.43	2.18*
Over 18 years	53	.48	2.48*
Spatial Visualisation			
All ages	116	.19	1.43
Under 13 years	40	.02	0.12
13-18 years	20	.18	1.52
Over 18 years	56	.23	2.00*

As the word 'adolescence' generally refers to ages over 12, all above presented findings might imply that gender differences in spatial abilities gradually commences to emerge at the beginning of the secondary school years. Subsequently, these abilities flourish throughout high school years, and during the university years show further progress. If the difference between the sexes is getting larger analogous to this emergence, and these changes occur during the school years; how many mathematics teachers are aware of this?

After giving different definitions of spatial ability, its sub-divisions and magnitude of gender differences in this ability, the first appearance age of gender differences has been examined. The following section will deal with attempts at explaining gender differences in spatial abilities.

[2] Note: Adapted from Voyer, D., Voyer, S. & Bryden, M. P. (1995). Magnitude of sex differences in spatial abilities: a meta-analysis and consideration of critical variables. *Psychological Bulletin*, *117*(2), 250-270. Adapted with permission.

Attempts of Explanation

The possible reasons of gender differences in spatial abilities have also been widely investigated by many of psychologists. Due to the fact that a majority of researchers who conducted the studies in spatial ability are psychologists, educators have some difficulties. These difficulties are mainly in investigating likely solutions of the gender difference problem in spatial ability among students. On the other hand, it seems unreasonable to reject the help of the studies conducted by psychologists. This section aims to provide, at least, a point of view to the educators to be used in a search for a remedial solution for gender differences in spatial abilities.

Several lines of the explanations for gender differences in spatial abilities converge on a number of points: biological explanations, life-experience-based explanations and process-based explanations. It seems quite legitimate to acknowledge that none of the explanations can be *the only* reason for these differences. Instead, different combinations of them are most likely reasons for these differences.

Biological Explanations

Biological mechanisms of humans have been widely investigated to explain GD in SA. Hypotheses mainly focus on two points. First hypothesis is hormonal changes particularly in puberty. Second one is genetic factors. However, no clear explanation has been made.

Waber (1977) claimed that timing of puberty is an important determinant of the sex differences. Her proposal was based on the assumption that females are early-maturers in comparison with males. She investigated maturation time, differences between sexes and within the same sex. Independent of sex, late-maturers performed on SA tasks better than early-maturers. Many researchers, like Petersen (1976), accepted and quoted her study.

Considering changes in spatial ability and sex hormones, both come to peak about 18 and then begin to decline. This figure is considered further evidence of linkage between gender differences in spatial abilities and sex hormones (Petersen, 1976). Recent studies reported interesting results on this issue. Some studies confirmed testosterone levels are found to be related to spatial ability in males and females. (Gouchie & Kimura, 1991; Kimura & Hampson, 1994; Bell & Saucier, 2004; Goyette et al., 2012) However, the idea that testosterone would relate “directly” to spatial performance is not confirmed (Nowak, 2012). It seems that, there is some support to this view by subsequent studies, and that it is not completely refuted (Halpern, 2000, p 170).

Genetic factors have also been hypothesised as one of the causes of the difference in SA between sexes. Supporters of this view, like Stafford (1961) and Hartlage (1970), put forward that spatial ability is determined by a genetic code on X chromosome. It is shown that spatial abilities are inheritable (Vandernberg & Kuse, 1979; Wittig, 1979)

Furthermore, in his review McGee (1979) provides evidence that spatial abilities are the most inheritable ones compared to verbal, quantitative, reasoning and memory abilities. At this point, explaining the whys and the wherefores of the claim that these abilities are carried by a recessive gene on X chromosome is out of the scope of this paper. However, for the basics of this idea, McGee (1979) and for current approaches, Halpern (2000) can be consulted.

In brief, despite the inharitability of spatial abilities being supported, neither X gene theory nor timing of puberty approach gets enough support from research.

Life–experience Based Explanations

It is a widely acknowledged fact that expectations from boys and girls are very different. In the very early ages, it is unusual to see a mother who takes her girl to buy a toy from the ‘boy’s toys section’. In this early stage girls are expected to play with ‘girl’s toys’. Boys generally play with the toys that promote spatial abilities like building blocks, balls and darts. Sherman (1967) suggested that different sex role patterns might result in differential childhood experiences, which affects the development of visual-spatial abilities. According to her, boys spend more time in such activities that promote these abilities than girls do. In fact, boys probably spend more time in ball games than girls do. These games are likely to require being accurate in estimating the destination of the ball, which is defined as spatiotemporal judgement by Halpern (2000, p 101). Throwing games like dart throwing may be regarded as the same category in terms of its requirements. Newcombe, Bandura & Taylor (1983) found an association between participation of spatial activities and spatial abilities. Baenniger & Newcombe (1989) conducted a meta-analysis to test this hypothesis. They found that prior participation in spatial activities is correlated with higher spatial test scores, and yet the magnitude of this effect is very small. Boys and girls differ in their selection of (masculine vs. feminine) toys, following television shows, playing computer games, and their outdoor activities and that this may provide differential opportunities for the development of visual-spatial skills (Cherney & London, 2006). Some researchers paid special attention to use of computer games and it has been found to be beneficial for improving spatial skills (Terlecki & Newcombe, 2005; Cherney, 2008). It can be concluded that it is reasonable for boys to outperform girls in the tests which measure spatial abilities because they participate in more activities promoting spatial abilities than girls do. This point of view is also in agreement with the other findings regarding the time of first appearance of spatial abilities. In this sense, what can be done is probably to provide materials encouraging different activities for children from the nursery schools onwards in order to prepare their minds for their education life.

Process-based Explanations

Spatial abilities are generally measured by tests. It is found for the same items of a test that different strategies are employed (Kyllonen et al., 1984). Differences in problem solving processes are widely accepted reasons in explaining gender differences in spatial ability tests. Thomas & Lohaus (1993) found in verticality and Water-level task that strategies of poor performers are 'clearly' different from those of good performers. As regards sex differences in spatial abilities, three possibilities have emerged: difference in (a) acquiring strategies, (b) selecting a proper strategy, (c) efficiently applying the strategy (Linn & Petersen, 1986). These possibilities will be discussed below.

First possibility is difference between males and females in acquiring strategies. If males are better in acquiring and developing new strategies and also adding them to their repertoire, they have the advantage of the chance to use different strategies in different tasks. If we consider Sherman's (1967) finding that boys spend more time in activities that promote these abilities than girls do, it can be seen that this point of view is consistent with life-experience based explanations. On the other hand, a question arising from that assumption about females is "why is there such a 'constraint' on the ability to acquire solution strategies? " (Linn & Petersen, 1986)

Another suggested possibility is that if both sexes acquire the strategies equally, there can be a difference in choosing the proper strategy in the process of solving the same problem. In this sense, there can be two main approaches.

First, it is possible in a test that girls employ the same strategy for questions which seem the same, whereas boys, because of their above outlined life experiences, may choose a more appropriate solution strategy. An example drawn from Kail et al. (1979) illustrates this point. They examined 104 undergraduate students with a mental rotation task and found that 30% of all females rotated items slower than males. Regarding this study, they stated that:

> One possibility is that these women used a different strategy of mental rotation than the rest of sample. That is, two general strategies of mental rotation can be distinguished. In 'holistic' mental rotation, the entire comparison stimulus is rotated simultaneously into congruence with target. A second strategy is one in which the individual features of comparison stimulus are rotated separately. That is, the mental rotation process is applied iteratively until all features of a stimulus have been rotated (Kail et al., 1979).

They indicated that some of the females showed a slower rate of rotating objects because females may have employed the latter strategy in which features of objects have been rotated sequentially. This strategy-choosing mistake may be because of the lack of experience that has been examined in the life-experience based explanations section.

Secondly, the conclusion of Gallagher & De Lisi (1994) has offered another possibility. They found that female students used generally conventional strategies, which are taught in classes. They also found that girls are better on well-defined problems and the problems in which the solution is straightforward. These results suggested that girls seemed to rely mainly on methods that are taught in schools. However, boys employed unconventional strategies and they are better on unusual and out of class problems. On the other hand, spatial ability test items, particularly mental rotation questions, where males are clearly superior, are generally unusual. These types of questions are not usually found in the school curriculum, and they require unconventional solution strategies. Therefore, the nature of spatial ability tasks might be regarded as another reason for gender differences in spatial abilities.

Another process-based explanation for sex differences in spatial abilities is in relation to efficiency of strategies. A recent study exemplifies this idea. Hosenfeld, Strauss and Koller (1997) investigated the underlying basis of males' superiority in spatial tasks. They questioned whether it was because of choosing a superior (holistical) strategy or because of the more effective usage of this strategy on 403 students. Results indicated that greater effectiveness in applying the strategy is the reason for males' superiority. Further support to this view comes from Linn & Petersen:

> Females and males do not appear to differ in ability to select the best strategy. Rather, they may differ in repertoire of strategies available for them. Tasks that require a single specialised strategy may reveal gender differences because the most efficient strategy is less well developed in females than in males (Linn & Petersen, 1986, p 92).

DISCUSSION AND REFLECTION

In recent years, spatial abilities gained more importance parallel to advances in sciences. In his book Macfarlane Smith (1964) states, as if he was writing today, that:

> A review of literature suggests that spatial ability becomes increasingly important in more advanced mathematical studies in which greater emphasis is placed on analytical and abstract thinking and on problem-solving... spatial tests may have value for selecting good workers in certain fields, such as mathematics and physical sciences (p. 295).

Spatial ability is an important skill in mathematics achievement but it is clear from this study that boys generally outperformed girls on spatial-visual tasks. We also demonstrated that gender differences in these abilities could be eliminated to a certain extent. There are, however, several questions arising from the findings and the ideas explained throughout the paper. These issues should not be neglected, rather they are quite important to take into consideration. First of all, if we consider there is a robust relationship between mathematics achievement and spatial ability, whilst keeping in mind the present mathematics teachers' approaches to the students; to what extent are mathematics teachers paying attention to development of this ability. Are we changing our teaching methods, if there are some students who are spatially talented but not verbally, in our class? Moreover, we should take into account that this ability, as above outlined in the biological explanations section, might be inherited. This basically means there are likely to be many spatially gifted people in each generation. This being the case, the importance of the problem should be recognised. Secondly, may be the question should be ‘Are the mathematics teachers aware of these spatially talented students?’ The answer ‘no’ is coming from Clausen-May & Smith (1998). Their handbook is probably the most significant source aimed in order for teachers to understand the significance of this ability. They stated that:

> …history shows that children with exceptionally strong spatial abilities and relatively weak linguistic abilities are likely to suffer at the hands of their teachers…spatially gifted people have continued to experience educational difficulties up to the present day…their thinking potential was never recognised or developed, even if they did very well academically (ibid, p 2-3).

An example drawn from the same study illustrates the point well. Martin is a graduate of materials technology in his 30s. All through his school life he “struggled” to be a successful student in mathematics but his case was realised after beginning to fail final year examinations. He was diagnosed as ‘dyslexic’ by a psychologist. The term ‘dyslexic’ refers to an illness that causes problems in verbal skills, namely spelling and reading. On the other hand, his IQ was calculated as 138 and he has been found to have ‘extremely good spatial ability’. He might be an extreme example but his statements are quite striking for the teachers:

> The current teaching styles do not take account of this and students like myself find that we continually underachieve purely because we are unable to demonstrate our full ability. Current teaching methods both stifled me and destroyed my confidence in my own intellect (ibid, p. 4-5).

He eventually graduated. How about some students who are not identified by their teachers? They are probably regarded as “backward” or “lazy” because of their difficulty in expressing themselves. Gohm et al. (1998) investigated spatially gifted high school students. They chose 1960 spatially talented students among 40,000. They pointed out that relative to the students gifted in mathematics, the spatially talented students are not fully utilising their academic capabilities, received less college guidance from school personnel and they are less motivated by teachers. Gohm et al. (1998) noted, in explaining this result, that because in school admission tests verbal abilities are emphasised, teachers might have paid attention mostly to verbal abilities.

Thirdly, if the school admission tests are privilege those with better verbal abilities how can we expect teachers to notice and identify spatially talented and spatially poor students? If spatial ability of the students does not get enough attention from teachers of various subjects such as physics, chemistry, technical drawing and especially mathematics, how can we expect next generations to realise their full potential? Teachers of these subjects should be trained to notice students with high and low spatial ability. Only then teachers will become well aware of students with this ability, and this will result in awareness of gender differences. Otherwise gender related differences in students’ spatial ability is more likely to receive very little attention.

The fourth important point that should be realised is the curriculum side of the issue. One of the mathematics teachers might claim that they are hindered by mathematics curriculum. It is a quite crucial dimension of the issue. In the light of all these findings it seems reasonable to ask that “What do we have in our curriculum to diminish the difference?” Moreover, it would be of interest in studies to ask the question, “What kind of modifications should be made to the curriculum to enable teachers to recognise the significance of students’ spatial ability?” It seems legitimate to look at this point of view expressed by Del Grande (1987), who states that:

> Builders of geometry curricula should take into account the development of the child’s understanding and space and process of visual information. Those involved with the teaching of primary mathematics must be aware of the spatial abilities of the students they teach and attempt to adjust instruction to those abilities. As research provides more information, those teachers will be able to do a more effective job of adapting instruction to the needs and abilities of their students. Spatial perception activities will be effective only if they are integrated into a well-rounded program and take into account the child’s total development (p. 135).

Any further statements on curriculum issues such as how should curriculum be restructured and what needs to be included or excluded to foster teachers awareness of individual differences in students’ perception and visualisation of subjects in 2 and 3 dimensional space would be out of the scope of this paper.

CONCLUSION

In this study a broad literature has been scanned. In this process, many studies were omitted; therefore, they did not presented in references section because of two reasons. First, some were irrelevant due to their purely psychological perspective, and the second, the restricted limits of this paper.

Although it is recognised by many psychologists and some of the educators that spatial ability is one of the important requirements in many subjects, this ability has not been given the attention it deserves. This being the case, gender differences in spatial abilities are either ignored or not fully recognised. In this sense, the present paper has been written to contribute to the awareness of educators. However, further researches required especially in such areas as teachers’ perception, awareness of and ability to identify spatially underdeveloped students, teachers’ knowledge and beliefs about possible remedial courses, and the ways in which gender related performance differences can be resolved.

What can be inferred from this paper is that the key to this issue lies in many education systems, which is mainly based on verbal/numerical abilities. There appear to be need for qualified teachers who consider students abilities and adjust their teaching techniques depending on students’ abilities. The question that teachers and teacher educators should probably ask is “How many Albert Einsteins disappeared in the hands of unqualified teachers up to present?" and, furthermore, "What would be if they were not?" It seems reasonable to suggest that changes should probably commence in the minds of teachers and those in charge of preparing the curriculum for the future generations.

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Constructivist Approach to Embodying Motion Problems in Mathematics

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ABSTRACT: *Research has shown that students have difficulties distinguishing position-time graph of a moving object and its path of motion. This study reflects on this deficiency and proposes three-block interdisciplinary lessons situated within the constructivist approach. It is hypothesized that providing students with tangible learning media such as experiments and having them experience abstract mathematical ideas helped eliminate this deficiency. Although the authors did not embrace the idea within a formal experimental research study, a case study targeting this deficit was conducted. The results from this case study piloted in one central Texas high school addressing a group of (N = 41) pre-calculus students revealed promising results. This paper outlines the contents of the lesson treatments conducted with these students. It also provides students' responses to a qualitative question that sought to find out if they could differentiate between an object's path of motion and its corresponding position time-graph. Analyses of these responses along with suggestions for further research are included.*

Key words: *Position-time graph, Path of motion, Constructivist learning, Linear motion.*

DESIGNING THE LEARNING ENVIRONMENT

Motion problems, traditionally taught in science classes are often embedded in mathematics curriculum and serve as means of applying mathematics in real life situations (National Council of Teachers of Mathematics, 2000). Lacking a scientific contextualization in mathematics classes these problems are often reduced to disjoint procedural rules that students are to memorize (Smith, 1996). Thus in order to support conceptual understanding, a need for providing students with a rich content emerged. The conceptual framework that guided the content design was supported by a constructivist perspective in which learning is perceived as a process of experiencing dissonance and working to resolve these dissonances by building viable explanations (von Glasersfeld, 1995). Constructivists believe that learners construct their own reality or at least interpret it upon their perception of experience. As such, an individual's knowledge is defined by constructivists as a function of one's prior experiences, mental structures, and beliefs that are used to interpret the environment around. What someone knows is grounded in perception of the physical and social experiences which are comprehended by the mind (Jonasson, 1991). There are multiple strengths of this learning theory. One of them is situating the learning in realistic settings and consequently placing learner's ability as a major factor for the learning to occur. In return, the learner, while experiencing the knowledge and constructing its own interpretation, stores it in the long-term memory allowing the knowledge to be available to apply in new situations. Another modern view on constructivist learning theory is presented by Clark & Mayer (2011) who de facto describe learning as a change due to learners' experiences and the knowledge as the magnitude of the change. The proposed in this paper constructivist learning environment links its contextual content to physics. Several studies have shown that integrated curricula provide multiple opportunities for more relevant and more stimulating experiences for learners (Frykholm & Glasson, 2005; Koirala & Bowman, 2003). Furthermore, integrated curricula have also helped learners acquire deeper understandings, recognize relevance and become interested and motivated in school (George, 1996; Mason, 1996). Consequently, positive learning attitudes have led to students' higher academic achievement (McBride & Silverman, 1991).

