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Guest editors
Man – Keung Siu
Constantinos Tzanakis

Volume 3 No. 1-2 2004

CYPRUS MATHEMATICAL SOCIETY
ΚΥΠΡΙΑΚΗ ΜΑΘΗΜΑΤΙΚΗ ΕΤΑΙΡΕΙΑ
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This issue is based on papers presented in the *Topic Study Group TSG 17* of the 10th *International Congress on Mathematical Education* (ICME 10), Copenhagen, Denmark, 4-11 July 2004

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FROM THE EDITORS

History of Mathematics in Classroom Teaching - Appetizer? Main Course? Or Dessert?

During the 10th *International Congress on Mathematical Education* (ICME 10) held in Copenhagen on 4-11 July 2004, several activities were devoted to the relations between the history of mathematics and the learning and teaching of mathematics. Among these activities Topic Study Group 17 (TSG 17), titled *The Role of the History of Mathematics in Mathematics Education*, was organized by A. El Idrissi (Morocco), S. Kaijser (Sweden), L. Radford (Canada), M-K. Siu (China, co-chair) and C. Tzanakis (Greece, co-chair). The four sessions in this group were attended by about 70 people from more than 20 countries and focused on *integrating the history of mathematics in the learning and teaching of mathematics*, in an effort to make clearer the meaning of a *historical dimension in mathematics education* and to deepen the understanding of its various aspects. The final programme consisted of 13 presentations with follow-up discussions among the participants. Relevant material on the presentations has been made available on the TSG 17 web page in the form of extended abstracts, full texts, related papers or links to other websites, so that prospective participants were able to download material of interest to them and study it in advance. This Special Issue of the *Mediterranean Journal for Research in Mathematics Education* is a collection of the edited and revised version of most of the papers presented in TSG 17.

The aim of TSG 17 is to provide a forum for participants to share their teaching ideas and classroom experience in connection with the history of mathematics, in the spirit of the 10th ICMI (*International Commission on Mathematical Instruction*) Study on the role of the history of mathematics in the learning and teaching of mathematics, which led to the publication in 2000 of *History in Mathematics Education: The ICMI Study*, edited by John Fauvel and Jan van Maanen, and to learn about work that has been done since then.

Introducing a *historical dimension in mathematics education* involves three different areas: *mathematics, history* and *didactics*. Implicit to the papers collected here is the key issue, *viz* to find and elaborate on a harmonious, balanced and effective interrelationship among these three scientific areas in a way that is enlightening and fruitful in mathematics education. The papers approach this key issue in different ways, focusing on at least one of the following four points:

(i) To consider in detail ***epistemological issues*** relevant to the relations between mathematics, history, mathematics education and other disciplines, which although long-standing, still remain at least partially unsettled.

This is important in order to understand better to what extent introducing a historical dimension in mathematics education is possible, legitimate, or beneficial for the learner and the teacher. Three papers attempt to consider such issues. *F. Furinghetti* gives an outline of the different views on the role of history in mathematics education and identifies two main lines of approach: history as a vehicle to reflect on the nature of mathematics as a socio-cultural process, and history as a possible way to conceive and understand mathematical objects. *G. Bagni* discusses some epistemological issues related to the historical analysis of a mathematical topic, necessary for achieving an effective and correct use of historical data in mathematics education, and presents some theoretical ideas that underline the primary importance of the correct social and cultural contextualisation. *K. Zormbala & C. Tzanakis* consider the old, still unsettled, question of whether and to what extent the ontogenetic development parallels the historical development in mathematics, and, based on both a historical analysis and empirical data, report on observed analogies that in principle could be beneficial for mathematics education research.

(ii) To enrich ***teachers' education*** at all levels, both by introducing courses in (particular aspects of) the history of mathematics and its relation to other disciplines, and by letting them become acquainted with historically inspired material that can be, or has been used in the classroom

In this way, teachers may hopefully begin to think of a historical dimension in teaching as a possible path for improving mathematics education at all levels, and may develop confidence and trust in this endeavour. Reporting on recent work done in this direction is the subject of two papers. *G. Waldegg* reports on junior high school mathematics teachers' collaborative work on problem-solving, and *M. Barabash & R. Guberman–Glebov* report on a sequence of activities in prospective mathematics teachers' education.

(iii) To construct and develop appropriate relevant ***didactical material,*** which can either be used directly in the classroom or constitute resource material for mathematics teacher

Such material should aim to motivate and guide the teacher, improve the teaching approach, or understand better students' difficulties and their idiosyncratic ways in learning mathematics. The need for and importance of such material has been emphasized in the ICMI Study Volume referred to above (Chapter 7, pp.212-213). Three papers focus on this point. *D. Taimina* discusses a collection on kinematic mechanisms developed by F. Reuleaux in the 19^{th} century. *J. Tattersall & S. McMurran* give an account of the use of the (recreational) mathematical problems published in the *Educational Times* in the Victorian era. *R. Charette* recounts his experience in producing historically motivated teaching capsules in elementary geometry.

(iv) To present ***particular examples*** and the underlying rationale, as an illustration of how history may contribute to the improvement of mathematics teaching in one way or another -- by exciting the students' interest, enriching their view of mathematics, or deepening their awareness of what mathematics really is.

Supported by empirical data, *C-I. Fung* gives two examples enlightened by historical materials, to illustrate the didactical importance in designing and investigating "substantial learning environments", that engage students in the process of mathematising. *M. Helfgott* presents two examples in optics and chemical kinetics to illustrate the deep interrelation between mathematics and the physical sciences, and how rich and fruitful teaching ideas this can generate.

In the course of discussion in TSG 17 it became clear that enough has been said on a "propagandistic" level, that rhetoric has served its purpose. The presentations were all about actual implementation of the conviction that history of mathematics can enhance the learning and teaching of mathematics. To provoke discussion, the use of history of mathematics in the classroom has been likened to an appetizer, a main course or a dessert, which caters respectively to motivation, content or enrichment. It came out of the discussion that, unlike the gastronomic analogue, a more fitting way is not to regard the use of history of mathematics in the classroom as in such separate compartmentalized categories. In fact, it is even debatable whether the phrase "using history of mathematics" should be used! The word "integrating" may be better, and the word "permeating" is even better. It is important to note that the main emphasis should be placed on the part of learning and teaching of mathematics. This is not to devalue the worth of history of mathematics as a subject in itself. Far from that, history of mathematics is as worthwhile and as scholarly as any other serious academic pursuit. But the aim of this group lies with the role history of mathematics can play in mathematics education so that we stress the *historical dimension* in mathematics education. The gastronomic analogue is, however, not completely out of tune. As all great cooks love cooking but do not merely follow recipes, so should we love history of mathematics and teaching mathematics but do not merely follow recipes. We like to have more didactical material with a historical dimension available for sharing, but we do not merely wait for others to prepare such ready-made material as recipes to follow. More good cooks will generate more recipes and in turn more good cooks. More good teachers will produce more good didactical material and in turn more good teachers. In this connection it does well to mention here several helpful resources, which were mentioned during the discussion going on in TSG 17:

(i) activities of one of the oldest study group affiliated to ICMI, viz the *International Study Group on the Relations between the Histroy and Pedagogy of Mathematics* (abbreviated as the HPM Group), with its website at http://www.mathedu-jp.org/hpm/index.htm or http://www.clab.edc.uoc.gr/hpm/;

(ii) a new online magazine of the *Mathematical Association of America* called *Convergence* available at http://convergence.mathdl.org, edited by V. Katz and F. Swetz;

(iii) a recent publication in CD version from the *Mathematical Association of America* (http://www.maa.org) titled *Historical Modules for the Teaching and Learning of Mathematics*, edited by V. Katz and K.D. Michalowicz.

As a final remark, we would like to point out that, despite its importance, history of mathematics is not to be regarded as a panacea to all pedagogical issues in mathematics education, just as mathematics, though important, is not the only subject worth studying. It is the harmony of mathematics with other intellectual and cultural pursuits that makes the subject even more worth studying. In this wider context, history of mathematics has yet a more important role to play in providing a fuller education of a person.

Acknowledgement: We would like to thank all the speakers in TSG 17 for working on their presentations, including the three who did not finally provide a revised manuscript to be published in this Special Issue. *J. van Maanen*, from his rich personal experience with students and school teachers, gave a succinct but quite comprehensive account and discussion on the kind of questions coming from different quarters that seek information on history of mathematics, or ask how historical material can be integrated into the learning and teaching of mathematics. His talk would be helpful in

the setting up of an FAQ website for those who are interested in the interplay between history of mathematics and mathematics education, and we look forward to seeing it written up and published elsewhere. *O. Yevdokimov* presented some examples from an e-learning textbook on Euclidean geometry, which elaborates on historical problems and contains historical information. His presentation is contained in a published article collected in the *Proceedings of the HPM 2004: ICME 10 Satellite Meeting* (edited by F. Furinghetti, S. Kaijer, A.Vretblad, Uppsala University, 2004, pp. 457-465). *A. Garciadiego* planned to give a presentation on a more advanced example, that of the emergence of the concept of a well-ordered set in set theory, so as to illustrate how history can illuminate the context in upper-division undergraduate mathematics, but at the last minute he had to cancel his trip and did not participate in ICME 10. We would also like to thank all the other members of the Organizing Team of TSG 17 for their continuous encouragement before, during and after ICME 10, their willingness to thoroughly review the submitted papers, and their invaluable contribution in organizing a most fruitful Topic Study Group. Finally, our most sincere thanks go to the Editors of the *Mediterranean Journal for Research in Mathematics Education*, especially the Editor-in-Chief, for giving us the possibility to realize the publication of this collection of papers, which we hope will be of interest to the community of mathematics teachers, mathematics educators and mathematicians.

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History and Mathematics Education: A look around the World with Particular Reference to Italy

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Abstract[1]: *Having served for four years (2000-2004) as a chairperson of the Study Group HPM (The International Study Group on the Relations between the History and Pedagogy of Mathematics) affiliated to ICMI, I have collected a good amount of information on the role of the history of mathematics in mathematics education. As a general view I may single out two main streams in the interpretation of this role: (A) History for reflecting on the nature of mathematics as a socio-cultural process, (B) History to construct mathematical knowledge. Stream (A) includes the idea of history as a means to humanize mathematics and to promote mathematics in the classroom. Stream (B) concerns the core of the problems related to teaching/learning mathematics. Mathematics education helps to outline reasons why researchers and practitioners think that history is a useful tool in teaching and learning.*

Keywords: Construction of mathematics knowledge, history of mathematics, humanistic mathematics, mathematics education.

INTRODUCTION

The idea of considering history of mathematics in relation to mathematics education is not new. In this connection, Fried (2001) mentions the paper by Barwell (1913), but evidence of this relationship is even older, see (Cajori, 1894;

[1]This is a modified version of a paper originally published in W.-H. Horng, Y.-C. Lin, T.-C. Ning & T.-Y. Tso (editors), *Proceedings of Asia-Pacific HPM 2004 Conference: History, culture and mathematics education in the new technology era* (National Taichung Teachers College, Department of Mathematics Education, Taichung, Taiwan, 2004), 2-17.

Gebhardt, 1912; Heppel, 1893; Loria 1899; Zeuthen, 1902). Miss Barwell was teaching students of the Training Department of Alexandra College (Dublin); she writes that she had also introduced "a little of the story of mathematical growth" (p. 72) to girls aged sixteen and seventeen. In general, the authors of that period were mainly referring to teacher education[2] and, moreover, there was not research on the practical realization in school of the ideas put forwards in theory. Starting from the 1960s, studies on the relationship between history and mathematical education became more structured according to a scientific pattern; also pupils of different school levels were considered.

A landmark in the evolution of this kind of studies is the activity of the Working Group organized at the ICME-2 (1972, Exeter) on 'History and Pedagogy of Mathematics'. The work of this group was continued in the following ICME-3 (1976, Karlsruhe), and in the same year the Executive Committee of ICMI approved the affiliation of the new Study Group, under the title *The International Study Group on the Relations between the History and Pedagogy of Mathematics, cooperating with the International Commission on Mathematical Instruction*[3]. In the same year another permanent study group was set up, *The International Group for the Psychology of Mathematics Education* ("PME"). The birth of these groups provides evidence of the healthy development of mathematics education as a scientific discipline. It also shows the existence of a variety of backgrounds for mathematics education so that this discipline is appealing, but also complex and problematic.

Having served for four years (2000-2004) as a chairperson of HPM I collected a good amount of information on what is going on in this field. In particular, I have in mind the issues of the *Newsletter* of HPM (editor Peter Ransom), which in this period was regularly published three times per year both in the paper version and in the Study Group HPM's website[4].

To clarify the place that history may take in the classroom I distinguish two themes[5]:

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[2]An interesting exception to this trend of focussing on teacher education is the Italian journal addressed to students *Il Pitagora* issued from 1895 to 1918, which considered the history of mathematics as a means for attracting students of secondary school to mathematics, see (Furinghetti, 2000).

[3]At present this title is usually shortened to "HPM". For a full account of the group's birth and development see the history of HPM by Fasanelli and Fauvel (2004).

[4]http://www.mathedu-jp.org/hpm/index.htm or http://www.clab.edc.uoc.gr/hpm/

[5]Further details of this distinction are in (Furinghetti, 1997).

(A) *History for reflecting on the nature of mathematics as a socio-cultural process.*

(B) *History for constructing mathematical objects.*

Stream (A) includes the idea of history as a means to promote mathematics in the classroom in order to humanize mathematics. Stream (B) concerns the core of the problems related to teaching/learning mathematics. Other divisions of the reasons why educators are interested in the history of mathematics are illustrated in the literature, see (Fried, 2001) for one. It seems to me that, generally speaking, they have strong commonalties with my rough schematization.

In the various papers, conferences, books I came across in recent years, I grasped attempts to answer the following questions:

- For a teacher or for a student is it advisable to know the history of mathematics?

- If yes, how much history one has to know?

- And how one has to know history?

I am aware that dealing with these questions requires a distinction to be drawn between the case of teachers and that of students, but I feel that the basic issues are common in the two situations. The main point is that I see the history of mathematics as an artifact[6], which may be introduced in the classroom as a mediator in the process of teaching/learning. Following Verillon and Rabardel (1995) the artifact - i.e. the concrete object, with its physical and structural characteristics, constructed for specific uses - becomes a "tool" - i.e. it incorporates its modalities of use, as they are interpreted by users - when users are able to finalize the use for their own aims. The central question I consider in this paper is "What makes the artifact (in our case history) a tool?"

In attempting to discuss some issues raised by the previous questions, I have in mind the objections I encountered when I presented to teachers proposals of introducing history in mathematics teaching. These objections are clearly listed in (Siu, 2004) and in (Tzanakis & Arcavi, 2000): the lack of time, teachers' scant

[6]The idea of historiography as an artifact comes to me from (Eco, 1994).

knowledge, students' poor interest and so on. In a different perspective Fried (2001) also points out serious and fundamental problems in the effort of combining mathematics education and the history of mathematics. My reaction to these objections is that I consider them very realistic; the missing point is that the use of the history of mathematics is an efficient mediator of mathematical knowledge not always and everywhere, but in 'convenient situations'. In what I call 'convenient situations' the teachers involved are supposed to have the necessary preparation to carry out the historical project. In particular, they have to believe that the history of mathematics may be a 'tool' in mathematics classes.

BACKGROUND

To write this paper I have considered the studies published in mathematics education journals, such as *Educational Studies in Mathematics*, *For the Learning of Mathematics*, and the special issues devoted to the history of mathematics in mathematics teaching (*Mathematics Teacher*, *Mathematics in school*) published in the years 1999-2003. Moreover I have carefully scrutinized the studies on history, epistemology and pedagogy carried out in Italy in the same period and I have found about 50 papers, many of them authored by secondary teachers. Also I have considered some innovative experiments of mathematics teaching carried out in my country to see possible relations with historical issues. Even if my overview is not so systematic and structured as that concerning geometry presented in (Gulikers & Blom, 2001), I was able to single out some examples suitable to discuss the role of the history of mathematics in mathematics education.

I focused on the Italian situation for two main reasons. The first, obvious, is that the Italian situation is the one I know best. The second reason is that the tradition of setting mathematics education in a historical or epistemological perspective is well rooted in Italy[7]. The historian Gino Loria was a pioneer in advocating the use of history in mathematics teaching, especially in relation with teachers' training, see (Loria, 1899). He followed the idea - widespread in those years in the mathematical community - that mathematics teachers need to revisit elementary mathematics from an advanced standpoint. Felix Klein (1896), who deserves great consideration both as a mathematician and as a mathematics

[7] A trace of the interest for history is still present in Italian national official programs: the historical approach is explicitly mentioned in the premises to the list of mathematical content (Recommendations for teachers). Moreover in many in-service mathematics teacher training courses organized by the Italian Ministry of Education history is contemplated as a part of the program.

educator of teachers, indicated the history of mathematics as an efficient means to perform this revisiting. In those years (the turn of the century) there was an active debate on how to apply history in the classroom. Under the influence of the recapitulation law - i.e. ontogeny recapitulates phylogeny - stated by the German biologist Ernst Haeckel in 1874, it was advocated that teaching sequences in mathematics should follow human development in mathematics. This theory had its apotheosis in the famous book by Benchara Branford (1921). This work was known in Italy: in the library that is Loria's bequest to the University of Genoa there is a copy of the 1921 edition with the author's dedication to Loria. Nevertheless, neither Loria, nor other Italian authors make explicit reference to the recapitulation law when they advocate the resort to history in mathematics teaching.