This manuscript describes the creation of a block of three interdisciplinary instructional lessons on motion for high school mathematics which provide students with scientific contexts that highlight motion conceptualization followed by motion mathematization in the forms of algebraic functions. Intertwined in constructivism and the content domains of mathematics and physics, this activity can serve several educational purposes such as (a) construction of new knowledge, (b) enhancing applications of new concepts in real life situations, and (c) presenting students with opportunities for exercising the process of mathematical modeling. This activity can be introduced in a high school mathematics class that encompasses the concepts of linear and quadratic functions. Due to its extensive kinematics component, it can also be conducted in a physics class.

One of the common misconceptions in mathematics class is students' lack in distinguishing position-time graph and path of motion. There is substantial research body regarding students' lack of comprehension of techniques of graphing motion (Cooney & Wilson, 1993; Kaput & Roschelle, 1997; Nemirovsky & Monk, 2000). The findings showed that students have difficulties reading and interpreting graphs of motion. Several researchers have pointed out the tendency of students to interpret the position-time graph as a picture of the actual path of an object's motion (Leinhardt, Zaslavsky, & Stein, 1990). A sound understanding of kinematics of motion is essential to succeed in calculus classes. Reviewers' of AP Calculus courses frequently highlight students' difficulties in contrasting velocity versus speed, and displacement versus distance (College Board, 2009). Pre-calculus courses, deeply enriched by exercising various function representations, provide great opportunities for addressing this misconception. Constructing mathematical and science knowledge involves not only manipulating symbols and equations; it also involves coordinating and interpreting mathematical relationships using specialized language of science (National Research

Council, 2000). In order to satisfy these goals, we propose to deconstruct fragmented ideas about motion and formulate a foundation that will enable the learners to first scientifically classify and then mathematically embody various types of motion. The theme of this activity provides opportunities for further coordination of mathematical symbols and their physical interpretations.

LEARNING OBJECTIVES

Presented lessons encompass multiple high school mathematics and physics learning objectives. The ultimate goal of these units is helping students achieve higher scores on mathematics and science exams as well as enhance their college readiness. The following are the areas of a particular concern:

Focus on processes. One methodological stronghold in the mathematics standards (National Council of Teachers of Mathematics, 2000) is focusing on developing critical thinking processes and skills. Students analyze given tasks and design strategies to mathematize them. Students need to recall processes and mathematical relations to express objects' positions in algebraic forms; however, a significant shift to validate the derived functions makes students revisit the process and correct if needed.

Fostering scientific thinking. While working on developing their approaches, students will discuss various strategies. In doing so, they will embrace scientific argumentation, including hypothesizing and drawing conclusions. They will predict certain behavior of their designs and test them. Consequently, students will immerse in an inductive inquiry process that will lead them to formulating the final products as described by the National Research Council (NRC, 2000).

Enhancing graph reading and interpretation. An emphasis will be given to graph interpretations which will reflect recent recommendations on placing more emphasis on constructing and reading graphs on high-stakes testing (Monk, 2003; Vogler & Burton, 2010). By incorporating mathematical apparatus, this activity provides great opportunities for mastering these cognitive skills. While constructing graphs from real data, and then interpreting them, students connect a real environment with mathematical abstractions. Consequently, it is intended that students' internal images of abstract mathematical symbolism will solidify into meaningful knowledge. Engagement is catalyst of students learning (NRC, 2000). Research shows that engaging students in meaningful work increases the rate of learning new concepts (Hancock & Betts, 2002). An engaged learner is inspired to accomplish the desired goals even in the face of difficulty (Schlechty, 2001). Building on that, we concluded that the degree, to which students engage in this activity, will not only help increase their achievement but will also have a significant impact on the quality of their final projects. High school students find it especially motivating when the learning environment relates to what they experience in everyday life and when the environment provides them with opportunities to apply knowledge learned in school (Shapiro, 1994). We incorporated this finding into building blocks of the constructivist activity by introducing and engaging the learners to the world of discovering patterns of motion through the conceptual lenses of mathematics and physics.

Fostering the skills of measuring scientific quantities. A well-designed scientific activity provides opportunities for practicing the skills of measuring, reading, and writing scientific quantities such as time, position, mass, or temperature. The applications of these skills, as well as identifying the degree of accuracy and precision were also incorporated into this activity. Albeit, measuring skills are traditionally of a science domain, measurement has received significant attention in the newly developed common core standards across all subjects, including mathematics (Porter, McMaken, Hwang, & Yang, 2011). This activity also reflects this recommendation.

GENERAL OVERVIEW AND LOGISTICS OF THE LESSONS

The fundamental questions that we asked before designing the lessons were the following:

1. Will a constructivist learning environment help students understand the differences between position-time graph and path of motion?
2. Will the students correctly elaborate on the differences between these two representations?

Stringed by the questions, the treatment of this study was developed. During lesson one, the students will contextualize and mathematically describe an object's path of motion and describe an object's velocity as positive or negative. Lesson 2 will introduce the learner to the process of motion quantification, thus the concept of velocity and its calculations will be exercised. The concluding lesson 3 will have the learner apply concepts, path and position time graph to describe motion presented by a simulated environment.

Materials and Equipment. During the first lesson of the unit, students would need to be equipped with battery powered toy-cars as shown in Figure 1, measuring tapes, and compasses.

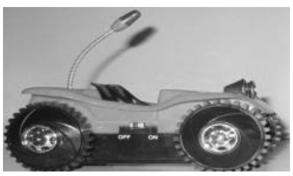

Figure 1. Battery powered toy-car.

Lesson 2 will be done outside of the classroom. Students will roll basketballs along the hallway and measure its positions and time them. The devices necessary to quantify the motion are stopwatches and tape measures. Graphing calculators and rulers can be used to enhance the process of transferring data into motion graphical representation. The third lesson of this unit utilizes computer simulations. These simulations are available for free on the internet and can be downloaded from http://phet.colorado.edu/en/simulations/category/physics/motion. Consequently, access to an internet connection will be required. A moving toy-car would also be helpful to demonstrate motion in various directions during one of the lessons.

LESSON 1: INTRODUCING THE CONCEPT OF FRAME OF REFERENCE AND POSITION

Content Introduction. The purpose of this lesson is the have the learner appreciate the need for a frame of reference to describe motion. Velocity will be mentioned, yet formal quantification of velocity will be introduced during the second lesson**.**

In order to describe motion and precisely construct a position-time graph for the motion, a frame of reference is needed. To highlight this necessity, the instructor draws on the board a picture of a walking student (see Figure 2) or of a car and posits the following question: How can one determine the position (location) of the student?

Figure 2. Student walking in the direction of east.

Students can be given some time to discuss their answers in groups. The learners will be puzzled because a frame of reference is not initially mentioned and it is needed. Being immersed in this dissonance, referring to von Glasersfeld (1995), students will figure out that there is a missing element (the XY axes) with respect to which location can be measured. After a short discussion, the teacher draws a frame of reference (XY axes) with labels of north, east, south, and west. Further discussion concludes that an east – west line, as it is shown in Figure 3 is sufficient for the motion decoding.

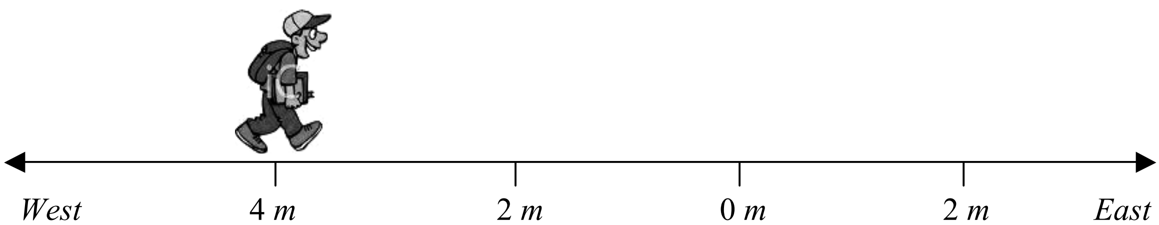

Figure 3. Application of a one-dimensional frame of reference to determine student's position.

The instructor points out that that the $x = 0\ m$ location represents the position of an observer (not shown in Figure 3). With this introduction, the teacher introduces the notation of an object's position and expresses the position of the object using a formal physics symbolism: $x = 4$ *m, west* or $= - 4$ *m.* Note that the description must be relevant to what students see on the board. The teacher then recalls traditional motion problems contexts found in mathematics textbooks, pointing out that most of them referred to a positive position, and positive velocity representing forward direction of object's motion. Thus the resulting position-time graphs were usually sketched in the first quadrants of the Cartesian plane. Yet, these are not the only possible graphical representations of motion, the object can move to the left of the point of reference or down to the point of reference (vertical motion). Thus the students need to see a broader picture of these ideas in order to construct graphs that will reflect these cases. In order to further conceptualize the use of negative quantities to describe motion, especially velocity, the teacher demonstrates a toy-car moving forward and then moving backward. The teacher asks, how can one depict (draw) the paths of motion and describe the movement of the car that would move forward and backward? What are the differences between sketching a forward and backward movement on the paths of motion and the position-time graph? Students might be grouped and asked to come up with their ideas. Teachers need to listen to their definitions and correct them or suggest another avenues if needed. It is expected that the students will conclude that there is a need for a frame of reference to begin the analysis and that there is a need to differentiate between movement in the positive direction (positive velocity) and negative direction (negative velocity). The remaining class time can be spent on applying this idea in practice. This is the time when students construct their own understanding of the concepts introduced by the teacher.

Creating Realistic Environment for Knowledge Construction. The purpose of the activity is to place the learners in a realistic setting and have them experience the process of mathematizying the object's path of motion. The following is a brief description of the activity that follows. The learners need toy-cars, tape measures, and compasses. They can be given worksheets with instruction that contain the directions. More examples can be prepared at the discretion of the teacher.

There are a few scenarios given. For each of the scenario:
1. Establish the car's initial position according to given conditions.
2. Have the car move according to the given direction of motion. You will use a compass to identify the directions, ex. north, south, east or west.
3. Describe the velocity as positive or negative (use positive for east and north and negative for south and west).
4. Determine the car's final position and the total distance travelled.
5. Using a one dimensional frame of reference; east –west or north-south, draw the path of motion of the car and label the initial and final positions.

The car will move for 10 seconds in the south direction starting from the origin.
The car begins its motion at 2m, east and moves in the east direction for 8 seconds.
The car begins its motion at 4m, north and its moves south for 10 seconds.

The extensive visualization of the idea of path of motion coupled with transferring it into graphical representation should help students build internal images of the concepts. Students realize that negative values of position, velocity (and eventually) acceleration are essential quantities of motion description. Students will also realize that non-directional-scalar quantities such as speed and distance are not sufficient to describe motion.

LESSON 2: CATCHING A ROLLING BASKETBALL–REAL DATA ACTIVITY

The main purpose of this activity is to have students quantify the rate of object movement thus its velocity as well as explore the organization of a data taking process that will lead to mathematization of observed motion. The purpose of this experience is not only to provide the students with a constructivist learning environment but also it is to bring reality into making abstract math ideas more tangible. The teacher informs the students that they will observe the motion of a rolling basketball, then decide about taking data which should lead them to finding the velocity of the basketball, and then to formulating a position function for the motion. An integrated task of this activity is also to sketch the path of motion of the rolling basketball. By this inclusion, the students will solidify the differences and similarities between both these representations. This task is the key element of the process of the misconception avoidance.

Activity conduct. The teacher divides the class into groups of seven students. Each group receives 5 stopwatches, a tape measure, and a basketball. The teacher explains that each group must create a set of at least five coordinates for the moving basketball with time and position as the variables. Students then brainstorm how to setup the activity so that they are able to create the data. They also need to decide what quantities to set as independent and dependent. If this activity is to be conducted with a group of students that have a rare exposure to these types of prior experience, the instructional supports listed below can be provided for the students.

One possible summary of the setup of the activity is as follows:

1. Measure a distance of 10 meters along a hallway
2. Mark an initial point of rolling (on the floor) and call it a frame of reference
3. Mark the position of 2m, 4m, 6m, 8m, and 10m along the 10m segment
4. 5 students with stopwatches will stand at the positions indicated above
5. One student will stand at a distance longer than 10 m to catch the rolling ball

Possible arrangement of the data collection process is illustrated in Figure 4.

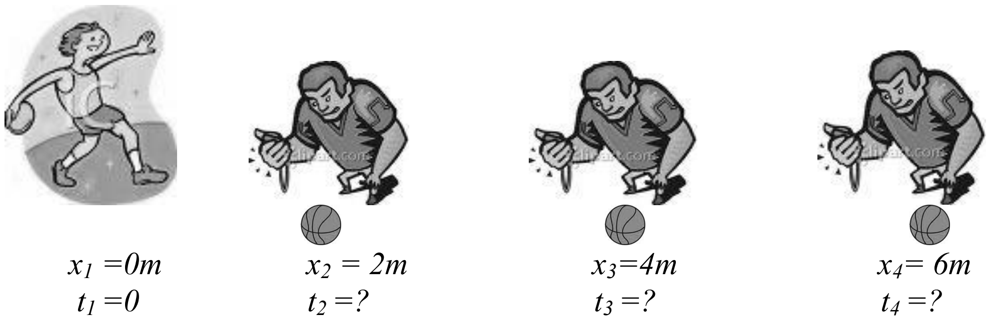

Figure 4. Diagram representing the process of taking data. Source: http://www.clipartguide.com/_pages/1386-0901-2000-4631.html.

The commentary accompanying the scenario follows:

1. Thrower pushes the basketball so that it rolls along the 10 m truck
2. Each student presses the stopwatch *on* when the ball is at x = 0 m
3. Each student stops the stopwatch when the basketball passes him/ her
4. The thrower collects data (time instants) from each group member

The students can be asked to rotate (being thrower) so that each has his/her own data. The teacher facilitates the process of data collection and corrects students' procedures if needed.

Teacher's guidance. Some students might ask why multiple data sets are necessary thus why having only two data sets, for instance (t_i, x_i) and (t_f, x_f) is not satisfactory to quantify the motion? Students who experience these doubts might be asked to research about the importance of gathering multiple data sets to mathematically model system behavior. Alternatively, the teacher might redirect the student's question to posit a doubt of inability of knowing an accurate curve shape if only two data sets from the experiment are generated. If further clarification is needed, the teacher can explain the possibilities of connecting two points are endless, thus to narrow the choices more intermediate data points are needed.

LESSON 3: MOVING MAN–ACTIVITY USING SIMULATED ENVIRONMENT

Lesson 2 was designed to provide students with tangible experimentations and to transfer gathered data into a symbolic mathematical function. Students need to realize that when object moves with a constant velocity, the magnitude of the velocity (thus also the direction of motion) and object initial position are sufficient to mathematize the motion. Consequently, due to a constant rate of change represented by the velocity, a linear mathematical model can be applied to quantify the object's position. Students learned that the steepness of the graph represents the object's velocity. During Lesson 3, students will apply their experiences in new situations. They will validate the knowledge and assure its transferability. As a medium of learning, a computerized simulation of moving man will be utilized. This environment was selected because of its effective motion and corresponding graphs display. Students can hypothesize graphs, and the simulation can be used to verify their hypotheses.

Introduction of the simulation. The teacher arranges the students in groups and assigns one computer to each group, if possible. If this arrangement cannot be furnished, the students can work from their desks and the simulation can be displayed on the classroom screen. The teacher and students need to locate http://phet.colorado.edu/en/simulation/moving-man. This simulation can be displayed in two versions: as *Introduction* and as *Charts*. The interface illustrated in Figure 5, shows its introductory phase. The man can walk according to a prearranged position, velocity, and acceleration. The teacher displays various motions focusing students' attention on the correspondence of negative velocity and negative direction of motion. This interface does not display a graph, however changing the interface to *Charts* by activating the radio buttons on the top left of the screen enables corresponding graphical representation of the motion that are position, velocity and acceleration graphs. In order to avoid overload of information, it is suggested that the teacher presents first the *Introductory* phase and then *Charts.*

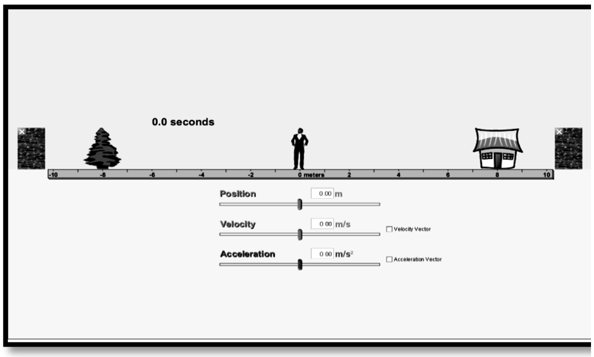

Figure 5. Screenshot of introductory phase of the simulation. *Source*: PhET interactive simulations, University of Colorado, http://phet.colorado.edu.

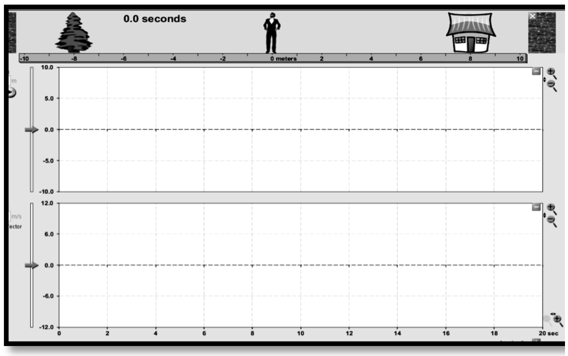

Figure 6. Screenshot of charts phase. *Source*: PhET interactive simulations, University of Colorado, http://phet.colorado.edu.

The teacher can demonstrate the properties of the simulation and focus students' attention on generating graphs. For the purpose of the lesson, only the position-time graph will be activated.

Inducing transferability of observed motions into its mathematical forms. The student's task will be to write algebraic functions depicting various motions and sketching their position graphs. Students will realize that the motion can be described in a one dimensional horizontal frame indicating positive to the right and negative to the left. Reflecting the horizontal path, the position function depicting the motion can be called *x(t).* Students realize that identifying the path of motion is not sufficient to describe the motion. One wants to learn the object's location at any given time instant using its mathematical form. The teacher assigns certain values for the initial position, velocity, and let the man walk (or run). Special attention can be given to analysis of negative position and velocity. The following combinations can be exercised:

1. Negative position (man will be standing by the tree), positive velocity, and zero acceleration (students are comfortable now with using negative values to describe position)
2. Positive position (man is standing by the house), negative velocity and zero acceleration (students realize that to have the man walk toward the tree, his velocity must be negative).

The teacher concludes that the task for this activity will be deriving mathematical equations (linear and possibly in a piece-wise form) and sketching them. The task will also require that students sketch the paths of the man in motion and then verify their graphs with the ones generated by the simulation. The teacher refers to what students learned during the Lesson 2 and links the knowledge with the new motion representations. Further discussion will generate the following conclusions:

If the motion is uniform, (constant velocity) then the linear function in the form of $y = mx + b$ can be used. For the purpose of the activity, the variables can be renamed as follow:
Dependent variable y can be replaced by the final position (x_f)
Independent variable x can be renamed to time (t)
Y-intercept of the linear function b can be renamed to initial position of the man (x_i)
Value of the slope of the function m will be represented by the velocity of the man.

Summing all of the modifications the students will use $x_f = vt + x_i$ instead of $y = mx + b$. For instance if the man's initial position is $x = -8$ m and velocity $v = 1$ m/s, then the position function is $x_f = 1t - 8$ and its position-time graph will show as depicted in Figure 7.

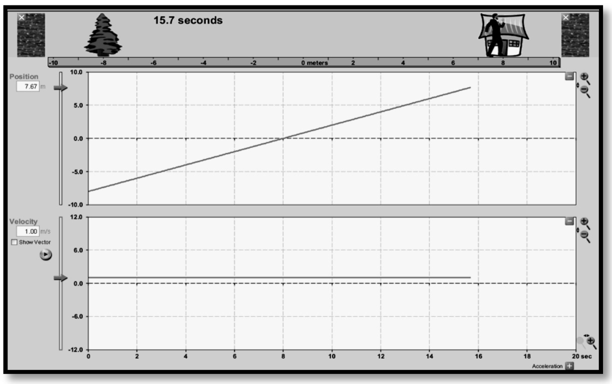

Figure 7. Screenshot of physics simulation *Moving Man*. *Source*: PhET interactive simulations, University of Colorado, http://phet.colorado.edu.

The students realize that although position-time graph is represented by a straight line with a slope of 1, the velocity-time graph appears as a horizontal line v(t)=1 with a slope of zero.

DISCUSSION OF STUDENTS' RESPONSES AND GENERAL CONCLUSIONS

While creating these lessons, we first identified students' misconceptions in distinguishing between an object's path of motion and position versus a time graph with associated misinterpretations. This has also been revealed by other researchers, for instance Leinhardt and colleagues (1990). Having identified these, we focused on creating a learning environment that would strengthen the conceptual understanding of both ideas and consequently eliminate these deficiencies. The students were presented with diverse constructivist learning environments frequently being asked to contrast an objects' path of motion and position time graph. In an attempt to gain insight into the effect of the units on students' abilities to differentiate between position –time graph and path of motion, a qualitative question targeting the idea was addressed The content of the question was the following:

Are there any differences between position-time graph of a moving object and its path motion? If your answer is yes, which representation provides more information about the motion?

Thirty six out of forty one pre-calculus students in this study stated that the representations are different. Below are some of these responses, verbatim:

1. Both representations differ. Position time graph shows displacement and position plus velocity can be derived at given time while path only shows distance with not time. Position – time graph shows more.
2. The difference between a position/time graph & the path of motion is that the position vs time graph can show acceleration and velocity and the path of motion all in one. The position vs. time graph provides more information about the motion.
3. Yes, they are different. The difference is that a position time graph lacks having both directions a vertical and horizontal (only has one) but has time instead. A position time graph provides more because it has respects to time.
4. Path of motion and position-time graph are different. The difference between potion time graph and path of motion is that position – time shows the object's motion in a function form. Position – time graph show s more information, because you can find the object's velocity and acceleration at different points or average velocity.
5. A position – time graph shows only what position an object is at after a certain time while the path of object's motion shows how or in what way they move from one position to another. Therefore, the path of an object's motion provides more information about the motion.
6. Position – time graph shows where the object is at a specific time. While the other which direction the object is actually going. The position – time graph provides more information about the object's motion.
7. A position – time graph is more accurate because you can get velocity & acceleration.

The students associate position-time graphs with a more sophisticated motion representation. They realize that being given position – time graph, one can derive its corresponding velocity and acceleration functions which are not accessible if only the path of motion is given. The students formulated internal images of the graph features and consequently supported their explanations using these images. It seems that the constructivist learning environment had a deep effect on solidifying the distinctions between the fundamental motion representations. How did students evaluate this learning environment? They praised the simplicity of the activities and enjoyed the diversity of the motion representations which consequently engaged them into active learning.

On the mathematics portion of the high-stakes test in 2010 not only did all of these students succeed but about 90% reached a commended performance status. Questions referring to graph analysis were answered correctly by almost all of these students. In the light of these encouraging results, we anticipate conducting a formal experimental research study that would employ not only qualitative analysis but also statistical apparatus that will allow for quantification of the results and provide confidence intervals for targeting populations of interests.

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Improving Achievement in Trigonometry by Revisiting Fractions Operations

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ABSTRACT: *This paper presents the results of a quasi-experimental study that included undergraduate trigonometry students at a private university in the southwestern United States. The present study examined if a review of operations on fractions prior to learning trigonometric identities would improve students' success on a trigonometry test. The course instructor administered a pretest consisting of ten fraction questions. Students identified their errors and made corrections to the pretest. The instructor then taught the chapter on trigonometric identities and administered the chapter test and fraction posttest. The findings suggest that having students review prerequisite skills before teaching advanced topics might aid in improving mathematical skills.*

Key words: *Trigonometry, Fractions, Mathematics education.*

IMPROVING ACHIEVEMENT IN TRIGONOMETRY BY REVISITING FRACTIONS OPERATIONS

The need to increase the number of students who are pursuing careers in science, technology, engineering, and mathematics (STEM) fields has been recognized (Kuenzi, Matthews, & Managan, 2006). Fulfilling this need will require an increase in the number of students who successfully complete the calculus sequence. Trigonometry is one course in this sequence, and the ability to operate on fractions is an important skill needed to succeed in trigonometry. The purpose of the present study was to examine if a review of fractions prior to teaching trigonometric identities would improve students' success on the chapter test. Such success in trigonometry could increase the number of students pursuing careers in STEM fields.

Science, Technology, Engineering, and Mathematics (STEM)

As noted by the National Science and Technology Council (2000), people rely on STEM fields “to increase the nation’s productivity and economic well-being, advance healthcare, improve the environment, help ensure national security, and help educate our youth" (p. 4). Building a strong workforce in STEM is crucially important for the nation’s productivity and economic security (George, Neale, Van Horne, & Malcom, 2001; Riegle-Crumb & King, 2010).

Research suggests that the lack of STEM workers in the United States has been limiting economic growth, and businesses have looked to guest workers to fill the STEM void. The United States has the potential to fill all of the positions for technical jobs, but as noted by George et al. (2001), “if a strong U.S. STEM workforce is to be ensured, it is imperative that this nation understand how to encourage and develop the STEM talent of all U.S. citizens, including all racial/ethnic groups, men and women, and persons with disabilities” (p. 4).

Interventions

Efforts for increasing STEM majors have included interventions focused on changes in curricula (Wicklein, 2006) and teacher training (see Bybee & Loucks-Horsley, 2000; Nugent, Kunz, Rilett, & Jones, 2010; Zhang, McInerney, & Frechtling, 2010). For example, Wicklein (2006) rationalized the need “for creating an engineering-design focused curriculum for technology education” (p. 29). Zhang et al. (2010) found that professional development training provided by STEM faculty resulted in K-12 teachers who were more confident and that K-12 teachers were learning more than content from the STEM faculty. An example of another intervention was the Model Institutions for Excellence (MIE) program, which increased the number of Hispanic STEM majors by providing "faculty development, laboratory renovation, student scholarships, intensified student services, and undergraduate research” (Merisotis & Kee, 2006, p. 288). Other strategies that have led to an increased number of successful STEM majors have been to encourage student collaborations and out-of-class meetings (see Springer, Stanne, & Donovan, 1999) and collaborations between universities and K-12 public schools both at the student level (DeGrazia, Sullivan, Carlson, & Carlson, 2001).

Fractions and Algebra

Success in courses beyond algebra will be a requirement for obtaining a degree in a STEM field. Success in trigonometry, which is the focus of the present study, is contingent on success in algebra. A prerequisite to success in algebra is fraction knowledge. The discussion on the relationship between fractions and success in mathematics primarily has focused on algebra or high school mathematics in general rather than on trigonometry. However, algebra is a prerequisite course for trigonometry.

Therefore, we focus our discussion on the available literature that discusses the importance of fractions for learning algebra.

The National Math Advisory Panel acknowledged that “fractions, and particular aspects of geometry and measurement—are the Critical Foundations of Algebra” (U. S. Department of Education, 2008, p. xvii). The importance of fraction knowledge in the early grades has been found to be a predictor of success in high school mathematics achievement. Siegler et al. (2012) found from their analyses that “elementary school students’ knowledge of fractions and of division uniquely predicts those students’ knowledge of algebra and overall mathematics achievement in high school, 5 or 6 years later” and this held true “even after statistically controlling for other types of mathematical knowledge, general intellectual ability, working memory, and family income and education” (p. 1). The importance of fractions in leading students from arithmetic to algebra was elaborated by Wu (2001) who stated that

> The proper study of fractions provides a ramp that leads students gently from arithmetic up to algebra. But when the approach to fractions is defective, that ramp collapses, and students are required to scale the wall of algebra not at a gentle slope but at a ninety degree angle. (p. 10)

While arithmetic focuses on computation of numbers, algebra is more general.

Brown and Quinn (2007a, 2007b) and Rotman (1991) also discussed the importance of operating on fractions and success in algebra. In a study consisting of students enrolled in elementary and intermediate algebra, Brown and Quinn (2007b) found a statistically “significant relationship between an individual’s ability to understand and perform fraction operations and his or her test scores in algebra” (p. 10). In 1991, Rotman reported on an empirical investigation between the relationship between arithmetic placement scores and success in algebra with students enrolled in developmental mathematics courses at a community college. While he found that "there appeared to be some connection between <u>some</u> arithmetic skills and performance in algebra, but the data is [sic] not convincing" (p. 5), Rotman (1991) went on to contend that "fraction concepts deserve to be singled out" (p. 8).

Algebra teachers have indicated that students enter algebra unprepared with the biggest weaknesses in word problems, “rational numbers and operations involving fractions and decimals”, and study habits (Hoffer, Venkataraman, Hedberg, & Shagle, 2007, p. 10). We would expect that because fractions are covered in the early grades that students would have obtained and retained their ability to operate on fractions prior to entering a college mathematics course. Brown and Quinn (2007b) noted that someone could argue the elementary algebra students’ poor performance on their fraction test could be explained “that by ninth and tenth grade, students have simply forgotten what they have previously learned regarding fractions, since the subject was formally taught in fourth, fifth, and sixth grade” (p. 10). They then note this argument “points to a lack of meaningful follow up in grades seven and eight, but it fails to address the problem” (pp. 10-11). In our study, we sought to determine if a brief review of fractions prior to teaching trigonometric identities would help address students' prior failure to retain their ability to operate on fractions.

Transfer of Learning

Many factors have been identified that influence one's ability to transfer knowledge. For example, Bransford, Brown, and Cocking (2000) noted, "transfer is affected by the degree to which people learn with understanding rather than merely memorize sets of facts or follow a fixed set of procedures" (p. 55). The National Research Council (2001) emphasized students must have acquired *procedural fluency*, which is "the knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently" (p. 121). Procedural fluency with fractions operations has become a foundational skill on which students then can build a deeper understanding of other areas of mathematics.

The context in which information is presented enhances the student's ability to transfer knowledge (Boaler, 1993; Bransford et al., 2000; Shepard et al., 2005). For example, Bransford et al. (2000) noted "transfer is also affected by the context of original learning; people can learn in one context, yet fail to transfer to other contexts" (p. 62). Shepard et al. (2005) further elaborated that "transfer refers to the ability to use one's knowledge in new contexts. Transfer is obviously a goal of learning. What good is knowledge if it can't be accessed or applied?" (p. 288).

Researchers Perkins and Salomon described transfer as *low-road* or *high-road* (Perkins & Salomon, 1988; Salomon & Perkins, 1989) and as either *forward-reaching* or *backward-reaching*. In low-road transfer, the skill became automated and had similarities to the new context. An example would be using one's skills as a skateboarder to learn to snowboard or surf. High-road transfer was described as a deliberate or intentional transfer, such as when one applied graph theory in designing computer science applications. As noted by Perkins and Salomon (1988), "in forward-reaching high road transfer, one learns something and abstracts it in preparation for applications elsewhere" whereas "in backward-reaching high road transfer, one finds oneself in a problem situation, abstracts key characteristics from the situation, and reaches backward into one's experience for matches" (p. 26).

We have not focused in depth on the topic of transfer in our review of literature, but did want to acknowledge the importance of transferring knowledge because trigonometry builds upon mathematical skills, such as fraction operations, that were learned in previous grade levels. One underlying belief of the reform efforts in mathematics education has been that "as students develop a view of mathematics as a connected and integrated whole, they will have less of a tendency to view mathematical skills and concepts separately" (National Council of Teachers of Mathematics, 2000, p. 65). The present study serves as an example of how knowledge concerning one concept, in this case fraction operations, might become too compartmentalized by the learner to be recognized and used in another situation.