Starting from the turn of the century in Italy, school mathematics was strongly influenced by the movement of rigor and the discussion on the foundations of mathematics, as well as, by the rising of logic as a branch of mathematics. In those days national programs and several new textbooks were influenced by this new fashion and interesting discussions about pros and cons of a strictly rigorous approach in mathematics teaching can be found in mathematics journals for teachers of that period. A good balance of different orientations was reached by Federigo Enriques. He was not only a great mathematician leading the Italian geometric school, but also a leader in the discussion on mathematics education. He edited an important book for teachers *Questioni riguardanti la geometria elementare* (Zanichelli, Bologna, 1900), (soon translated into German on Klein's initiative). With the title *Questioni riguardanti le matematiche elementari* this work had many editions (first 1912, second 1914, third 1924-27) and contributed to the mathematical culture of generations of teachers. In his approach to mathematics education problems, Enriques stressed the importance of epistemology and history. This approach, together with his interest in problems of knowledge, is already evidenced in the talk delivered in the first important international meeting of the newborn ICMI, held in Milan (Enriques, 1911). This orientation of combining history and epistemology characterizes a part of the Italian studies in mathematics education, as shown in two surveys (Bagni, Furinghetti, & Spagnolo, 2004; Speranza & Grugnetti, 1996).

'HUMANIZING MATHEMATICS' AND HISTORY

Very often the declared motivation supporting the use of history in mathematics teaching is the wish to 'humanize mathematics'. The expression 'humanize mathematics' has not a clear meaning and its explanation involves not only mathematicians[8], but also philosophers. In the paper (Tymoczko, 1994) - which is the text of the plenary talk delivered at ICME-7 (Québec) - the fact is pointed out that in the past there has been little communication between the world of philosophy of mathematics and that of mathematics education. As two exceptions may be considered those of Wittgenstein (who, indeed, started his career as a school teacher) and Polya. Tymoczko (1986 and 1994) considers among the various aspects characterizing the mathematical activity, those concerning teaching. He gives interesting insights, but his approach is affected by his philosophical background. He is outside research in mathematics education and has not the feeling of what actually happens in the classroom (if I remember well, this was one of the objections made to him in the discussion after his intervention at ICME-7).

In his talk, Tymoczko firstly attempted to discuss some mistakes of philosophers. They were wrong in neglecting the real world as a basis of mathematics, because this discipline is an integral part of the 'common sense.' Mathematics has meaning as activity of a community. Seeing pure mathematics as the essence of mathematics has distorted the philosophical perspective on it. Tymoczko (1994) states that we may think of a civilization without pure mathematics (he mentions Egyptians and Babylonians) and thus he reverses the traditional view of the question: he is not interested in investigating how from pure mathematics utilitarian (distinguished by him from applied) mathematics follows but, rather, how from utilitarian mathematics pure mathematics is generated. The crucial point of the philosophical discussion is whether mathematical objects exist. For those who face the problem by thinking of pure mathematics as independent from human activities the answer is problematic; for those who look at mathematics as a part of human activity the answer is affirmative (mathematical objects exist as dogs, flowers,... exist.)

As for the world of education - which for Tymoczko belongs to the domain of applied mathematics, in contrast with that of pure mathematics as thought by philosophers - the important point for Tymoczko concerns the concept of *humanistic* mathematics. To the question "What made mathematics one of the humanities?"[9] he answers:

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[8]In his plenary talk at the International Mathematical Congress of Kyoto, Manin (1991) stated that he wanted "to restore a certain balance between the technological and the humanitarian sides of mathematics." (p. 1666)

[9]He is referring to the project launched by Alvin White (Harvey Mudd College, USA) who wanted mathematics taught in a human way.

> Certainly not the mere fact that humans did it? Human do science too. [...] Pure mathematics is ultimately humanistic mathematics, one of the humanities, because it is an intellectual discipline with a human perspective and a history that matters. There is no answer to the question: What is important in mathematics, once for all? We can only ask what is important in mathematics to human beings, with given abilities and limitations at a given point in their mathematical development." (Tymoczko, 1994, p. 334)

He acknowledges that philosophers were wrong in considering only pure mathematics, but at the same time he ascribes to educators the opposite mistake of concentrating too much on *useful* mathematics. This approach does not introduce students to the mathematical discipline or, better, to humanistic mathematics; his idea is that "To introduce students to humanistic mathematics is to introduce them to a human adventure, an adventure that humans have actually partaken of in history" (*ibid*em, p. 335). Tymoczko gives an example of classroom activity aimed at making mathematics a humanistic discipline, that is to present the historical development of third degree equations. Tymoczko's idea of linking humanistic mathematics to history is widespread. Maybe, a non-declared intention of Tymoczko is an interpretation of the role of history as a means to introduce students to hermeneutics. This is another widespread idea: I found it in the presentation notes of the Italian mathematics journal for students *Il Pitagora* sent to potential subscribers in 1895, see (Furinghetti, 2000).

Coming to the issue discussed in this paper about the use of history in mathematics teaching, I have noticed that humanizing mathematics is often associated with anecdotes, stories, and vignettes. Not only schoolteachers but also lecturers in graduate schools and universities use these devices. This fact emerged from the answers in the Mailing List HISTORIA MATHEMATICA to a request posted by a doctoral student who was developing his dissertation on the use and efficacy of historical vignettes in mathematics teaching. A few university lecturers answered that they introduce activities such as telling anecdotes, showing stamps representing mathematical subjects, and so on. Their justifications resound those offered by secondary teachers performing the same activities: they rely mainly on the affective factors, which intervene in teaching/learning process. One of these factors is the personal pleasure in dealing with history. I have noticed that teachers are satisfied and feel they have reached the aim of humanizing their classroom; see (Bidwell, 1993).

Other teachers act differently. For example, Percival (2001) attempts to show the "humanist, 'human-made' side of arithmetic" (p. 21) to her students aged twelve and thirteen. This is realized in the classroom by linking history of mathematics and mathematics itself through a social approach and multicultural co-operation. The mathematical subject dealt with is elementary arithmetic. In other papers this subject has been treated through ancient documents, but in this experiment we have an artefactual approach which appears to be extremely suitable to the author's purpose: students make their own constructions of objects (such as Babylonian tablets) and documents imitating those studied, and use ancient calculating devices, albeit in modern reconstruction. Thus concrete activity parallels conceptual construction of the concept of numerical systems and their manipulation.

A Debated Question: Is It Possible To Teach The History Of Mathematics In Secondary School?

In the international scene there is little evidence of activities in which the history of mathematics *per se* is introduced in mathematics teaching. Objection 1 ("I have no time for it in class!"), reported by Siu (2004) in his list, epitomizes the cause of this trend. Because this widespread absence I have found very interesting two experiments carried out in Italy, where a few elements of the history of mathematics *per se* were taught to secondary students. Both experiments were planned by the teachers developing them in class.

The first experiment is described in (Testa, 2001). Students of an Italian high school with humanistic orientation (*Liceo classico*) were engaged in a historical investigation on the following subjects: Archimedes' life and death, contributions by the ancients to the study of optics, Foucault's pendulum. They were provided by the teacher with suitable addresses of websites, secondary sources, primary sources. The teacher tried to encourage students to work autonomously; he only guided them in the Internet surfing. Students worked collaboratively in groups. Secondary sources (popularizing books, treatises on history) were used as a hint for outlining the content of the materials produced, but students were encouraged to read historical-scientific documents (mainly some passages from Plutarcus) in the original version. This work can be considered an example of the use of the history of mathematics to foster students' acquisition of competencies and skills useful to other school disciplines, e.g. historical method, writing narratives, selecting information.

The second experiment concerns a project aimed at delivering elements of the history of mathematics inserted in the mathematics course to secondary students, see (Demattè, 2004). The assumption of this project is that to know history is not just to have mere information about facts, but rather, it is to have a frame in which the isolated pieces of mathematics learnt in school acquire a cultural meaning. I mean that school mathematics is often perceived by students as a set of notions which are scantly linked together: the history of mathematics provides the links. In principle all researchers agree with this assumption, but in practice there are few realizations of it. In the experiment about which I am reporting, the teacher taught grade 9-10 students a few elements of the history of mathematics, i.e.:

- notions about the work of famous mathematicians, about moments of the mathematical development, about the links of mathematics with civilizations, culture, and other sciences

- interpretation of mathematical texts through the use of competencies acquired in the mathematics course and in other courses. For example, a linguistic analysis was performed to understand old documents.