PURPOSE

Increasing the number of STEM majors will require an increase in the number of students who successfully complete trigonometry. In order for this to occur, students must be able to proficiently work with trigonometric identities, which will also require fraction knowledge. We would hope that students will be able to transfer their knowledge of fractions to trigonometric identities. We believe that some students might not be competent operating on fractions because they have not retained the information and that a review, albeit a brief review, might help students retrieve these skills. The present study examined if a review of operations on fractions would improve students' success in trigonometry. The research questions were: (1) Does reviewing operations on fractions in a trigonometry course improve students' performance on a trigonometric identity test? and (2) What factors do trigonometry students who have had a review of fraction operations perceive as having explained improvements in their class performance compared to classes from a previous semester?

METHOD

This study consisted of undergraduate students enrolled in trigonometry in the spring semester at a private university in the southwestern United States: 67 students in the experimental group and 75 in the control group. The experimental group was enrolled in the Spring 2010 semester. The control group was enrolled in the Spring 2008 semester. The sample was primarily White males (72% males and 85% males, 88% White and 84% White, experimental and control, respectively). Other ethnicities in the experimental group included three Hispanic, one African American, and three international students (i.e., China, India, and Dutch). Other ethnicities in the control group included four Hispanic, three African American, and four Asian American students.

The experimental group consisted of eight freshman, 44 sophomores, six juniors, and nine seniors. The majority in the experimental group were aeronautical science majors (n = 43), followed by engineering and engineering technology (n = 14) and kinesiology and education majors (n = 10). The control group consisted of 47 freshman, 14 sophomores, 11 juniors, and three seniors. The majority in the control group were aeronautical science majors (n = 35), followed by engineering and engineering technology (n = 25) and biology, kinesiology and education majors (n = 11). There were four students that had not declared a major.

The experimental and control groups were similar on those characteristics and both were enrolled at different years at the same private university and taught by the same professor. This was the only professor on the campus to teach trigonometry in the spring semesters. A different professor taught classes each fall semester. Therefore, we did not identify any evidence towards selection bias from students' selection of instructors or course section times in the spring semester. However, bias might exist by not including students in the study who were enrolled in the fall semester. Possible selection bias was that the sample was limited to one private university and one instructor. Because the studies were conducted in different spring semesters, students in the control and experimental group were not able to compare notes regarding the need to review fractions. Therefore, social threat was not a problem in the study. The instructor was cognizant of the need to teach the course in the 2010 spring semester with the same teaching style and grading system as the 2008 spring semester.

Trigonometry

The sample consisted of students enrolled in a trigonometry course. Students pursuing STEM fields are required to complete trigonometry, which is a prerequisite to calculus. Both Algebra II and geometry are prerequisites for trigonometric concepts. Trigonometry topics have been taught as a stand-alone course or encompassed in a precalculus course. Trigonometric identities are a major component of a trigonometry class. As noted by Spector (2011),

> the significance of an identity is that, in calculation, we may replace either member of the identity with the other. We use an identity to give an expression a more convenient form. In calculus and all its applications, the trigonometric identities are of central importance. (p. 1)

An example of a trigonometric identity is $\sin^2 t + \cos^2 t = 1$.

Procedure and Instrument

Procedure

The experimental group was administered a 10 question fraction operation test at the beginning and the end of the semester. The fraction test is provided in Appendix A. Calculators were allowed for simplifying problems 6 - 10. The problems were similar to the types of problems that would be encountered when solving the identities. After completing the pretest, the professor reviewed the fraction quiz with students during class time. Students identified their errors and made corrections. Then the chapter on identities was taught in the same manner as in previous semesters. A chapter test was administered that consisted of approximately 35 trigonometric identities. The students were allowed to write the identities on 3 x 5 cards and bring the cards to the exam; therefore, memorization of identities was not a deterrent in their course grade. Subsequently, a fraction posttest that was parallel to the fraction pretest was administered as well as the trigonometry exam. The control group had been administered an equivalent trigonometry test.

This study somewhat modeled the 5-R Intervention program, which consists of five steps (see Clements & Ellerton, 2009; Ellerton & Clements, 2011). Students first have a “reality check” and “realize that they did not know what they thought they knew” (Ellerton & Clements, 2011, p. 403). Step two involves the teacher addressing the errors and misconceptions. Step three of the 5-R Intervention program was not addressed in this study, which would have been a written reflection on their understanding. Step four was revisiting their understandings. Students were continually applying these skills to solve the trigonometric identities; hence, they arguably were revisiting the concepts. Step five consisted of an assessment to determine if they had retained their abilities to solve the fraction problems.

The instructor compared the chapter identity test scores from this administration to those recorded from the previous semester and found the test scores were higher on average for the experimental group. Then a qualitative component was included. Students in the experimental group were informed that their identity tests scores appeared higher than the previous semester scores. The students were then asked to conjecture about why they thought their grades were higher in comparison to students who had completed the test in previous trigonometry classes.

Instruments

Both the fraction test and trigonometry tests were developed by the professor who had a number of years of teaching experience both at the high school and university level. The professor had a PhD in mathematics education, had taught this trigonometry course for 24 semesters, and was the only professor at this university to teach this course in the spring semesters. The trigonometry test met the needs of the course objectives; therefore we considered content validity to be met. The fraction test was developed based on the skills the experienced professor deemed were necessary to solve or simplify trigonometry identities. In addition, the professor had confirmed with another university professor that the content and rigor were at the expectation level. The researchers confirmed the problems were relevant.

The entire test for trigonometric identities is provided in Appendix B. Examples of problems from the test on trigonometric identities include:

(a) Find the exact value of the expression sin (330° + 45°),
(b) For the following two expressions, write the expression in terms of a single trigonometric function: cos*x*cos 2*x* + sin *x* sin 2*x* and sin 7*x*cos 3*x* – cos 7*x* sin 3*x* and

(c) Given $\sin\alpha = -\frac{4}{5}$, α in Quadrant III, and $\cos\beta = \frac{-12}{13}$, β in Quadrant II, find

A. sin ($\alpha - \beta$), B. cos ($\alpha + \beta$), and C. tan ($\alpha + \beta$).

Calculators

In Texas, where the current study was situated, many high school teachers incorporate calculators within the high school curriculum because students are allowed to use graphing calculators on the state mandated mathematics assessment (Milou & Bohlin, 2003). Our focus was on the participants' ability to perform routine calculations. Therefore, we did not examine the impact of instruction or learning with calculators on student success.

Analysis

The study was quasi-experimental (Johnson & Christensen, 2008). Chapter test scores from previous semesters were utilized as the control group. Paired *t* tests with omega-squared effect sizes were utilized to examine differences in performance on the pre and posttest fraction operation test (Sheskin, 2004). Paired *t* tests involve correlated responses; therefore, Cohen's *d* effect sizes were not appropriate and omega-squared ($\acute{\omega}^2$) effect sizes were reported (Sheskin, 2004). According to Sheskin (2004), an $\acute{\omega}^2$ greater than .0099 but not greater than .0588 would correspond to Cohen's small effect size, which in our case we would consider noteworthy. An $\acute{\omega}^2$ larger than .1379 would correspond to Cohen's large effect size benchmark. *T* tests with Cohen's *d* effect sizes were utilized to determine if reviewing fractions would result in an improvement in students' ability to utilize trigonometric identities. Bivariate correlations were conducted to determine the relationship between posttest scores on ability to solve fractions and the chapter test scores.

The purpose of the qualitative data analysis was to provide additional insights regarding the increase in student test scores. Student comments were copied onto a spreadsheet to facilitate thematic analysis and retrieval (Richards & Richards, 1994). The comments were classified into broad themes, which were subsequently revised by the authors. Consensus was required throughout the process. Efforts were made during the analysis to ensure credibility, transferability, dependability and trustworthiness of the results (Lincoln & Guba, 1985).

RESULTS AND CONCLUSIONS

Quantitative Analysis: Differences Between Experimental and Control Groups

Noteworthy differences existed between the control and experimental group on the trigonometry chapter test containing identity problems ($t[139] = -3.76$, $p < .001$; Cohen's $d = -3.16$; $M = 70.75$, $SD = 23.11$; $M = 55.00$, $SD = 2.33$; experimental and control, respectively). Except for the administration and review of the fraction test, the control and experimental groups were taught in the same teaching and grading style by the same instructor. Table 1 contains the grade distributions for the trigonometry test.

Control Group: Fraction Test

On the paired data for the control group, there were noteworthy differences between the mean scores on the fraction pretest and posttests ($t[58] = -7.81$, $p < .001$; $\acute{\omega}^2 = .50$; $M = 62.03$, $SD = 31.72$; $M = 92.20$, $SD = 15.09$; pretest and posttest, respectively). Three students did not complete the fraction pretest and five students did not complete the fraction posttest. The instructor decided to award bonus points on the fraction pretest. However, bonus points were not included in the analysis and were not awarded on the posttest. For the paired data on the pretest, 39% made a 60 or less and 5.1% made a 100.

On the posttest paired data, 5.1% made a 60 or less and 64.4% made a 100. Of the students who did not complete the posttest because of absences, two made a C, one made an A, and two made an F on the identity trigonometric test.

The relationship between the chapter identity test scores and the fraction posttest scores was statistically significant with a small noteworthy Pearson correlation coefficient (r = .258, p = .043). The small effect size suggests that while addressing fractions might aid in improving students' test scores, being able to conduct fraction operations with and without a calculator was insufficient in helping *all* students improve in their ability to solve trigonometric identity problems.

Table 1

Grade Distributions of Experimental and Control Groups in Percents

	Chapter Test 3	
Grade	Control (n = 74)	Experimental (n = 67)
A	9.46	29.85
B	10.81	10.45
C	10.81	13.43
D	16.22	13.43
F	52.70	32.84

Qualitative Analysis: Students' Perceptions for Why They Performed Better

Students were informed that on average their tests scores were higher than previous semesters and were asked to conjecture why they thought the grades were better on the identities exam compared to other students from previous trigonometry classes. Table 2 contains the reasons students believed they might have performed better than the control group did in previous semesters. One student who identified the fraction test as helpful stated,

> I believe that the test score improved as a direct result of the fractions refresher quiz. Particularly the quiz administered prior to the test. I feel that such a quiz goes a long way towards eliminating simple syntax errors, and helps the students learn, relearn, or remember the processes without the added pressure of a grade.

Another student commented that "it was easy to focus on the trigonometry of the problems because we had already been given a refresher on how to combine fractions." A third student who found the fraction review helpful believed the scores were higher

> because the teacher reviewed the concepts of fractional multiplication and division. These concepts can be easily forgotten and confused with each

> other. I believe that had this not been overlooked before, that students would have had more success on that particular test.

Another student stated "I think the fraction test helped us a little because it maybe took care of some of the little easy mistakes that can be made on that test." Some students indicated they already felt comfortable with working with fractions. One of those students noted that while they had not had difficulties with fractions, the fraction test

Table 2

Disaggregation of Responses By Grade Distributions on Trigonometry Identity Test For the 67 Students Who Completed the Fraction Post-Test

Grade	Intervention	Unable to Identify Reason	Conditions That Were Consistent with Previous Semesters		Reasons That We Would Assume were Consistent with Previous Semesters	
	Reviewed Fractions (*n* = 22)	Did Not Provide a Reason (*n* = 12)	Instructor (*n* =14)	Quizzes (*n* = 5)	Practiced or Studied More (*n* = 15)	Other (*n* = 14)
A	7	3	6	3	6	1
B	4	0	1	1	1	4
C	2	1	1	0	4	3
D	2	3	2	0	1	2
F	7	9	4	1	3	4

Note. One student specified reviewed how to use the calculator to compute fraction operations. The five students who did not complete the post-test were not included in the counts for no responses. Students who states "Quizzes" did not specify which quizzes they were referencing. Students could have provided multiple reasons.

might have helped because some students have difficulty with fractions. This student stated that "Personally, I've never had an issue with fractions, but I know a few classmates have trouble with it."

Several of the trigonometry students identified the university instructor or the administration of quizzes as possible agents that positively impacted their test results. However, these differences were probably not related to the instructor because the instructor taught both the control and experimental groups and administered quizzes both semesters. Several students indicated that maybe they were inspired to study more than the control group because the instructor emphasized that the next test had in the past been difficult for a large percent of students and the instructor stated test averages tended to be lower for the identity exam compared to other exams. However, the latter was not a plausible reason because the instructor had made these comments each semester to emphasize to the students the need to devote sufficient time and effort to studying for this test. Therefore, the instructor's teaching and comments would be comparable for both the experimental and control groups. Responses in the *Other* category varied but included the following: trigonometry in high school (n = 4), no problems with fractions (n = 2), wanted to get good grades (n = 1), and had a tutor (n = 1). Some students with a response categorized in the *Other* category thought the class might be better prepared than previous classes and one student thought that the professor might have improved on her instruction.

DISCUSSION

Increasing the number of students who graduate with a degree in a STEM field will require an increase in the number of students who are successful in courses such as algebra and trigonometry. Prerequisite mathematics skills for trigonometry include geometry and algebra, and prerequisite skills for algebra include fraction knowledge (U. S. Department of Education, 2008). Students' placement in college-level courses is often determined by completion of prior mathematics courses or appropriate scores on placement tests. Therefore, by the time students enter a college-level trigonometry course, a presumption is that they have a specific set of prerequisite mathematics skills to succeed in the course. Each semester the professor who involved her students in the present study found that a lack of success on a trigonometric identity test served as a barrier for many students' ability to succeed in trigonometry. We hypothesized that their ability to operate on fractions impacted their ability to succeed on the test and that a review of fraction operations would result in an increase in average test scores. In an attempt to improve student learning and test this hypothesis, a review of fractions was conducted prior to learning trigonometric identities. The results indicated the review of fractions, on average, increased student success.

Changes in curricula and pedagogy have been suggested as strategies for increasing the number of STEM majors (DeGrazia et al., 2001; Wicklein, 2006). In the present study, minor changes in instructional strategies that included a review of fraction operations improved student success in trigonometry. The instructional changes in this study required approximately 30 minutes of class time, which involved two administrations of a fraction test and a brief review of the fraction items following the pretest. Therefore, only a minimal amount of class time was devoted to the intervention, and the intervention did not interfere with the instructors' ability to cover the required topics of the course.

In this intervention, students obtained their reality check on their abilities with fraction operations; the teacher addressed the errors; and students revisited these skills in the trigonometric section. Thus, this process affirmed the intervention program introduced by Clements and Ellerton (2009). In future studies, we recommend that students complete a written reflection (i.e., step 3) as was suggested by the 5-R intervention program. A full implementation of the 5-R Intervention program might improve students' abilities on other prerequisite skills needed to succeed in STEM classrooms.

Limitation

A limitation of the present study is the inability to generalize to the population of trigonometry students. The present study was conducted at one private university in the southwestern United States. Therefore, the sample might not be representative of the general population. We recommend that future studies should be conducted at various university educational settings.

Implications

Because success on the trigonometry identity tests improved for the experimental group in comparison to the control group, the intervention has the potential of increasing the number of STEM majors who are able to matriculate through trigonometry and into calculus. As noted by George et al. (2001), increasing the number of STEM majors is important to the economic growth of a country.

We hypothesized that this intervention would work for the students who had previously learned but did not retain fraction operation skills. In an ideal world, students would realize the skills they needed to review in order to be successful in higher-level mathematics courses. In the real world, some students do not necessarily understand nor can some students identify the necessary prerequisite skills for a mathematics course. Our intervention sought to address the failure of some students to retain mathematics skills learned in early grades and to do so in an efficient and timely manner. In other words, we sought to be part of the solution (see discussion by Brown & Quinn, 2007b).

Mathematics students in colleges and universities might not realize that the concepts they learned in elementary and middle school reappear in or are transferred to more sophisticated scenarios (Shepard et al., 2005) as they progress through mathematics courses. Fractions are first taught in early years and have been identified as a foundational skill for algebra (Brown & Quinn, 2007a, 2007b; Rotman, 1991; U. S. Department of Education, 2008; Wu, 2001). Fractions are imbedded in trigonometric identities.