About two months after the end of the history course, a questionnaire was administered to the students with the aim of verifying what they kept of and how they perceived the new subject. This questionnaire (available at http://www.iprase.tn.it) is made of 35 closed items. It may be fruitfully used in teacher training courses as a background to discuss the following questions:

- Which educational goals does the history of mathematics allow to reach?

- Is it realistic to design teaching sequences, which encompass the knowledge required for answering the items of the questionnaire?

- Which competencies are stimulated by the learning of elements of the history of mathematics?

- May history change students' attitude towards mathematics?

When I had the occasion to discuss these questions with teachers I noticed that they are keen to accept the mere teaching of history as a useful activity in the classroom according to their idea of what teaching mathematics means. If for them teaching mathematics means to transmit notions and routine skills, the mere teaching of history is rejected as a waste of time. On the contrary, the activity at issue appears fruitful if teaching mathematics is conceived of as an opportunity to take part in a global educational project in which students acquire the skill of working collaboratively in groups, of pursuing given goals by exploiting the available means, of learning to process information.

In the background of both experiments there is the assumption that history can actually contribute to changing students' attitude towards mathematics, see (Charbonneau, 2002). The reader may observe that I do not argue that the activities I have mentioned may contribute to humanizing mathematics. This was not in the declared objectives of the teachers involved in both experiments; their objectives were mainly those of integrating mathematics teaching in the global teaching of school disciplines by stressing the common objectives and skills. Nevertheless, we may say that they have made mathematics a humanistic discipline.

CONSTRUCTING MATHEMATICAL OBJECTS

I consider important the two questions examined before (humanizing mathematics, the history of mathematics as a school subject), nevertheless I am aware that the core of the discussion about the use of history in mathematics teaching is linked to its contribution to the construction of mathematical objects in class. Gray and Tall (2001) consider different types of mathematical objects:

> One is the *embodied object*, as in geometry and graphs that begin with physical foundations and steadily develop more abstract mental pictures through the subtle hierarchical use of language. Another is the *symbolic procept* which acts seamlessly to switch from a 'mental concept to manipulate' to an often unconscious 'process to carry out' using an appropriate cognitive algorithm. The third is an *axiomatic concept* in advanced mathematical thinking were verbal/symbolical axioms are used as a basis for a logically constructed theory. (Here the fourth type of concept might occur by distinguishing between those concepts evolving from embodied objects and those from encapsulated objects.) (p. 70)[10]

The adjective embodied evokes paradigms linked to embodied cognition. Núñez, Edwards and Matos (1999, pp. 48-49) write that: "Rather than positing a passive observer taking in a pre-determined reality, these paradigms hold that reality is constructed by the observer, based on non-arbitrary culturally determined forms of sense-making which are ultimately grounded in bodily experience." Even if the term 'embodiment' is used in a number of different ways within contemporary cognitive science, all researchers "share a focus on the intimate relation between cognition, mind, and living bodily experience in the world, that is, on the ways in which complex adaptive behavior emerges from physical experience in biologically-constrained systems." (p. 49) I deem that history, for the nature of the problems considered and the way their solution is approached, offers the environment where objects may be embodied (I use this term in a broad sense.)

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[10]This classification is discussed in (Inglis, 2003) and (Tall, 2004.)

An example of students' bodily experience is the measurement of inaccessible heights starting from historical documents, see (Gulikers, 2002-03) and (Foschi, 2003). The first paper refers to a project aimed at teaching similarity. To this purpose the author uses a Dutch book by Cardinael[11] (1610) in which a method to measure the height of a tower with the aid of a mirror is described. Another method to measure inaccessible heights is based on the old instrument known as the cross staff described in the geometry book of Pierre de la Ramée translated into Dutch in 1622[12]. In the paper the original texts of some problems are reported as well as the pupils' assignments.

The second paper deals with the theorem 19 in Euclid's "Optics" and with some related topics, such as the properties of similar right-angled triangles and the propagation and reflection of light. The central part of this paper originates from classroom lessons. In that experiment original sources were used (passages from *Ludi matematici* by Leon Battista Alberti, *Quaderni della fenice*, Edizioni Guanda, Milano, 1980). According to the author the interrelations between physical ideas and mathematical theoretical arguments were grasped by the students involved in the experiment.

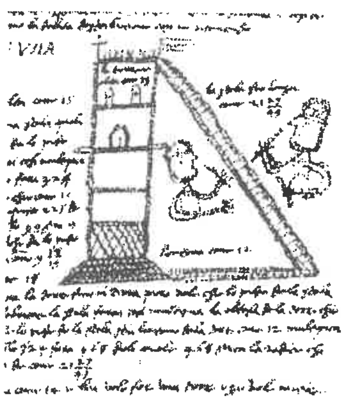

Figure 1. One tower and one ladder

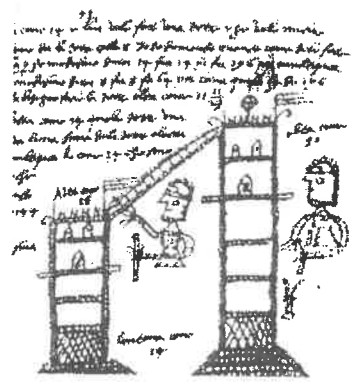

Figure 2. Two towers and one ladder

[11]Cardinael (S. Hansz. van Harlinghen), c. 1610., *Hondert geometrische questien en hare solutien*. Amsterdam.
[12]Ramus, P. 1622, *Meetkonst in xxvii boeken vervat*. Amsterdam.

The problems reported in the two papers about measuring inaccessible heights remind me of the problems of ladders and towers largely used in the Middle Age and later in the treatises on practical geometry. I take some nice examples used to introduce students to problem posing from an Italian book dating back to the end of the Sixteen Century[13]. The book in question is not a real mathematical text; it is rather a manual for solving practical problems of farmers, coopers, masons, and craftsmen. No general methods are given, but only methods and tricks conceived *ad hoc* for each different kind of problems. In Figure 1 and Figure 2 the images accompanying some problems of towers and ladders are reported. The text of the problems relating to Figure 1 is interesting and funny:

- Problem 1: Given a tower and a ladder, at which distance has the bottom of the ladder to be put so that the top of the tower is reached?

- Problem 2: Given a tower and its distance from the base of the tower, which is the length of the ladder that allows reaching the top of the tower?

- Problem 3: Given a ladder and its distance from the base of the tower, which is the height of the tower so that its top is reached with a given ladder?

Through these naïve problems young students may learn to distinguish between problems in context, which have a meaning, and those, which have not: the third problem has no meaning since nobody would build a tower to match the length of the ladder and the distance from the base. In the same spirit one may analyze the problem illustrated in Figure 3 that deals with the rule for finding the perimeter of a triangle. Unfortunately the sides measure 7, 8 and 15 so that the triangle in question does not exist.

[13]Abate Giovanni Agostino, 1992., *Giometria de figure quadre*. Savona: M. Sabatelli. - Printed version edited by G. Farris made on the Sixteen Century manuscript. -

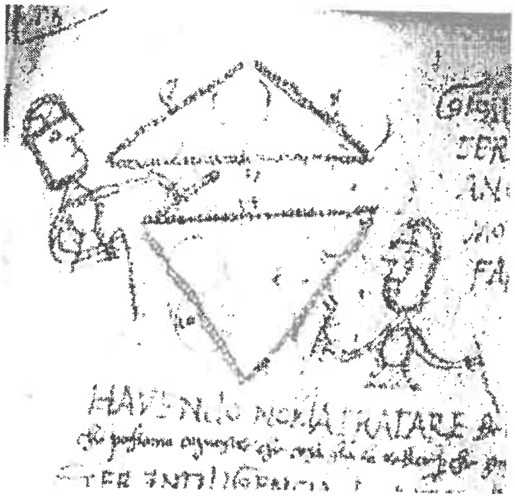

Figure 3. To find the perimeter of a triangle

The passage from the world of *embodied objects* to the world of *symbolic procepts* is fostered by metaphors. There are nice examples of this in history. As reported by Lǐ and Dù (1987), in the Pythagoras or Gōugǔ theorem the vertical gauge is called 'Gǔ', meaning a measure, or scale. The shadow of the gauge cast by the sun on a plane is called 'Gōu' and the hypotenuse is known as 'Xián', a string of a bow. So the right-angled triangle is called 'Gōugǔ shape'. We see that metaphors and language interplay in shaping the meaning of mathematical objects. This Chinese metaphor resembles the metaphors described in (Lakoff & Núñez, 2000), in particular as it does not reside in words; it is a matter of thought.