In this study, the undergraduate students improved on the identities assessment as compared to the control group. In addition, the mean posttest scores on the fraction test were on average 30 points higher than the posttest scores ($M = 62.03$, $SD = 31.72$; $M = 92.20$, $SD = 15.09$). A statistically significant correlation existed between the scores on the fraction posttest and the trigonometry test. Therefore, higher scores on the fraction test tended to result in higher scores on the trigonometric identity test. However, the findings indicated that mastering fractions did not necessarily ensure *all* students will assess well on trigonometric identities. The results from our study indicated that some students mastered the fraction skills prior to the administration of the trigonometric identity test, but then were nevertheless unsuccessful as determined by a failing grade. For example, only 5% of the students in the experimental group failed the posttest fraction test while 33% in the experimental group failed the trigonometry test. However, the 33% was a much smaller percent than the 54% of students who failed the trigonometry test in previous semesters. All of the students who failed the posttest fraction test failed the trigonometry identity test. Therefore, while ability to operate on fractions appeared to improve the success rates of many students, being able to operate on fractions was not enough of an intervention for some students. Other mathematics or cognitive skills deterred those students from succeeding. Future research needs to focus on how to help the remaining third of the class.

We believe in the importance of having students reflect on their success and failures. When students were asked to reflect on why their average was higher than previous semesters, they were not informed that students in previous semesters had not received the fraction intervention. Upon reflection, 33% of the students hypothesized that refreshing fractions helped them achieve a higher exam grade than the control group while 22% hypothesized that they performed better because they practiced problems or studied more. Therefore, some participants did not first identify the relevance of reviewing fraction operations while learning the identities. Some students indicated they thought the fraction operations might have helped other students but that fractions were not difficult for them. The professor later explained that they had received the added benefit of reviewing fractions.

The students' ability to apply fraction operations concepts to trigonometry identity problems might be considered an example of Salomon and Perkins' (1989) high road backward-reaching transfer in that students had to reflect on their knowledge of fractions "in the transfer context [trigonometry class] rather than in the initial learning context [middle school mathematics class]" (p.128). The instructor learned that refreshing students' skills on fraction operations improved their ability to work with the trigonometric identities and fraction operations and that only a minimal amount of time was required for reviewing these skills. The present study illustrates that even minimal change in curricula impact the improvement of students' mathematical abilities. The intervention probably worked for the set of students who had demonstrated mastery of these skills at an earlier point in their academic career but had not retained these skills.

Having students review prerequisite skills prior to learning advanced topics might improve student learning by removing one barrier to learning. Much of this review can be accomplished outside of class through an online testing system. In addition, such reviews require a minimal amount of class time, but can be a critical step in helping students transfer skills and concepts while assisting them in acquiring new knowledge in advanced mathematics courses.

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APPENDIX A

Trig Fraction Test (Simplify 1-5 without a calculator and 6-10 with a calculator.)

1. $\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$ = ____________________

2. $\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{3}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$ = ____________________

3. $\left(\frac{4}{5}\right)\left(\frac{8}{17}\right)-\left(\frac{-3}{5}\right)\left(\frac{15}{17}\right)$ = ____________________

4. $2\left(\frac{3}{5}\right)\left(\frac{-4}{5}\right)$ = ____________________

5. $\left(\frac{3}{5}\right)^2-\left(\frac{-4}{5}\right)^2$ = ____________________

6. $\dfrac{2\left(\dfrac{-3}{4}\right)}{1-\left(\dfrac{-3}{4}\right)^2}$ = ______________________

7. $\sqrt{\dfrac{1-\left(\dfrac{-12}{13}\right)}{2}}$ = ______________________

8. . $\sqrt{\dfrac{1+\left(\dfrac{-12}{13}\right)}{2}}$ =______________________

9. $\dfrac{1-\left(\dfrac{-12}{13}\right)}{\dfrac{5}{13}}$ = ______________________

10. $-\sqrt{\dfrac{1+\left(\dfrac{-\sqrt{3}}{2}\right)}{2}}$ = ______________________

APPENDIX B

Trigonometry Test

Chapter 3

1. Find the exact value of the expression.

$\sin(330° + 45°)$

In questions 2 & 3, write the expression in terms of a single trigonometric function.

2. $\cos x \cos 2x + \sin x \sin 2x$

3. $\sin 7x \cos 3x - \cos 7x \sin 3x$

Find the exact value of the given function.

4. Given $\sin \alpha = -\frac{4}{5}$, α in Quadrant III, and $\cos \beta = \frac{-12}{13}$, β in Quadrant II, find

a. $\sin(\alpha - \beta)$ b. $\cos(\alpha + \beta)$ c. $\tan(\alpha + \beta)$

In questions 5& 6 use the half-angle identities to find the exact value of each trigonometric expression.

5. $\tan 165°$ 6. $\cos \frac{5\pi}{12}$

Find the exact value of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$ given the following information.

7. $\cos \theta = \frac{4}{5}$ θ is in Quadrant IV.

Find the exact value of the sine, cosine, and tangent of $\alpha/2$ given the following information.

8. $\sin \alpha = \frac{5}{13}$ α is in Quadrant II.

Write each expression as the product of two functions.

9. $\sin 5\theta + \sin 9\theta$

In questions 10 & 11, write the given equation in the form $y = k \sin(x + \alpha)$, where the measure of α is in degrees.

10. $y = \sqrt{3} \sin x - \cos x$ 11. $y = 8 \sin x - 3 \cos x$

In questions 12 & 13, find the exact value of the given expression. If an exact value cannot be given, give the value to the nearest ten-thousandth.

12. $\cos\left(\cos^{-1}\frac{1}{2}\right)$

13. $\cos^{-1}\left(\sin\frac{\pi}{4}\right)$

Solve the equation for *x* algebraically.

14. $\sin^{-1} x = \cos^{-1}\frac{5}{13}$

In questions 15 & 16, solve each equation for exact solutions in the interval $0 \le x < 2\pi$.

15. $2 \sin x = \sqrt{3}$

16. $4 \sin x \cos x - 2\sqrt{3} \sin x - 2\sqrt{2} \cos x + \sqrt{6} = 0$

Bonus Question (10 Points)

Solve the equation for solutions in the interval $0 \le x < 2\pi$

$4 \sin^2 (x) + 2\sqrt{3} \sin (x) - \sqrt{3} = 2 \sin (x)$

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Articulation and Change of Senses Assigned to Representations of Mathematical Objects[1]

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ABSTRACT: *This article documents the phenomenon related to the difficulties encountered by some students to articulate the senses assigned to different representations of a mathematical object obtained by semiotic transformations of treatment. It is presented a description and an analysis of the process of assigning senses achieved by students regarding specific tasks, where is required making such treatments between representations*

This paper is situated in a semiotic context, and studies in general the relationship semiosis-noesis in the construction of mathematical knowledge by students from grades 9th and 11th of the secondary education (Colombia); this study, without being exhaustive, includes aspects of mathematical activity, communication of mathematical objects emerging and cognitive construction of mathematical objects.

Keywords*:* *Mathematical object, Semiotic representation, Treatment transformation, Meaning, Articulation of senses.*

INTRODUCTION

The language has been constituted for human being as a way of describing the world and understanding their productions, generating at the same time the need to culturally build meanings. From the statement from Bruner (2006), language is acquired as used, in interaction processes, where the functions and communicative intentions are established. Such acquisitions are quite sensitive to context.

The form of human life, as suggested by the author, depends on shared forms of discourse, on meanings and shared concepts[3], which are published in each culture, which enables human beings to negotiate differences of interpretation and meaning,

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[3] Perhaps it would be more appropriate to say that such meanings "are taken as shared."

recognizing that the symbolic systems used by individuals in the construction of meaning preexist, and are rooted in the language and the culture. It is through symbolic systems that humans construct and give meaning to the world.

People live publicly, that is, not in isolation, through meanings and interpretations that must be publicly accessible. If meanings are not shared with others, they are useless. As Bruner (2006) puts it, the culture and the search for meaning within culture are the true causes of human action. In other words, it is culture that shapes human life and mind, giving meaning to the action.

The relationship between human beings and the world, as suggested by Bagni (2009), has no observational cognitive character, but practical-comprehensive. In relation to the school context, as recognized by this author, students must give sense to sentences or propositions, to parts of speech that sometimes they barely know or ignore at all. They must give sense to a particular signs, must "make speak" these signs. So, learners must move in the plane of the active interpretation of signs, in terms of hermeneutics. What has been discussed so far, emphasizes the fact that is assumed in this study: the subjects of a group, in processes of interaction around a specific task, necessarily stem from interpretations different to words, signs and representations. It is in such interaction, that they make explicit the assigned senses and consensually build the required meanings for tackling the task.

The present investigation documents the phenomenon related to the difficulties faced by some students to articulate the *senses* assigned to semiotic representations of the same mathematical object[4], obtained by transformations of *treatment*, in other words, by transformations within the same semiotic system of representation (Duval, 1999). A description and analysis of the processes of senses assigning of nine students, six of grade 9th and three of grade 11th, was made. It was based on work done by them in three small groups in relation to specific tasks[5], in which the meaning assigned to certain semiotic representations is explored and it is required to make treatment transformations. A qualitative research approach is assumed, making a descriptive-interpretive analysis, using two theoretical perspectives: Ontosemiotic (Godino, 2003, Godino, Batanero, & Font, 2007) and Sociocultural (Radford, 2006).

The situations presented below allow highlighting the complexities associated to semiotic transformations of treatment, related to the difficulty in articulating the senses assigned to a mathematical object, which were reported by D'Amore (2006):

> *Situation 1.* Proposal to 5th grade students in basic education (in Italy, mean age 10 years). Calculate the probability of the next event: throwing a die to obtain an even number.

[4] In principle, the meaning is taken here as a partial meaning (Font & Ramos, 2005), associated more with the contextual and even temporal. Each context helps generate sense, though not all possible senses. The meaning of an object (institutional/personal), following Godino and Batanero's ideas (1994), is the system of practices (institutional/personal) associated with the field of problems from where such object emerge at any given time.

[5] In Colombia the school previous to the university is organized into eleven grades, grouped in three levels: Basic Primary Education (5 degrees, ages 6 and 10-11 years), Basic Secondary Education (4 degrees, between 11-12 and 14 -15 years) and Vocational Secondary Education (2 degrees, between 15-16 and 16-18 years).

> After working in small groups, with the teacher's orientation, students shared that while the possible outcomes when rolling a dice are 6 and those who make true the event are 3, the answer is 3/6. They also recognize that this probability can be expressed as 50%, while accepting the equivalence between 3/6 and 50/100, given by the teacher. Even some of the students recognize that speak of 50% means that *we have half of the probability to verified the event relative to the set of possible events* and, therefore, must be valid as response the expression ½, which is accepted and validated by the other students and the teacher, that is, the senses assigned to the mathematical objects are shared.
> Once the class session is concluded, the researcher poses to the students that the fraction 4/8 would also be an appropriate answer, taking into account that it is equivalent to 3/6. Students and teacher say they do not agree. The teacher of the course says that the fraction 4/8 *cannot represent the event because the faces of a dice are 6 and not 8.*

In this case, what explains this «change» in the assigned senses to representations of mathematical objects shared before? Or rather, why not "articulate" the different senses assigned to the representations? If 4/8 is a result of treatment of 3/6 well dominated by students and teacher, why the sense of mathematical object "probability of obtaining an even by number throwing a dice" is not "preserved" with 4/8?

> *Situation 2.* The sense assigned by a group of university students (in Italy) to the equation $x^2+y^2+2xy-1=0$ is "a circle" and the equation $x+y=\frac{1}{x+y}$ is "an amount that has the same value as its converse." They recognize that by transformations of treatment, they can pass from the first equation to the second, but the question, does the second equation represent a circle or not? Find answers as follows.
> *Student A: Absolutely not, a circle should have* $x^2 + y^2$.
> *Student B: If simplified, yes!*

While the first equation does not correspond to a circle, it is not this paper interest to tackle this aspect. The interesting thing here is that in the first case, "being circle" is associated with a certain expression, which is seen as an icon, and in the second, the semiotic transformation (of treatment) of the one who gives or not sense or to the expression; to perform the transformation generates a "change of sense." The mathematical object "circle" is accepted and related to the first equation, but it is not accepted by the second equation, although this one is obtained by treating the first by the students themselves.

From what is described in the above situations, the senses assigned to each of the specific representations of a mathematical object, apparently, have no connection with each other to enable their articulation. There is evidence in a variety of situations, at different levels of schooling (D'Amore, 2006), about a "change of sense" when a semiotic representation is transformed into another, within the same record of representation. Events similar to those described above have been made evident in the Colombian context by the author of this research.

THEORETICAL FRAMEWORK

Science, seen as human activity, allow interpretations for understanding the phenomena of the world, without necessarily assuming that there are privileged ways to do it. The problem has many facets and, there are different ways to interpret and address them; ways that has to be analyzed, contrasted, criticized, endorsed, or restated according to its relevance and effectiveness. This current research assumes a philosophical pragmatist approach, in which experience, typically human,[6] is necessarily referred.

From a pragmatist approach as Rorty's (1991), the *ideas* are not only taken as guide for *action*, but its validity and importance derived from the utility and effectiveness in a given situation or problem that satisfies the needs or requirements from a subject or society[7]. In particular, following approaches by this author, the *reality* is described by using languages, but not preexisting to them, it develops with them, born with them, makes sense with them. Human language is contingent -it can happen or not, can be one way or the other-, and reality is setting in, and through languages. Therefore, the descriptions of the world and the truth cannot be independent from human beings. The reality is a set of agreements between humans. The world, meanwhile, is a set of events, of facts rather than things. That is, there is nothing that can be considered as "objective reality", but human groups with different discourses and "objectivity" should be seen as a desire to persuade and agree unforced. There is no hierarchy between disciplines or discursive genres of science or of the humanities; *scientific* language is only one of the possible languages. Using Wittgenstein's terms, scientific language is only one possibility in the *language games*.

Semiotic Representations and Types of Transformations

In recent years have returned with some force studies about semiotic representations and their relationship with cognitive operation, among which stands out the one developed in the last two decades by Duval (1999, 2004). In certain everyday contexts, and in some fields of scientific knowledge, it is possible to access the objects directly through perception, the use of instruments or, indirectly, using representations of such objects. In other fields, access via representations is not only useful but mandatory; such representations are produced using different systems of representation of different nature[8].

In mathematics, in particular, learning of objects is primarily conceptual, which requires the appropriation of semiotic representations, in other words, representations by signs.

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[6] From the ideas of Wittgenstein (1958/1999), it is not about examining the use of words that subjects do in specific situations, but also about recognizing the existence of social rules of use of signs in language games in certain contexts.

[7] What is contingent, it is in a certain place and time and may not be useful in another time or in another situation. Temporality and usefulness validate each other, mutually in the field of contingencies (Dáros, 2001).

[8] As stated by Duval (2004), such systems may be *non-semiotic* -neural networks (such as the different forms of memory), physical instruments (such as microscopes and telescopes) -, or may be *semiotic* (by signs). Producing a semiotic representation is necessarily intentional. This is, for some authors, the difference between semiotics and semiology.

The subject does not come into direct "contact" with the objects, as these are not accessible perceptual or instrumental. While the ostensive references of such objects are not possible, it is necessary to use representations[9].

In the process of teaching and learning mathematics, it is essential the use of representations of objects in a variety of semiotic systems of representation, more specifically, in a variety of semiotic *registers* (Duval, 1999). In particular, it is necessary to seize opportunities to transform a semiotic representation of a mathematical object into another representation of the same object. Such transformations between semiotic representations occur within the same record of semiotic representation, called *treatments*, as well as between different registers, called *conversions* (Duval, 1999).

It is usually stated that cognitive problems are related to the conversion while any related to treatment is not usually seen as a relevant issue for the construction of the mathematical object. Duval (2004), for example, recognizes the conversion as one of the fundamental cognitive operations for the subject's access to a true understanding, explicitly highlighting the complexity involved in the recognition of a same object through completely different representations, as produced in heterogeneous semiotic systems, and focuses his gaze on the difficulties of learning mathematics in this process. Nevertheless, it does not highlight the complexity associated to transformations made within the same semiotic system of representation. However, in mathematics, treatment transformations between semiotic representations -within the variety of records used-, not only are essential but can be a source of difficulty in understanding processes of mathematics by students.

In the international context there are several research papers that specifically address issues related to the semiotic transformations, among which highlights those by Duval (1999, 2004, 2006); D'Amore (2006); Godino, Batanero, and Font (2007), Font, Godino, and D'Amore (2007), D'Amore and Fandiño Pinilla (2008), and Santi (2011).

Sociocultural Approach

Even if it is recognized the epistemic importance of language, as mediator of human activities, it is argued that only in terms of discursive practices cannot adequately describe the ways of thinking, understanding and conceptualizing. In the pursuit of knowledge, as suggested by Radford (2006), human beings speak, gesticulate, write, use artifacts, and grab objects appealing to a variety of culturally arranged semiotic systems. While the signs and artifacts used mediate knowledge acts, alter the cognitive ability to be affected by things, and make this capacity, and therefore knowledge, be culturally dependent.

Inspired in the anthropological and historical and cultural schools of knowledge, Radford (2006) suggests elements of a culture theory of objectification, supported by an

[9] These representations may be *discursive* -using natural language or formal languages- or may be *non-discursive* -through Cartesian graphs or geometric figures-.

epistemology and ontology unrealistic. This author rejects the mentalist conceptions of thought, going for a characterization because of its nature semiotically mediated, and its mode of being as reflexive praxis. Radford (2006) stresses the role of cultural artifacts (systems of signs, objects, instruments, etc.) in the social practices since they are constituent parts of thought and it not only aids, that is, humans think with and through these artifacts. It recognizes, in particular, that individuals require becoming aware of cultural objects, and the impact cultural meanings have in the way an individual thinks and knows the objects of knowledge, while they not only guide the activity in which this happens, but they give certain "form." To operationalize his theory, Radford (2006) introduced a fundamental concept of semiotic-cognitive nature, which he called as *objectification*, precisely this subjective awareness of the cultural object.

From a socio-cultural approach, knowledge is related to the activities of individuals in a particular context, in other words, culture is consubstantial from knowledge.

Ontosemiotic Approach

Since the proposal targeted by Godino (2003), the concept of system of practices is conceived as a set of significant practices to solve a problem area. Mathematical activity, organized in systems of operational and discursive practices, has an essential role in the generation of mathematical entities (cultural/mental). For this author, mathematical objects are conceived as emerging from a practices system, as complex entities progressively constructed that enriches and completes itself from reflective activity in resolving certain problem areas. Emphasizing that these are the result of human construction, they evolve and depending on the persons or institutions may be provided with diverse meanings, shifting the focus to the action of individuals in contexts[10], mediated by instruments.

For Godino (2003), the theoretical notions, *system of practices* and functional categories of *primary entities* or types of objects (language, situations, procedures, definitions, properties and arguments), the five *dual facets* (personal/institutional, ostensive/non-ostensive, copy/type, elementary/systemic, expression/content) from which these entities can be considered, as well as the notion of *semiotic function (*expression/content, every expression refers to a content) constitute an adequate possibility to analyze the human cognition[11]. For this work, the analysis focused on the dimension or facet *expression-content.* In this approach the semiotic transformations are an emerging aspect of a semiotic function that relates a representation R (antecedent), in the couple configuration-practice system of objects with a representation S, in another couple practices -configuration system of objects.

[10] *Context* is seen as a set of extra/inters linguistic factors, that support or determine the mathematical activity and, therefore, the form, adaptation and significance of objects placed in its game.

[11] This notion, as Godino remarks (2003), comes from Hjelmslev (1943), who called *sign function* to the relationship of dependence established between the parts of a text and its components, as well as between the components thereof; notion that was later described by Eco (1979) as *semiotic function*. See also: D'Amore & Godino, 2006, 2007; Font, Godino & D'Amore, 2007.

In the performing of any *mathematical practice*[12], subjects make use of basic knowledge, and it triggered a set of relationships between different types of objects (primary entities): problem-situations, language, definitions, procedures, properties, and arguments. In other words, personal mathematical practices activate a network of emerging and intervening objects (Figure 1), in other words, the *cognitive configuration* on action (Godino, Batanero, & Font, 2007).

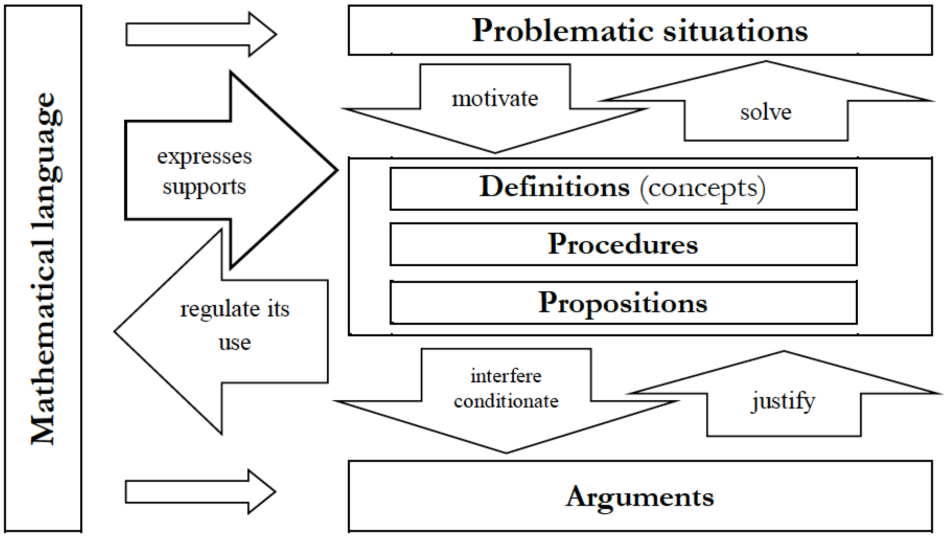

Figure 1. Configuration of Primary Object.

Objects, Meaning and Sense

Learning mathematics and mathematical activity itself, while require appealing to the transformation of signs within semiotic registers, culturally given, are inherently a semiotic activity. However, from a social and cultural approach, to understand the use of signs, we should take into account the reflective activity mediated that underlying the coordination of semiotic registers. Thus, everything associated with the meaning moves from *object* that the signs *represent*, to the practice that they enable and mediate (Santi, 2011). Therefore, we pass from the coordination of semiotic systems to the integration of systems of the practice from which meaning emerges, in other words, to cognitive configurations that are activated by such practical systems.

Thus, the meaning cannot be identified only by the relationship representation semiotic-object of reference. To account for the complexity of mathematics as a cultural and individual effort, it is not enough to reduce learning and mathematical thinking to a coordination of different representations with a common denotation. The way we get to

[12] From the EOS, mathematical practice is regarded as any action or manifestation, not only linguistic, formed both in solving mathematical problems and communication of other found solutions, in order to validate or generalize to other contexts and situations or problems (Godino, Batanero & Font, 2007; D'Amore, Font & Godino, 2007).

think and know the objects of knowledge is framed by cultural meanings that go beyond the content of the activity, within which occurs the act of thinking. That is, the meaning attributed to a mathematical object depends on both; the subject and the context in which it is addressed, and therefore, it is somewhat flexible, dynamic, and moving[13]. The meaning of an object, as it is assumed by Radford (2006), is attributed by culture and has an existence that transcends the subject, it is more stable. One could say that the meaning is more decontextualized and general[14]. Nay, the sense is relative to several sensory and semiotic modalities, and it is associated more to the pragmatic, while the meaning is associated more with cultural semantics. The sense of an object can be considered as a contextual meaning of that object.

Sense of a Primary Mathematical Object

Given a primary mathematical object, the *sense* of such object is the content of the semiotic function that has such primary object as an expression of the semiotic function (Figure 2).

Expression	*Content*
Primary object	Sense of the primary object

Figure 2. Sense assigned to a primary mathematical object.

A single parent object can have different senses (Figure 3). For example:

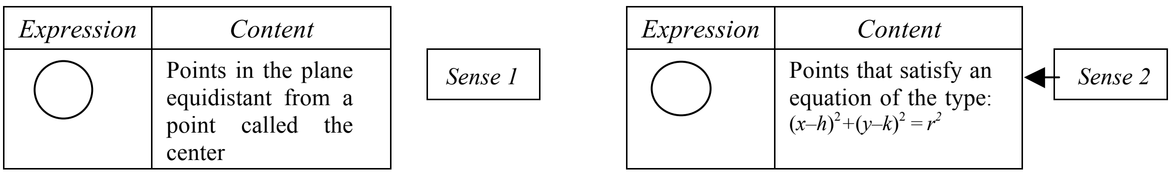

Figure 3. Different senses of an object, which is institutionally expected to construct apprentices.

Articulation of Senses

There is a joint of senses when it is established a semiotic function between two different senses of the same primary mathematical object. This is, when one of the senses (content) of the primary object becomes the expression of a new semiotic function whose content is another sense of such object. Thus, for example, in the case of Figure 3, the joint produced may be synthesized in a diagram as follows.

[13] The subject can be an individual, a group of individuals or an institution. The sense can be assigned individually, be the result of a negotiation within a community of practice, or acceptance of an institutional nature.

[14] For him, mathematical objects are "fixed patterns of reflective activity (...) embedded in the world of constant changing of the social practice mediated by devices" (p. 111), where such devices can be objects, instruments, systems of signs, etc.

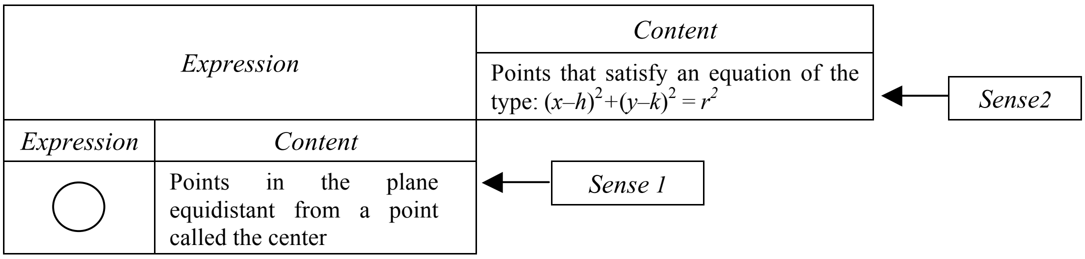

Figure 4. Joint of senses.

This joint is the result of the concatenation of the two-presented semiotic functions. We have a primary mathematical object with two different senses assigned. In other words, there is a joint of senses when a new sign-function is established in which one of the two preceding semiotic functions plays the role of expression (Figure 4). Such combination of semiotic functions can be simplified into a single semiotic function that relates (articulates) both senses:

Expression	**Content**
Points in the plane equidistant from a point called the center	Points that satisfy an equation of the type: $(x-h)^2+(y-k)^2 = r^2$

Figure 5. Joint of senses (simplification).

Thus, the joint of senses means that contents of semiotic functions previously considered as different (no explicit relationship) are now considered, in some ways, equivalent (explicit relationship). From what stated above, two primary objects (especially two representations), which are considered syntactically equivalent, while one of them is obtained from another as a result of a treatment process, they can be associated to the *same sense* (the same content), that is, also retaining the semantic equivalence. When the sense assigned to a semiotic representation does not articulate with the sense assigned later to another semiotic representation obtained from this one by treatment while the originally given sense is "abandoned" and a new sense is assumed, it will be said that there is a *change of sense.*

METHODOLOGICAL DESIGN OF RESEARCH

This research is part of an approach of qualitative research, from the type descriptive-interpretative, analyzing in real context of the described phenomenon, related to the change of sense. It makes use of the structured interview based on task (Goldin, 2000), conducted in small groups of students in grade 9th (basic education) and grade 11th (Middle education), from five schools, two official schools, located in the outskirts of the city of Bogota, in areas of low socioeconomic status, considered vulnerable, and three private schools, located in not peripheral areas of the city, where attend students of middle and high socioeconomic status.

Instruments

We worked with three semi-opened instruments, each one with a task associated with a specific topic: probability, equivalence of expressions and conical. The first two were proposed to students in grades 9^{th} and 11^{th} and the third only to grade 11^{th}. Each of them is with three items, and a similar design to what is shown below.

Questionnaire 2 (Equivalence of expressions):
Hereinafter, assume that n represents an integer either. Please answer in the order in which the points appear and go to the next only when you have fully answered the previous point.

(1) Say what does it mean or you assigned interpretation of the expression $3n$. Can be interpreted as *a number tripled.*

(2) State whether the following equality is valid or not: $(n-1)+n+(n+1)=3n$
   a. Mark with an **X** the answer you think is correct ***Yes*** **( )** ***No*** **( )**
   b. If yes, check equality, if not, give reasons why not met.

(3) Can the expression $(n-1)+n+(n+1)$ be interpreted as *a number tripled?*
   a. Mark with an **X** the answer you think is correct ***Yes*** **( )** ***No*** **( )**
   b. Explain or justify below with as much detail as possible, your response:

Gather of Information

In addition to the inquiry by developing tasks (proposed in the questionnaire) and the content of the notes taken by the researcher, there are transcripts of audio-taped interviews, which were made with each of the small groups, selected based in the responses to the proposed task in the different questionnaires.

Once recognized the importance for students to get involved in the development of the activity, the tasks were initially worked individually by each of the students from different courses (each with about 40 students) and then in small groups (2 to 4 students). Later, under the guidance of the teacher of the institution in charge of the course, there was a discussion of some of the responses given by small groups[15], which were selected by the teacher based on the observation made by him during the students working time.

From research in social psychology of learning, it is recognized that in such situations it is important to take into account that students must not only fulfill the proposed task, but also respond to the complexity of the social situation, in this case, to understand the expectations of the researcher and the nature of the problem, in addition to understand their role, and their peers' role in the interaction.

The selection of small groups to interview was conducted from the responses to the task, which contained three items, based on the following criteria:

[15] It recognizes the importance of who socialize a response, does it on behalf of the group; not only to realize the group process but also that his speech can be supported by the other group members and reduce some of the tension that usually can be generated when an outcome is defended individually.

(1) That in the first item, the investigation made by students was compatible with the "institutional meaning" (Godino & Tanner, 1994) assigned to that object.
(2) That in the second item at least one of the students can recognize explicitly the syntactic equivalence between the given expressions (arithmetic or algebraic), that is, that at least one student can perform a treatment process that allows him to get one of the expression from the other (making use of the properties into the arithmetic or algebraic register).
(3) That in the third item at least one of the students who recognized the syntactic equivalence in the previous item of expressions, answers negatively.

From such small groups of students four were selected, one for each of the four institutions, i.e., 16 small groups in total, taking into account the availability and the interest shown by the students to participate in the interviews. Of these 16 small groups were finally selected 3, one for each proposed task, whose oral and written production constitutes the main source of data reported and analyzed in this research.

Interviews were conducted in a different space from the classroom where were present only respondents (one group at a time) and the researcher. For interviews was prepared the following general script:

- Explicitly confirm the names of those interviewed at the beginning of the interview and request that, when possible, students mention their name in the interventions made (to record the voices, in addition to the researcher's notes).
- Check if group members share at least one interpretation from the worked expression(s) in the proposed task(s).
- Check if group members show a mastery of the transformations of treatment required for getting one of the expressions from other(s), i.e., if they recognize a syntactic equivalence between them.
- Investigate the possibility for respondents to recognize that the interpretation given to one of the expressions can be assigned to another expression syntactically equivalent to it, that is, if they can or cannot articulate the assigned senses to the referring expressions.
- Check if changes occur with respect to interpretations initially given to worked expressions, that is, if a change is generated in the sense assigned to the primary mathematical objects.

Although the researcher had a script for the development of the interview, incorporated some retrospective questions, and even some hints to supplement the inquiry and was attentive to recognize situations of interest for the purpose of research, asking specific questions designed to get more information.

Data Analysis

It was made an analysis of the individual production of each student, from the three small selected groups, in relation to the task, although the focus was on the production that was generated during the interview with the groups, in which was possible the intervention of each student, the interaction between them and, in particular, the choice to agree or disagree with the statements of the members of each small group in relation to the item(s) of the proposed task(s). Group work was privileged, while it is explicitly recognized the interest in enabling students to both, share views and consider the statements of others about the work done on the task, as well as the possibility to recognize a "shared sense" in the development of the task.

It is assumed that the verbalization of processes of thought and action provide important information, not only from written materials, such as those obtained by instruments of inquiry (tasks or questionnaires), but also from interaction processes, like those generated in the work with small groups or by interviews, in the context of a given task. The interview to small groups, unlike individual, offers more opportunities of interaction that allow recognizing the different interpretations made regarding mathematical objects involved in such task and identify the senses assigned to the expressions, in addition to recognize some reasons that make possible or not the assignment of senses and the articulation of these. It also offers a less formal and tense environment for each of the respondents, due to the interviewer's attention is not permanently focused on the work of a single individual. Respondents can interact with each other, welcome or call into question the claims and arguments presented by their peers, as well as having the opportunity to meet, analyze and have additional elements relating to the proposed tasks and the arguments initially considered.