As a final example I report a case in which the construction of a mathematical object (the concept of maximum) in the classroom occurs through the embodiment of object (via perception on the computer screen) and metaphorical language. I shall link this case with a passage by Fermat[14].

In the laboratory of mathematics, in which students use the dynamic geometric software Cabri, the following problem was assigned: "How does the area of a rectangle vary when the perimeter is constant? Take, for example, 12 as the value of the perimeter." Grade 10 students engaged in the experiment; they were rather good in using Cabri, but did not know calculus. In their classroom, exploration was regularly used as a method of approaching conjectures and proof; the work was

[14]For a full account of the experiment see (Paola, 2004). It is remarkable that the experiment was set in a class of computer science, without any reference to history.

carried out in groups. They began by drawing with Cabri a segment line *HK* of length 6 (half perimeter) and took a point *P* on it (*HP* and *PK* represent the lengths of the sides of a rectangle of perimeter 12.) After that, they took the segment *AP* congruent to *HP* on a straight line, and the segment *PB* congruent to *KP* on the line perpendicular to *HP* at *P*. The figure was completed to obtain a rectangle of perimeter 12. In this first phase the students saw ('perceived') that when *P* moves on the segment *HK* the rectangle changes. Afterwards they perceived that the area depends on the length of *AP*. This dependence could be established in a more precise way by the use of a spreadsheet. Some students were able to produce conjectures on the trend of the variation. At the end the rule was shown through the command 'trace' of Cabri, see Figure 4.

Thus the students were led to conclude that the greatest area was reached when the two sides are equal, i.e. the rectangle is a square. For them it was difficult to give an explanation of this fact. But a group of students tried the following informal explanation:

> Look at that, teacher: if I point in the middle and after I shift a little to the left and a little to the right, the area decreases.

This is not a proof, and not even an explanation, but this sentence gave the teacher a hint to explain how to pass from the pure perceptive phase to a phase of using symbols as a tool for solving problems. In this phase the teacher worked in the world of symbolic procepts. He said:

> How can we translate into symbols - which means to work in an effective way - the expression "to shift a little to the left and a little to the right from the middle point"? The answer is $3 - x$ and $3 + x$. Then the area of the rectangle is $(3 - x)(3 + x)$, that is $9 - x^2$.

At this point the students acknowledged easily that the greatest area is reached when $x = 0$. The metaphor of "shifting to the left and to the right" is a bridge between perceptive situations and symbolic conceptual situations.

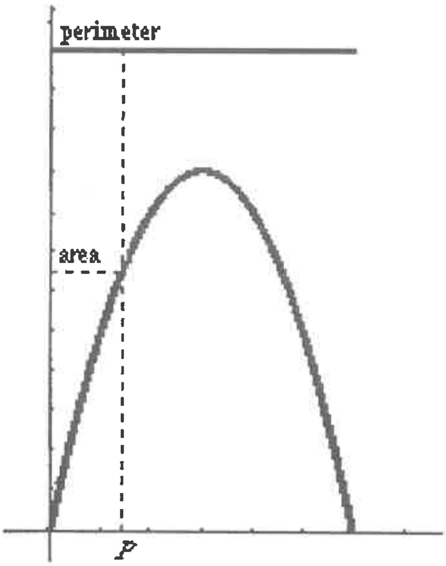

Figure 4. Figure made with Cabri by students for the problem of finding the greatest area

In considering this episode from the historical perspective taken in this paper we note a fascinating resemblance with the following passage from *Methodus ad disquirendam maximam et minimam*, in which Fermat sets his method for evaluating maxima and minima[15]:

> The whole theory of evaluation of maxima and minima presupposes two unknown quantities and the following rule:
>
> Let a be any unknown of the problem (which is in one, two, or three dimensions, depending on the formulation of the problem). Let us indicate the maximum or minimum by a in terms which could be of any degree. We shall now replace the original unknown a by $a + e$ and we shall express thus the maximum or minimum quantity in terms of a and e involving any degree. We shall adequate [*adégaler*], to use Diophantus' term, the two expressions of the maximum or minimum quantity and we shall take out their common terms. Now it turns out that both sides will contain terms in e or its powers. We shall divide all terms by e, or by a higher power of e, so that e will be completely removed from at least one of the terms. We suppress then all the terms in which e or one of its powers still appear, and we shall equate the others; or, if one of the expressions vanishes, we shall equate, which is the same

[15] I report Fermat's passages as reprinted in (Fauvel, 1990, pp. 28-30).

thing, the positive and negative terms. The solution of this last equation will yield the value of a, which will lead to the maximum or minimum, by using again the original expression.

Here is an example:

To divide the segment AC at E so that AE x EC may be a maximum.

We write $AC = b$; let a be one of the segments, so that the other will be $b - a$, and the product, the maximum of which is to be found, will be $ba - a^2$. Let now $a + e$ be the first segment of b; the second will be $b - a - e$, and the product of the segments, $ba - a^2 + be - 2ae - c^2$; this must be adequated with the preceding: $ba - a^2$. Suppressing common terms: $be \sim 2ae + e^2$. [Dividing all terms: $b \sim 2a + e$][16]. Suppressing e: $b = 2a$. To solve the problem we must consequently take the half of b.

We can hardly expect a more general method.

Final Remarks

In the period 2000-2004 the world production in the field of history and pedagogy of mathematics was - obviously - much richer than I was able to illustrate. In particular, having focused on the pedagogical side, I missed out the issue of discussing theory and methods in history of mathematics, see (Rubin, 2001), which is the other side of the coin. In my overview my concern has been to go on in my search for the pedagogical meaning when linking history and mathematics education.

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[16]The sentence in square brackets is not in (Fauvel, 1990): it is my translation of the sentence "*et, omnibus per E divisis, B adaequabitur Abis + E*" (Fermat, 1891, tome.I, p.134). The same sentence is translated as "*Divisant tous les termes: b ~ 2a + e*" in the French version (Fermat, 1896, tome III, p. 122).

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Prime Numbers are Infinitely Many: Four Proofs from History for Mathematics Education

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Abstract: *In this paper we discuss some epistemological issues related to the historical analysis of a mathematical topic in order to achieve an effective and correct use of historical data in Mathematics Education. In particular we present G. Brousseau's epistemological obstacles perspective and L. Radford's socio-cultural approach, and underline the primary importance of the correct social and cultural contextualization of historical references. Finally, we propose the comparison of some different strategies used by mathematicians in different historical periods in order to prove the infinity of prime numbers and we discuss some educational consequences with reference to the theoretical frameworks presented.*

Keywords: epistemological assumptions, history of mathematics, infinity, prime numbers, teachers education.

HISTORY AND DIDACTICS: THEORETICAL FRAMEWORKS

One of the pedagogic uses of the History is to link psychological learning processes with historical and epistemological issues (Radford et al. 2000); it is a very important topic of research in Mathematics Education and the debate about it is open (see for instance: Heiede, 1996).

With regard to the interaction between History and Didactics, different levels can be considered: the presentation of anecdotes can be useful in order to strengthen pupils' conviction (as noted in: Radford, 1997); higher levels bring out multidisciplinary relationships and metacognitive possibilities (Furinghetti & Somaglia, 1997). These levels do not reflect just practical educational issues, but also imply important epistemological assumptions: for instance, the selection of historical data is epistemologically relevant, and several problems are connected with their interpretation, these having to do with our cultural institutions and beliefs (Gadamer, 1975).

From the historical point of view, frequently a new concept is encountered by Mathematicians in operative steps, like problem solving or proving activities; this new concept will then go on to be framed theoretically many years or centuries later and will finally assume the features that we (nowadays!) consider typical of real mathematical objects (Giusti, 1999). A similar evolution can be pointed out in the educational field: frequently the first contact with a new notion takes place in operative steps. A. Sfard notes that the development of "abstract mathematical objects" is the product of the comprehension of processes (Sfard, 1991; with reference to the notion of function see for instance: Slavit, 1997).

A parallelism between historical development and cognitive growth brings us to consider relevant epistemological problems: is it correct to present History as a path that leads to modern theories? What is the role, played by social and cultural factors that influenced historical periods? Knowledge cannot be considered in absolute terms: as we shall see, it must be understood in terms of cultural institutions (as underlined in: Radford, 1997).

Let us briefly present some theoretical frameworks.

- According to G. Brousseau's "epistemological obstacles" perspective (in: Brousseau, 1983, the author quotes: Bachelard, 1938 and Piaget, 1975), a goal of historical study is finding systems of constraints (*situations fondamentales*) that must be pointed out and studied in order to understand existing knowledge, whose discovery is connected to their resolution (Radford, Boero & Vasco 2000, p. 163). Obstacles are clearly subdivided into epistemological, ontogenetic, didactic and cultural ones (Brousseau, 1983 and 1989).

Such subdivision points out that the sphere of the knowledge can be considered in isolation from other spheres. This perspective is characterised by other epistemological assumptions (Radford, 1997): the reappearance in teaching-learning processes, in the present, of the same obstacles encountered by mathematicians in the past; and the exclusive, isolated approach of the pupil to knowledge (Brousseau, 1983).

Epistemological assumptions needed by the aforementioned perspective are relevant. Let us underline that it is impossible, nowadays, to see historical events without the influence of our modern conceptions (Gadamer, 1975; see moreover:

Brousseau, 1983, pp. 191-192); so we are forced to consider the following dilemma: should we abandon historical references and their educational uses, in order to avoid polluting them with our conceptions of the past? Otherwise we must accept our modern point of view and take into account that, when we look at the past, we connect two cultures that are "different [but] they are not incommensurable" (Radford, Boero & Vasco, 2000, p. 165; see moreover: Furinghetti & Radford, 2002).

- According to L. Radford's socio-cultural perspective, knowledge is linked to activities of individuals and, as we noted above, this is closely related to cultural institutions (Radford, 1997); so knowledge is not built individually, but within a wider social context (Radford, Boero & Vasco, 2000, p. 164). The role played by History must be interpreted with reference to the different socio-cultural situations considered and moreover gives us the opportunity for a deep critical study of the historical periods.[1]

However, in our opinion, there is no contradiction between Brousseau's and Radford's perspectives. Their aims are clearly different. As a matter of fact, Brousseau, too, underlines the importance of historical contextualization; nevertheless his position is presented with analytical and classificatory purposes, while according to Radford knowledge cannot be separated in pure knowledge and sociological one: knowledge is social through and through (Bagni & D'Amore, preprint).

PRIME NUMBERS ARE INFINITELY MANY

The comparison of some different strategies used by mathematicians in different historical periods in order to prove a theorem can be interesting with reference to the theoretical frameworks previously outlined (Dhombres, 1993). We shall consider Proposition IX-20 of Euclid's *Elements* which states that prime numbers are infinitely many (the original statement is in a potential sense: Ribenboim, 1980, p. 3):

Prime numbers are more than any assigned multitude of prime numbers

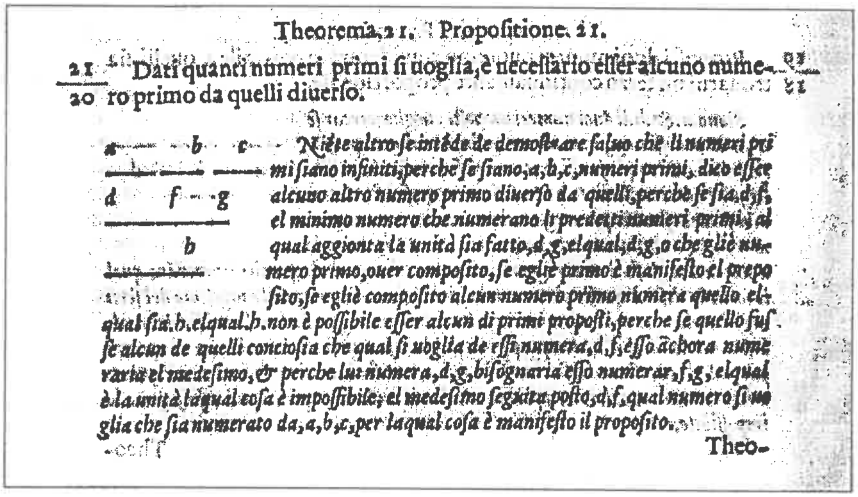
Here we consider four proofs of this celebrated theorem (there are a lot of different possibilities, so our choice is epistemologically relevant, see for instance:

[1]Another important approach is P. Boero's "voices and echoes" perspective (see for instance: Boero et al. 1997 and 1998).

Ribenboim, 1980, Aigner & Ziegler, 1998, Bagni, preprint), by Euclid, Euler, Erdös and Fürstenberg:

I. ***Euclid*: 300 B.C.** (see for instance the classical editions: N. Tartaglia, 1569, p. 171; F. Commandino, 1619, p. 118).

Let A, B, C be prime numbers. We are going to prove that there are more primes than A, B, C. Take the least common multiple D of A, B, C and add the unit to D. Then the number E = D+1 is either prime or not. Let E be prime. Then the primes A, B, C, and E have been found: they are more than A, B, C. Next, let E not be prime. Therefore it is divisible by some prime number (according to *Elements* VII, 31). Let it be divisible by the prime number F. Let us prove that F is not the same with any of the numbers A, B, C. If possible, let it be so. Now A, B, C divide D, therefore F, too, divides D. But it also divides E. Therefore F divides the remainder, the unit: this is absurd. Therefore F is not the same as any one of the given numbers A, B, C and by hypothesis F is prime. So the primes A, B, C, F have been found: they are more than the assigned multitude of prime numbers A, B, C.


Theorema.21. Propositione.21.

21 Dati quanti numeri primi si uoglia, è necessario esser alcuno nume-
20 ro primo da quelli diuerso.

a b c d f g h

Niete altro se intēde de demostrare falso che li numeri primi siano infiniti, perche se siano, a, b, c, numeri primi, dico esser alcuno altro numero primo diuerso da quelli, perche se sia, d, f, el minimo numero che numerano li predetti numeri primi, al qual aggionta la unità sia fatto, d, g, elqual, d, g, o che gliè numero primo, ouer composito, se egliè primo è manifesto el proposito, se egliè composito alcun numero primo numera quello, elqual sia, h, elqual h, non è possibile esser alcun di primi proposti, perche se quello fusse alcun de quelli conciosia che qual si uoglia de essi numera, d, f, esso āchora numeraria el medesimo, & perche lui numera, d, g, bisognaria esso numerar, f, g, elqual è la unità laqual cosa è impossibile, el medesimo seguita posto, d, f, qual numero si uoglia che sia numerato da, a, b, c, per laqual cosa è manifesto il proposito.

Theo-

Figure 1

Figure 1: In this edition of Euclid's *Elements* (Tartaglia, 1569, p. 171), the visual representation of numbers can be referred to Greek *Geometric Algebra*; the proof is expressed in a verbal register (the register available at the time; see more in Barbin, 1994).

II. ***Leonhard Euler*: 1737 and 1748** (Ribenboim, 1980, pp. 7-8 and 155-157).

Let us consider, since $|x| < 1$: $\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n$. By putting either $x = \frac{1}{2}$ or $x = \frac{1}{3}$ we have:

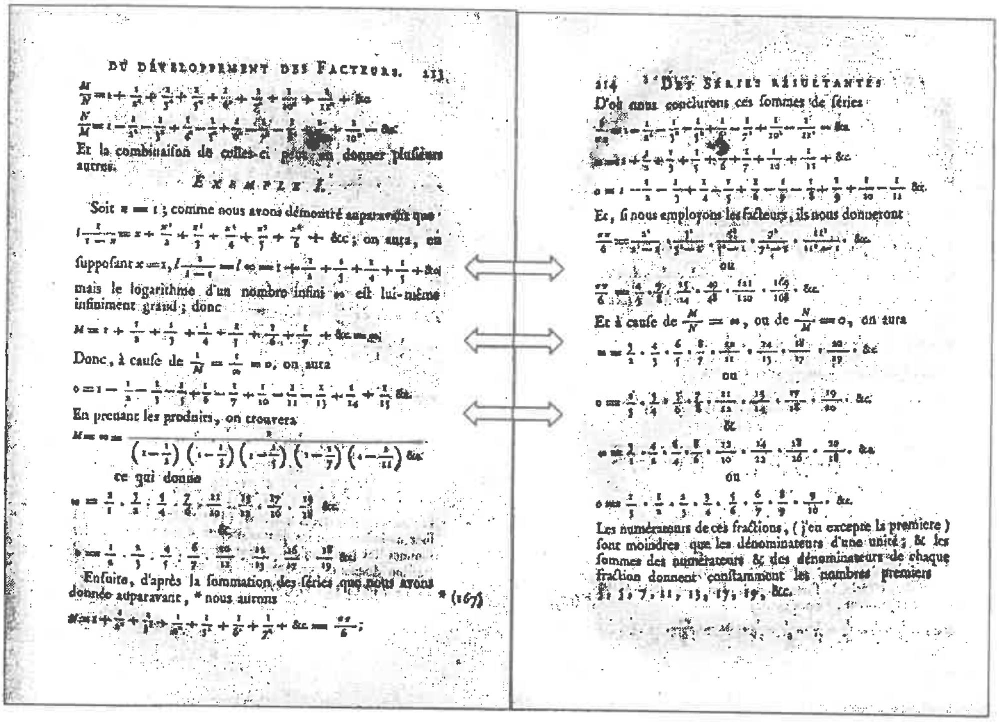
$\frac{1}{1-\frac{1}{2}} = \sum_{\alpha=0}^{+\infty} \frac{1}{2^\alpha}$ and $\frac{1}{1-\frac{1}{3}} = \sum_{\beta=0}^{+\infty} \frac{1}{3^\beta}$ so we can write:

$$\frac{1}{1-\frac{1}{2}} \cdot \frac{1}{1-\frac{1}{3}} = \sum_{\alpha,\beta}^{0,+\infty} \frac{1}{2^\alpha} \frac{1}{3^\beta}$$