The analysis of the student productions reported in this research is made from various perspectives, oriented from three theoretical proposals, with different levels of use thereof. First, the structural-functional approach from Duval (2004), used primarily to identify the field of research, owing to that allows making denominations that allow to see and to describe the phenomenon under study. Moreover, sociocultural approaches from Radford (2006) and onto-semiotic from the group of Godino (2003), used to explain and understand the student productions, because their facts not only recognize cognitive facts but also cultural and historical. For these two authors, the use of formal systems of signs is an emerging phenomenon of the systems of practices social and cultural framed. To perform the analysis of the transcripts of the interviews, it is done a thematic segmentation, which is itself segmented in each of the interventions of the participants in the interview, making an enumeration.

DEVELOPMENT OF THE RESEARCH

It is presented, with some detail, the work done by two of the interviewed small groups, in two of the proposed tasks; the task about probability and the task related to conical, both worked with students from grades 9^{th} and 11^{th}.

Task About Probability

This section describes some aspects of the work of a group of 9th grade students (Questionnaire 1), from the CHA institution, integrated by Pablo (E_4), Daniel (E_5) and Jonathan (E_6). In his individual work, Pablo establishes three semiotic functions. One between the expression 50% and content "favorable cases on possible cases" another between the expression 50% and the content 3/6, and another between the expression 3/6 and "favorable cases on possible cases," that is, it can articulate the senses assigned. However, for him the probability cannot be 4/8, because if so, raises, *the given information would be wrong and the dice would have given 8 faces*. In the Figure 6 it is presented the cognitive configuration of primary mathematical objects attained by Daniel, obtained from work both individually and in group.

Problem (Tsk – Questionnaire 1)

Bellow, we will be referring to a traditional six-sided dice. Please answer in the order in which the points appear and go to the next only when you have fully answered the previous point.

1. What is the probability **to obtain an even number by rolling a dice**?
   (a) The probability is:
   (b) Briefly explain how did you make the calculation of the probability:
2. Is there another way to express the probability obtained in the previous point?
   (a) Mark with an **X** your answer **Yes ( )** ***No*** **( )**
   (b) If yes, show which way would it be. If not, explain why there would not be other way of expressing this probability.
3. Can it be affirmed that the fraction 4/8 is the probability to obtain an even number by rolling a dice?
   (a) Mark with an **X** the answer you consider correct **Yes ( )** ***No*** **( )**
   (b) Justify below with as much detail as possible, your response:

Language
Divisor
50%
3/6
4/8

Definition
Favorable cases on possible cases
Procedure
Aplication of definition

Argument
Tesis: 4/8 is not the probability
Reason: The dice is divided into 6 instead of 8 "fractions"

Figure 6. Initial configuration of Daniel (E5).

As a group (G-2), students found the probability to obtain an even number by throwing a dice, which they represent in two different ways, using the numerical expressions 50% and 3/6. However, none of them recognize that the fraction 4/8 can be interpreted as such probability. That is, they fail to articulate the senses assigned to the previous numerical expressions. In the small group work these students maintained the interpretations individually made regarding the proposed task. The reason why they do not accept that the fraction 4/8 is the requested probability, is *that the number of sides of the dice is 6 and not 8*, due to the sense assigned to the fraction 4/8 is "anchored" to the dice, to the concrete object referred to in the proposed task. The data obtained is summarized in the Table 1.

Table 1

Grillage Synthesis (Initial) –Pablo, Daniel y Jonathan– Grade 9th, School CHA

	Pablo (E_4)	Daniel (E_5)	Jonathan (E_6)	Group
Recognizes several ways to represent the probability	Yes	Yes	Yes	Yes
Articulates senses assigned	No	No	No	No

Interview G-2

The segments of the transcript of the interview are reported below in three columns. In the first one is numbered each intervention, in the second one the person involved (interviewer: P and students: E_k) and in the last one the accompanying text. These students, in their individual questionnaires and in the small groups' questionnaire had recognized more than one way to represent the probability of the event in question, proposing expressions like 50%, 3/6 and ½. Nevertheless, all of them started saying that the fraction 4/8 does not express the probability of such event.

[1] P Jonathan … What do you say Jonathan?
[2] E_6 Well, … as the main theme is a dice, it is recognized that the dice has six sides
[3] E_6 …
Yes, if we take the pairs out it would be three... 4/8 then is not as representative
[4] E_6 of the dice, since the dice has neither eight faces, other four even numbers.
[5] P Looking at it from the way I see it, the four eighths is not accurate, then ...
[6] E_5 That´s it
I understand. What do you say Daniel?
[7] P Well ... if it's a 6-sided dice, the division is three, three of the probability [refers
[8] E_4 to the fraction 3/6], it can't be the reference number four [refers to the fraction 4/8].
[9] E_4 All right… and Pablo?
Well, I think there is no..., not, because I think it wouldn't work, I think the fraction would be misconceived to how the problem should be solved, based on the sides of the dice.…
A dice is never going to have eight sides ... I think that.

The students focus on the proposed situation, specifically in the dice as object. While Jonathan (E_6) recognizes that the fractional 3/6, ½ and 4/8 are equivalent: *they can give the same*, insists that 4/8 *is not accurate* ([4]). He explains that fractions have a "generator", which in this case would be ½, but focuses on the role of the denominator. Later, as they interact, they start to partially modify their opinion, and begin to recognize that it is possible to accept the answer 4/8, but there are still several questions. For example, Pablo states:

But then, to be the fraction of the dice it wouldn't work because it is not specific what is sought ... 3 to the even numbers and 6 to the sides of the dice, instead in 4/8 it would not be clear.

[10]	E_4	Well ... what I say is that the fraction ... yes, the same thing, that the fraction itself work but it´s misconceived in detail, why? Because it gives bad information about the probability and about the sides of the dice.
[11]	E_4	If they specify and give for example a 3/6, it would be a detail already, what happens is that the fraction...
[12]	P	When you say, "it would be a detail already" what do you referred to?
[13]	E_4	Rather, the fraction used is not well specified but why? Because ... 4/8 is basically the same as 3/6, right?
[14]	P	Mmju *[I understand, continue]*.
[15]	E_4	… But then, to be the fraction of the dice it wouldn't work because it is not specific what is sought ... 3 to the even numbers and 6 to the sides of the dice, instead in 4/8 it would not be clear.

After listening to the ideas presented by their peers, particularly by Jonathan (E_6), Daniel (E_5) returns the statement [4] and tries to overcome the difficulty on not accuracy, proposing a dice with more sides that includes numbers on their faces with decimal figures. For him, it is physically easy to build a dice that work for this purpose. However, sometimes hi mentions an 8-sided dice and sometimes that it can be 6-sided as long as the sum can be 8, if you change the given usual numbers of the dice with decimal numbers. He fails to clearly express how this dice would be, which he recognizes different but possible:

[16]	E_5	Decimal numbers, for example 2.5 plus 2.5, 4 so it may result, … to get 8, two point five, point six, bla, bla, bla... *[etc.]*.
[17]	P	Oh, ok! … that the numbers of the dice are not from 1 to 6, but others.
[18]	E_5	Exactly.
[19]	P	… but, assuming that you don't have different dices and you know that you have been ask probability of getting an even number with a traditional dice, without changing the dices...
[20]	E_5	No, because there is no accuracy.
[21]	E_5	Rather, it would be illogical that one would say… ehhh; there is a 6-sided dice, which is divided into four eighths.
[22]	P	Mmju… *[I understand, continue]*.
[23]	E_5	Unless you can split it, because… it is a dice.
[24]	E_5	Well, for example, yes, physically it is very easy, we make a 8-sided dice, but… rolling decimal numbers that the sum of all is 8… and that at the same

[25] P time can be 6, yes? … Imagine this dice, the sum is 8, but the sides are still 6.
[27] E_5 Yes, and, how would it be?
Well, for example… I don't know … a different dice, how can I say it? For example, that the numbers change or … that work with decimals.

Subsequently, to investigate a little more about the potential "anchor" to the initial situation, the dice as physical object ([21] and [24]), the interviewer (P) decides to ask indirectly, by going to an argument of another student, who claimed that fractions like 3/6, 4/8 or 15/30 are equivalent to half, and so any of them could represent such probability as well as, for example, 10/20:

[28] P What would you say? Would you agree? Or would you see a problem in
[29] E_6 this statement?
[30] P I would agree.
[31] E_5 Daniel, I see yo thoughtful ... *[Smiles].*
Not exactly when… I don't contradict that it is the half and that it'd be the
[32] E_5 same… *[in his gestures and his voice tone there is evidence of doubt].*
… But if you search accuracy, if it is a dice of 6 sides, I would work with
[33] P the right numbers; 3 of 6.
[34] E_5 Three of six … *[continue]*
… Because it wouldn't be clear that, for example, I call 4/8 in a 6-sided
[35] P dice, I don't think, no! … Then it is an 8-sided dice, yes?
[36] E_4 What do you say Pablo?
[37] E_4 Well, I also agree that ten twenty *[fraction 10/20]* is the same as 3/6.
... But if I'd formulate the …, the question… , if for example you are
[38] E_4 asked Can be stated that the fraction 10/20 is the probability that rolling a dice it is obtained an even number?
[39] E_4 You would be blocked, because … how come 10/20? Since when have a dice 20 sides, you know? then [that is] what I don´t get *[laughs a little]*… What happens is that in the common sense of the people … of the … of everyone, that would not be understood.

Finally, the interviewer asks again to Daniel (E_5), who has been quietly listening intently to his classmates, if now he would accept the argument of the student:

[40] E_5 Right now, yes … After discussing all of this.
I mean, for someone common, no, but…
[41] P And what made you change your mind?
[42] E_5 Because…, here for example, I answer in the first question 50% *[points to the first item of the questionnaire]* … from 100 it would be the same, a half
[43] P *[he means that 50 is a half of 100]*
[44] E_5 So you say, 4/8 would be a half, so it's the same, it doesn't matter.
Equivalent yes … but not looking at the sides, or at the dice, but at the half.

During the interview, after several interactions with Pablo (E_4) and Jonathan (E_6), his group mates, and the interviewer (P), Daniel (E_5) recalls that in its Questionnaire, he had responded that the probability is 50%, which is *half*, and recognizes that 4/8 is also half; then he accepts that the fraction 4/8 is *equivalent* to 50% ([42]). Namely, when he manages to decenter from the object, of the sides of the dice, and focuses its attention on the formal expressions representing half, he gets to recognize that the fraction 4/8 expresses the desired probability and so articulate senses assigned ([44]). Meanwhile, Pablo and Jonathan do not accept the last argument given by Daniel and although the interviewer says that the issues raised by him seems to be a good argument, they do not change their minds and insist that even if the fraction 4/8 is *equal* 3/6, this fraction is not the desired probability. In fact, for Pablo (E_4) the dice should have as many sides as the digit in the denominator ([38] and [39]).

Cognitive Configuration of Primary M*athematical Objects*

Below is presented the cognitive configuration achieved by Daniel (E5) after the interaction process during the group interview, in relation to the work from the task of probability (Figure 7). In this diagram, by a solid line, are pointed the semiotic functions initially established by the student, between an expression and a content. By a dashed line are pointed new semiotic functions, shown during the interview in the interaction with peers in the small group.

In his individual work, Daniel had established three semiotic functions (Figure 6). During the interview he explicitly establishes a new semiotic function between 4/8 (expression) and 3/6 (content), although he initially suggested that in the specific situation of the dice, *if it is sought precision* he rather work with the right numbers that are: 3 of 6 ([32]). Then, after about three minutes listening intently to Pablo and Jonathan's interventions, and the interviewer's questions (P), he does "separate" from the given concrete situation of the dice and establishes a new semiotic function; this time between the expression 4/8 and content "number of favorable cases divided number of possible cases", accomplishing a joint between the different senses assigned ([40]).

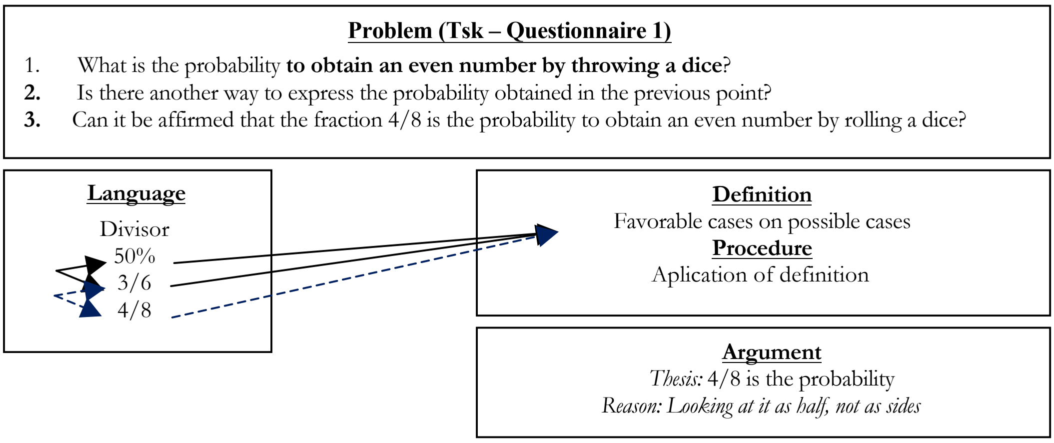

Figure 7. Final configuration of Daniel (E5).

To summarize. The process of interaction during the group interview led to changes in the initial interpretations made by students. On the one hand, the three of them explicitly recognized the equivalence between fractions 3/6 and 4/8, and even that any fraction equivalent to 3/6 could represent the desired probability in the task, but their "anchor" to the object, to the dice and the number of sides, do not allowed them to articulate the senses assigned to such terms and, therefore, Pablo and Jonathan do not accept that 4/8 is this probability. Daniel (E5), meanwhile, managed to "separate" from the concrete situation and recognizes that the fraction 4/8 represents the probability and achieves to articulate the senses assigned to different numerical expressions. Table 2 summarizes the information obtained.

Table 2

Grillage synthesis–Pablo, Daniel y Jonathan– Grade 9th (School CHA)

	Pablo (E_4)	Daniel (E_5)	Jonathan. (E_6)	Group
Recognizes several ways to represent the probability	Yes	Yes	Yes	Yes
Articulates senses assigned	No	Yes	No	No
Change (recognition of equivalence between 3/6 y 4/8)	✓	✓	✓	✓
Change (articulation of senses)		✓		

Task About Conics

This section describes some aspects of the work of a group of students of the institution CHA, grade 11th (G-1), composed of Maria Elvira (E_1), Daniel A. (E_2) and Daniel D. (E_3), in relation to a task about conics. In the individual work was evident that the sense assigned by each of the students to the equation $x^2+y^2+2xy-1=0$ was the circle[16].

In their individual work, Maria Elvira (E_1) established three semiotic functions, one between the expression $x^2+y^2+2xy-1=0$ and the content "a circle" another between the expression $x^2+y^2+2xy-1=0$, and the contents (equation) $x+y=\frac{1}{x+y}$, and the third semiotic function between the expression $x+y=\frac{1}{x+y}$ and the content "circle." So, Maria Elvira achieved to recognize the *syntactic equivalence* between these equations, by making the transformation of treatment required for getting an equation from the other, and to articulate the senses assigned to them.

Daniel A. (E_2), meanwhile, established a semiotic function between the expression $x^2+y^2+2xy–1=0$, and a content "circle", but does not recognize equivalence between the

[16] Although the quadratic equation $x^2+y^2+2xy-1=0$, does not represent a circle, but a "degenerated conic" (two parallel lines), for purposes of the analysis that presented this section, is it not relevant if the equation is erroneously interpreted by the students as circle.

two equations. He states that in one of them appears a product (the term $2xy$), which then becomes a sum ($x+y$), "which is meaningless". Also he states, that in one of them the variables are squared and in the other do not, "Square root is taken to the entire equation, which cannot be".