On the right we have 1 (by $\alpha = \beta = 0$) and the inverses of positive integers having only prime factors 2, 3. If prime numbers were finitely many, $p_1, p_2, \ldots, p_m$:

$$\frac{1}{1-\frac{1}{p_1}} \cdot \frac{1}{1-\frac{1}{p_2}} \cdot \ldots \cdot \frac{1}{1-\frac{1}{p_n}} = \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n}^{0,+\infty} \frac{1}{p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}}$$

where on the right we have the *harmonic series*. But the quantity on the left would be finite while the harmonic series diverges: this is absurd.



DU DÉVELOPPEMENT DES FACTEURS. 213

Et la combinaison de celles-ci peut en donner plusieurs autres.

EXEMPLE I.

Donc, à cause de $\frac{1}{N} = \frac{1}{\infty} = 0$, on aura

En prenant les produits, on trouvera

ce qui donne

Ensuite, d'après la sommation des séries que nous avons donnée auparavant, * nous aurons

214 DES SÉRIES RÉSULTANTES

D'où nous conclurons ces sommes de séries

Et, si nous employons les facteurs, ils nous donneront

Les numérateurs de ces fractions, (j'en excepte la premiere) sont moindres que les dénominateurs d'une unité; & les sommes des numérateurs & des dénominateurs de chaque fraction donnent constamment les nombres premiers 3, 5, 7, 11, 13, 17, 19, &c.

Figure 2

Figure 2: Some of Euler's notations and procedures (Euler, 1796, I, pp. 213-214) would not be considered "rigorous" by our modern standards; but formal correctness must be investigated in its own conceptual context (Radford, 1997)

Moreover Euler proved that the series $\Sigma 1/p$, p being primes, diverges (this proof can be seen in: Tenenbaum & Mendès France, 1997, pp. 23-24; Bagni, preprint).

III. ***Paul Erdös*: 1938** (Erdös, 1938; Aigner & Ziegler, 1998, p. 6).

Erdös, too, proved that the series $\Sigma 1/p$, p being prime numbers, diverges. Let $p_1 = 2 < p_2 = 3 < p_3 < \ldots$ be the prime numbers (in increasing order).

If the series $\Sigma 1/p$ converges, then there would be a positive integer k such that:

$$\sum_{i\geq k+1}\frac{1}{p_i}<\frac{1}{2}.$$

Let us call $p_1, \ldots, p_k$ *small primes* and p_{k+1}, p_{k+2} *great primes*.

Let N be any positive integer; we can write: $\sum_{i\geq k+1}\frac{N}{p_i}<\frac{N}{2}$.

Let N_b be the number of the positive integers $n\leq N$ divisible for (at least) one great prime and let N_s be the number of the positive integers $n\leq N$ divisible only for small primes. We shall prove that there is N such that $N_b+N_s < N$ and this is absurd (in fact: $N_b+N_s = N$).

Let $\left\lfloor \frac{N}{p_i} \right\rfloor$ be the number of positive integers $n\leq N$ that are multiples of p_i.

From $\sum_{i\geq k+1}\frac{N}{p_i}<\frac{N}{2}$ it follows: $N_b \leq \sum_{i\geq k+1}\left\lfloor \frac{N}{p_i} \right\rfloor < \frac{N}{2}$.

With regard to N_s, above we underlined that every positive integer n can be written in a unique way as the product of a square-free number q and of m^2; let us write every $n\leq N$ having only small prime divisors as $n = a_n b_n^2$, a_n being square-free. So every a_n is a product of different small primes and there are exactly 2^k different square-free parts. Moreover, since $b_n \leq \sqrt{n} \leq \sqrt{N}$, there are at most $\sqrt{N}$ square parts, so: $N_s \leq 2^k\sqrt{N}$.

Being $N_b \leq \sum_{i\geq k+1}\left\lfloor \frac{N}{p_i} \right\rfloor < \frac{N}{2}$ true for every N, in order to achieve the final *reductio ad absurdum* we have to find a number N such that $2^k\sqrt{N} \leq \frac{N}{2}$ i.e. $2^{k+1}\sqrt{N} \leq N$; it is $N = 2^{2k+2}$ so with reference to this N we would have: $N_b+N_s < N$.

IV. ***Harry Fürstenberg*: 1955** (Fürstenberg, 1955; Ribenboim, 1980, pp. 11; Aigner & Ziegler, 1998, p. 5; Fürstenberg's ideas are taken up again in: Golomb, 1959).

Let **Z** be the set of integers, $a \in \mathbf{Z}$, $b \in \mathbf{Z}$, $b>0$ and: $N_{a,b} = \{a+k \cdot b: k \in \mathbf{Z}\}$.

We shall call the set A *open* if A is $\varnothing$ or if for every $a \in A$ there is a positive integer b such that $N_{a,b}$ is a subset of A; it is well known that every union of open sets is open.

If A_1, A_2 are open, $a \in A_1 \cap A_2$, $N_{a,b_1} \subseteq A_1, N_{a,b_2} \subseteq A_2$, so: $a \in N_{a,b_1 b_2} \subseteq A_1 \cap A_2$: so every finite intersection of open sets is open.

Therefore the described family of open sets induces a topology in **Z**.

Let us note that every non-empty open set is infinite. Moreover every $N_{a,b}$ is closed, since: $N_{a,b} = \mathbf{Z} \setminus \bigcup_{i=1}^{b-1} N_{a+i,b}$ so $N_{a,b}$ is the complement of an open set.

Every integer different from 1 and -1 has at least a prime divisor p so it belongs to $N_{0,p}$; therefore: $\mathbf{Z} \setminus \{1, -1\} = \bigcup_{p \in \mathbf{P}} N_{0,p}$, P being the set of prime numbers.

If **P** were a finite set, then $\bigcup_{p \in \mathbf{P}} N_{0,p}$ would be a finite union of closed sets, and so closed; hence $\{1, -1\}$ would be open.

Clearly this is absurd, because we stated that any non-empty open set is infinite.

A Comparison Of The Quoted Proofs

First of all, let us underline that all these arguments are *proofs*, with the usual meaning nowadays ascribed to this word. Euclidean *Elements*, for instance, are placed after the passage from the empirical Greek Mathematics to the deductive Mathematics, in a socio-cultural context based upon the distinction between real knowledge and opinions (drawn by Parmenides; see: Szabó, 1978) and a social intellectual habit consisting of a particular style of argumentation (Radford, 1996 and 1997). As a matter of fact, as far as Euclid's proof is concerned, the use of the *reductio ad absurdum* can be related with the "Being/non-Being" ontological structure of the considered period (Radford, 2003, p. 70; with reference to Euclid's proof see moreover: Bernardi, 2002, p. 208; the revised version by E.E. Kummer, presented in 1878, is discussed in: Bagni, preprint). In Table 1 we summarize some features of the quoted proofs.

Author and date	**Proved statement**	**Logical structure**	**Conception of infinity**	**Mathematical context**
Euclid (300 b.C.)	Prime numbers are more than any assigned multitude of rime numbers	Infinity of primes follows by *reductio ad absurdum*	Exclusively potential infinity	Basic Arithmetics
Euler (1737 and 1748)	Prime numbers are infinitely many (he proved moreover that the series $\Sigma 1/p$, p being primes, diverges)	*Reductio ad absurdum*	He considered the infinite series $\sum_{i=0}^{\infty} \frac{1}{p^i} = \frac{1}{1-(1/p)}$ With reference to series, infinity is potential	Use of some analytical notions
Erdös (1938)	The series $\Sigma 1/p$, p being primes, diverges	*Reductio ad absurdum*	Explicit use of infinite series	Basic Number Theory
Fürstenberg (1955)	The set of prime numbers is infinite	*Reductio ad absurdum*	Actual infinity	Basic Topology

Table 1

Clearly the considered proofs are developed in different mathematical domains; but the crucial point is that they were put forward in very different historical and socio-cultural contexts. The celebrated proposition according to which prime numbers are infinitely many is just the hint, the early idea that stimulated different mathematicians, in different periods, to develop important mathematical contents (Bagni, preprint).