In the work done individually by Daniel D. (E_3), he established two semiotic functions, one between the expression $x^2+y^2+2xy-1=0$ and the content “circle,” and another, between the expression (equation) $x^2+y^2+2xy-1=0$ and the content (equation) $x^2+y^2+2xy-1=0$. He argues that the equation of the circle must have squared variables, and because in the equation $x + y = \frac{1}{x+y}$ they are not squared, then it cannot be circle, i.e., although he recognizes the syntactic equivalence obtained by treatment the two given equations, fails to establish a semiotic function between the expression $x + y = \frac{1}{x+y}$ and the content *it is a circle,* then, he asserts that this expression has not the variables squared. The cognitive configuration of primary mathematical objects initially achieved by Daniel D. (E_3) is obtained from the work both individually and in groups, it is as follows.

<table>
<tr><td colspan="2"><u>Problem (Task – Questionaaire 3)</u>
1. Say what it is, what does it represent, what does it mean or which interpretation you make from the following equation: $x^2 + y^2 + 2xy - 1 = 0$
2. Is the equation $x + y = \frac{1}{x+y}$ equivalent to the equation $x^2+y^2+2xy-1=0$?
3. Is the equation $x + y = \frac{1}{x+y}$ a… [circle] ?</td></tr>
<tr><td rowspan="2"><u>Language</u>
Circle
Variables
Equation
$x^2+y^2+2xy-1=0$
$x + y = \frac{1}{x+y}$</td><td><u>Definition</u>
A circle
<u>Procedure</u>
Application of properties
<u>Properties</u>
Multiplicative Inverse and Additive Inverse (implicit)</td></tr>
<tr><td><u>Argument</u>
Thesis: The equation is not a circle
Reason: the variables are not squared</td></tr>
</table>

Figure 8. Initial configuration from Daniel D. (E_3).

After, Daniel A. (E_2), who had not recognized the possibility of making transformations of treatment to establish the equivalence between the given equations, in his work with Maria Elvira (E_1) and Daniel D. (E_3), he managed to do it. Thus, as a group, these students shared the sense assigned to the equation $x^2+y^2+2xy-1=0$, that of being "a circle", and recognized a syntactic equivalence obtained by treating between such equation and the equation $x + y = \frac{1}{x+y}$, but they did not share the possibility to articulate the senses assigned to such equations.

In connection with the recognition of the transformations of treatment required to obtain the equation $x^2+y^2+2xy-1=0$ from the equation $x + y = \frac{1}{x+y}$, that is, from the syntactic equivalence between these two equations using properties of real numbers, as well as the recognition of semantic equivalence, as possibility of articulating senses assigned to them. We have the information that is summarized in the Table 3.

Table 3

Grillage Synthesis (Initial) –Maria Elvira, Daniel A. and Daniel D. – Grade 11th, School CHA

	Ma. Elvira (E_1)	Daniel A. (E_2)	Daniel D. (E_3)	Group
Recognizes syntactic equivalence	Yes	No	Yes	Yes
Articulates senses assigned to equations	Yes	No	No	No

Interview G-1

From this group, only two of the three students recognized in their individual work the syntactic equivalence obtained by treatment between the given equations. However, in their work as a group they were able to perform the required treatment to one of the equations (expression) for getting the other (content) and so, to recognize this equivalence, but they failed to articulate the assigned senses to the two expression. That is, they did not managed to establish a semiotic function between the expression $\mathrm{x + y = \frac{1}{x+y}}$ and the content "a circle"; while one of them (E_1) had achieved to establish such function, and therefore articulate the senses assigned to the two equations, as a group they focused on the "form" of the expressions in each equation, explaining that "in one of them the variables are squared and in the other are not."

[45]	P	Is the equation $x + y = \frac{1}{x+y}$ *a circle?* You had to say yes or no, you said no. I like to hear the arguments, What are the reasons why you say it's not a circle? Who would like to start? … Maria Elvira?
[46]	E_1	No, well no, I wouldn't say it is a circle, because when I relate it, is when there is a…, when ehh… $x+y$ is dividing or multiplying, I mean when there are two, because when adding twice then it would be $2x$, then no, it wouldn't fit and when the two variables are squared, well, that's the principal thing to make it a circle…
[47]	P	
[48]	E_2	And in the case of Daniel A.?

I say it is not a circle because if I pose this equation like that,
[49] E2 $x + y = \frac{1}{x+y}$ then I don´t see it as a circle
… because one know that the circle as basic equation is ehhh …
[50] P when both variables are squared, right?
[51] E2 Mmju … *[yes, I understand, continue]*.
… If you find that, you can make the procedure to… it is possible that you can determine from this … I mean, I do, if I move the
[52] P values, change the values, each to another, it is possible that I find the … the equation for the circle, but …
[53] E2 But, initially what … what you look at to decide whether or not it is
[54] P [a circle] What is it?
[55] E3 That, let's say, that the variables are squared.
… That they are squared … Daniel D., what do you say?
That Maria Elvira is right, if you see it like this *[points at the equation $x + y = \frac{1}{x+y}$],* you see it as a normal equation, but if you
[56] E3 think about this *[indicates the denominator on the right side of the equality]* on the other side are the squared variables, so like that you don't see it as a circle.
But the main point of a circle is that the two variables are squared.

It is evident that in this group predominates the perception of the equation as an icon associated with the "circle," characterized by having squared variables ([46], [48], and [55]), about the proof of the syntactic equivalence initially made by E1 and later worked as a group. Basically it reflects a change in the interpretation of Daniel D. (E_3), while Daniel A. (E_2) maintains its initial interpretation.

[57] P Well, but the answer would be no, because they are not *[squared*
[58] E3 *variables]*, or would it be yes?
It is *[a circle],* but you don't see it like that, I mean one does not assimilate it just like that, because one don't, like in the mental process
[59] E1 you don't go directly to multiply, *indicates the expression x+y in the denominator, to the right of the equation]*.
The thing is that when you see it just like that *[indicates the equation*
[60] PE1 $x + y = \frac{1}{x+y}$ *],* I mean when you don't, don't see variables explicitly
[61] squared, then you don't start immediately to think what that is…
Aha *[yes, I understand, continue]*
Instead… I assimilate it, when I see it dividing the same variable, then well … it is equalizer, I started to multiply right away, so that's how I … I assimilate it to squared or the same when I have them expressed the two of them … I mean the two variables multiply, then is when I do it right away.

Daniel A. (E_2) insists that in the first instance he would say that the equation ($x + y = \frac{1}{x+y}$) is not a circle, because "from what is seen," it is not explicit what it is, since one have to perform transformations "to move [...] the parts of the equation," the variables, which he does not normally do, while if you give him an expression like x^2+y^2+2xy-1=0 " you know right away it's a circle." However, Daniel D. (E_3) in relation to the above said, confirms that for him it is a "circle", that the problem is that he treats it different because "don't see this form that we were taught, as a circle has two square variables [...] on both sides of equality." It means, as a group reaffirm the iconic look that they make of the equations, particularly of the expression associated with a circle, as a result of a misconception, possibly derived from an interpretation associated with a *classroom history*, different to the one institutionally intended by the teacher.

Cognitive Configuration of Primary Mathematical Objects

Below is presented the cognitive configuration finally achieved by Daniel D. (E_3), after the process of interaction during the group interview, conducted in relation to the work from the task on conic (Figure 9). In this diagram, by a solid line, are pointed the semiotic functions provided by the student, between expression and content. By a dashed line, is indicated the new semiotic function, evidenced during the interview, in the interaction with peers in the small group. In his individual work, Daniel D. had established two semiotic functions (Figure 8). During the interview he explicitly establishes a new semiotic function between the expression $x + y = \frac{1}{x+y}$ and the content "it is a circle", thus, achieving a link between the two senses assigned ([58]).

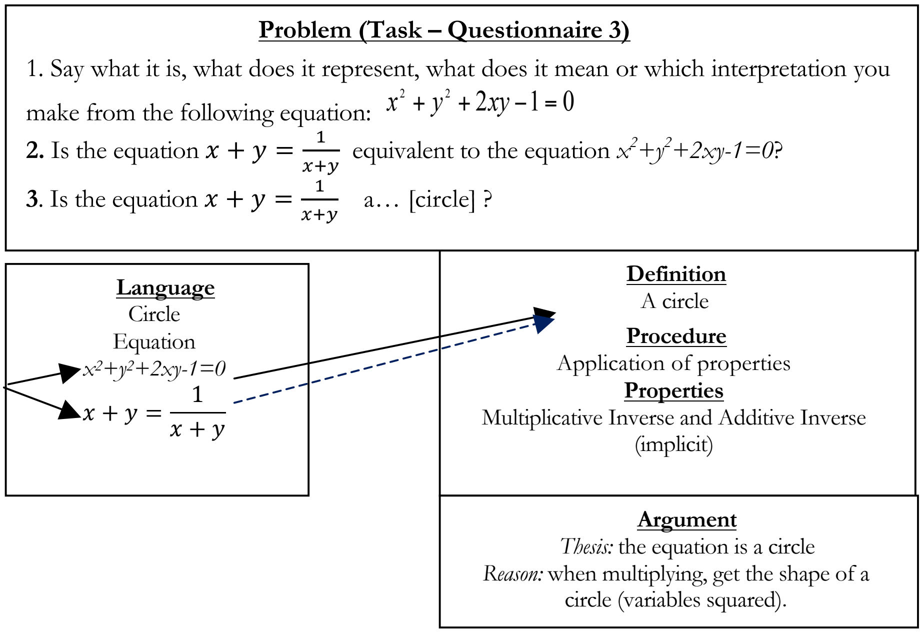

Figure 9. Final configuration of Daniel D. (E_3).

To summarize. As evidenced above, the individual work of each of the three grade 11th students, only one of them (Maria Elvira) could assign the same sense to the two equations. In the small group work, two changes were generated; on the one hand, Daniel A. accepted the syntactic equivalence between the two equations, as he acknowledged that by the changes in treatment he could obtain one equation from the other, and secondly, Maria Elvira welcomed arguments from her two mates regarding that, despite the performed transformations of treatment, the absence of one of them from squared variables, make it discarded the option of "view" as a "circle." Later, during the interview, the arguments given by each of them allowed them to share the sense initially assigned by Maria Elvira to the equation $x + y = \frac{1}{x+y}$, in addition to achieving an articulation of the different senses assigned to the two given equations. It is important to stand out, that for two of the students apparently continued prioritizing an iconic image of the circle, whereby both variables should be squared.

After the interview, and in relation to the recognition of the three students referred in this section of the syntactic equivalence between the equation $x + y = \frac{1}{x+y}$ and the equation $x^2+y^2+2xy-1=0$, as well as the recognition of the semantic equivalence, changes occur in the recognition of the syntactic equivalence and the articulation of the senses assigned to the equations. The data obtained is summarized in Table 4.

Table 4

Grillage Synthesis (Final) –Maria Elvira, Daniel A. and Daniel D.– Grade 11th, School CHA.

	Ma. Elvira (E_1)	Daniel A. (E_2)	Daniel D. (E_3)	Group
Recognizes syntactic equivalence	Yes	Yes	Yes	Yes
Articulates senses assigned to equations	Yes	Yes	Yes	Yes
Change (recognition of equivalence)		✓		
Change (articulation of senses)		✓	✓	✓

It is important to stand out that the process of interaction during the group interview led to changes in the interpretations made initially by the students; on the one hand, it made possible that one of the students (E_2), who by failing to focus his gaze on the form of the two given equations, recognizes the syntactic equivalence of these equations, and, second, that two of them (E_2 and E_3) achieved to articulate the senses assigned to them. Also, it made possible that E1 retake the interpretation she had done in her individual work, which had temporarily changed in the first work in small groups, to reaffirm the joint of the senses assigned to the two given equations.

RESULTS AND CONCLUSIONS

In the work done by students from 9th grade (last grade of basic education) and students from 11th grade (last year of pre-university education), regarding three specific tasks, it showed the difficulty that have many of them articulating several senses assigned to expressions associated with a mathematical object. Even though some of them recognize the syntactic equivalence obtained by treatment between two or more expressions, they are not always able to articulate the senses assigned to such expressions and may even change the initial sense assigned to one of them. The difficulties that students find to articulate the assigned senses to expressions can be grouped mainly into four groups, which are described below.

Iconic recognition of expressions. The sense assigned to the expressions in some cases is based on an iconic recognition thereof. For example, when considering that the "basic equation" of a circle is one in which the two variables are explicitly squared and are on one side of equality. In relation to the expression $(n\text{-}1) + n + (n+1)$, although several of the students were able to perform the treatment for getting the expression $3n$, they state that it is an addition and cannot be interpreted as triple a number. They suggest that each of these expressions incorporates procedures that differentiate them, although the second one is the result of the processing performed with the first. This reflects a cultural fact, the allocation of senses associated with each form of algebraic expressions[17]. Evidence is provided that these interpretations are entrenched, with some frequency, in school work.

It is important to stand out that similar investigations to those reported here have been conducted, informally, with university students taking courses related to training of mathematics teachers. For example, several students who were in fourth semester college degree in the area of math, recognized the equation $x^2+y^2+2xy\text{-}1=0$ as a "circle", but despite accepting the syntactic equivalence between the equation $x+y=\frac{1}{x+y}$ and the equation $x^2+y^2+2xy\text{-}1=0$, they did not recognize a "circle", because they did not "see" in the equation $x+y=\frac{1}{x+y}$ that the variables were squared.

Another case is that of a professor of secondary education, with university education in the area of mathematics and teaching experience of several years in grades 8th through 11th, who face the question: Can the expression $(n\text{-}1) + n + (n + 1)$ may be, represent, or be interpreted as, three times a number?, she raised initially, and categorically, that there was not a number three times, because "the triple is $3n$, ... while the given expression is the sum of three consecutive numbers"; later, once she made transformations of treatment to the given expression: $(n\text{-}1) + n + (n+1) = n + n + n + 1\text{-}1 = 3n$, she thought for a few seconds and with a surprised expression said: *"This strikes me as strange, I never thought about the possibility that the sum of three consecutive numbers could be three times a number, ... I never thought so".*

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[17] Many teachers, in their courses, insist on such facts, as they consider that stress in the form of expressions or equations constitute a "help" for their students.

Anchor to given situations. It is evidenced a tendency to make interpretations linked almost exclusively with the proposed situation, i.e. it is evidenced some "anchor" to the given situation in the task, like in the proposed case to find the probability of rolling a dice to obtain an even number. We found a strong forwarding to the specific object "dice." Thereby is recognized a cognitive problem associated with the use of *concrete models* in building of objects of school mathematics. Even if the concrete material can provide an effective support for the mathematical intuition, in some cases it may be an obstacle. Even though it is just a model for the teacher, the student may be a learning object (Maier, 1998).

Interaction and changes in the interpretation. The interviews with small groups evidenced the importance of interaction spaces, as opportunity to meet other people's arguments, questions, ways to organize their ideas, that allows strengthening or modifying the initial interpretations. It is important to pin down that the interaction options, particularly the semiotic functions explicit by some, are not necessarily recognized or assumed by their peers, so it does not always produce changes in their interpretation, in assigning new senses or joint thereof. Even if the interventions and arguments from other peers of the small group can influence changes in the interpretation of some of the members, when the arguments are not clearly accepted, such changes can occur for short periods.

Mathematical language difficulties. There were some difficulties that students find regarding interpretations of the given expressions and performing of treatments of such expressions, particularly in the algebraic context. One of the difficulties is related to the generalization from particular cases[18] and difficulties processing transformations of treatment of the algebraic expressions.

To summarize. We present evidence that confirms the phenomenon reported by D'Amore (2006) on difficulties encountered by students to articulate senses associated with expressions recognized by them as syntactically equivalent and elements that allow making explicit for causes of this difficulty articulating the senses, associated to three fundamental facts. One, that although students "manage" the basic properties of number systems that enables them to make the transformations of treatment required establishing the syntactic equivalence of the expressions, they find it difficult to associate senses different from the given expressions. Two, the tendency to anchor in specific situations arising in the context of the proposed task and, three, the "look" basically iconic of algebraic expressions. Also, it highlights the importance of the interaction processes as a key element to enable the articulation of senses assigned to syntactically equivalent expressions. There is not only some time to socialize and recognize the arguments made by others but also, and especially, to analyze the arguments presented by each other, which are not assumed uncritically.

[18] In this regard, Radford (2008) discusses the problem of naive induction. He reports that, from particular cases, students tested with formulas until you find the right formula that allows them to calculate any term of a given sequence, abduction processes are explained but as "guessing". Therefore, he states a need to distinguish between algebraic generalizations and arithmetic generalizations.

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