Let us examine some educational possibilities connected to the presentation of the quoted proofs. We have underlined that it would not be meaningful to state that they make reference to a similar epistemological obstacle; when Euler or Fürstenberg proved the infinity of prime numbers, they knew ancient Euclid's result and approached the problem according their own conceptions. So these proofs allow us to compare the different cultural contexts of the periods in which they were conceived, with reference to different cultural institutions and beliefs, and this is the fundamental issue.

For instance, let us present some remarks.

- First of all, proved statements are different: as a matter of fact, Euclid considers "a given quantity of prime numbers" (nowadays we should say: "a set of prime numbers"). Euler and Erdös prove that the infinite series $\Sigma 1/p$, p being primes, diverges; and this is sufficient (but not necessary) in order to state that prime numbers are infinitely many: it is interesting to underline that in the 18th century the focus is mainly operational. In the proof by Fürstenberg, the reference to the set of primes is explicit. So we can consider two different approaches:

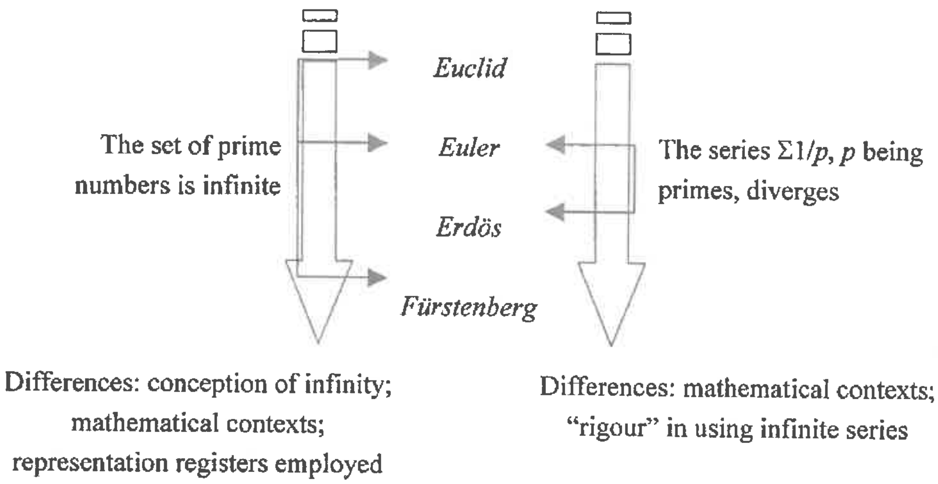


Table 2

- Of course the particular conception of infinity is a crucial element in order to comprehend the sense of the proofs mentioned: Euclid considers potential infinity, following Aristotle and according to the cultural institutions and the beliefs of his own time; Euler and Erdös make reference to a series, and so to a process, and Fürstenberg considers an infinite set in an actual sense.

- The connection between Mathematics and socio-cultural context is fundamental: for instance, Euler's approach by infinite series is not "tuned in" merely to applicative features of the scientific frame of mind in the 18th century (see: Crombie, 1995). In reality the influence of non-mathematical elements is complex and deep.

- Apart from different mathematical contexts, we noted that a very important difference between Euler and Erdös regards the "rigour". But formal correctness must always be investigated in its own conceptual context and not against contemporary standards, in order to avoid the imposition of modern conceptual frameworks on works based upon different ones: so Euclid and Euler *were* rigorous *in their own ways*. These remarks imply some relevant issues related to the educational use of original sources: when we consider Euler's proofs nowadays, for instance in classroom practice, we often *rewrite* them according to our standards: so, through this, really we are looking at the past through our "non-transparent lens" (Confrey & Smith, 1994, p. 173). As noted, probably this is unavoidable: but we must always keep it in mind.

- Representation registers are influenced by the historical periods considered (concerning the primary role played by semiotic aspects see: Radford, 2003, where *Cultural Semiotic Systems of Signification* are presented): however, with regard to Euclid's proof, it is important to take into account both the period in which the original argument was conceived (300 b.C.) and the period of the editions considered (Tartaglia, 1569; Commandino, 1619; see: Barbin, 1994). In Euclid, "placed within the Eleatan-Platonic mode of knowing" (Radford, 2003), we do not find visual methods used, for instance, in the sense of Pythagoreans (on the Greek *Geometric Algebra*, see: Kline, 1972); of course the status of visualisation in the 16th century is completely different (we suggest considering: Bombelli, 1572, in particular the 3rd Book) and it influences the quoted editions of *Elements*. Euler makes reference to diagrams and integrals in his proof of the divergence of $\Sigma 1/p$; later, the importance of symbolic registers seems to be progressively increasing. Of course a complete study would consider the various particular registers used, for instance, by Euler, by Erdös or by Fürstenberg; in fact there is no single register of a given kind: the nature of a register depends on the community of practice in question.

FINAL REFLECTIONS

A wider research will provide a more detailed analysis of the proofs mentioned with reference to their respective socio-cultural contexts and to their comparison (Bagni, forthcoming-b). We now summarise some reflections introduced above:

a. Euclid's proof must be considered in relation to Greek intellectual habits.

b. Euler's approach must be considered in relation to the socio-cultural situation of the 18th century.

c. The comparison between Euler and Erdös allows us to underline that rigour must be evaluated in its own conceptual context.

d. Different notions of infinity, for instance in Euclid and in Fürstenberg, are related to different social and philosophical contexts.

e. Different representation registers must be considered with reference to the communities of practice in question, both in the period in which the original works were written and in the period of their editions.

More generally, in the first paragraph we stated that the History of Mathematics gives us important educational opportunities:

- the possibility of a metacognitive reflection;
- the possibility to achieve a wide comprehension of historical periods.

These possibilities are indivisibly linked: in fact the transfer of some situations from History to Didactics cannot be stated just by analogy, rather it needs a wider cultural dimension that must take into account non-mathematical elements too (Radford, 1997). Of course the perspective presented would require good epistemological skill on the part of teachers and pupils: however, in our opinion, an "internalist" History, and hence a conception of the development of Mathematics as a pure subject, isolated from non-mathematical "external" influences, is hardly useful in Mathematics Education (Grugnetti & Rogers, 2000, p. 40; Bagni, to appear). From this point of view, with reference to aforementioned educational opportunities, the former can be justified by the latter.

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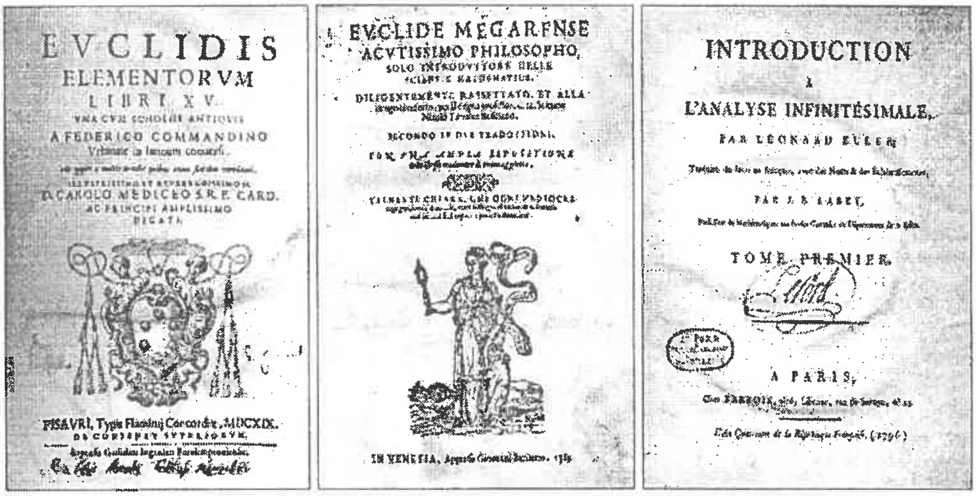

Figure 3

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The Concept of the Plane in Geometry: Elements of the Historical Evolution Inherent in Modern Views

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Abstract: *In the late 19th century, Hilbert introduced the concepts of point, straight line and plane as primitive notions of his axiomatic system of Geometry. Their interrelations are specified via the axioms of his system. Since then, the Mathematics Education community adopts some simplified version of Hilbert's system and considers that these primitive notions can be easily understood. Therefore, no attempt is made to describe them. A question of didactical interest that naturally arises is whether and to what extent these notions are understood mathematically and in particular, what are the points of view that are encountered. We have conducted an empirical research with non-mathematics university graduates to whom we posed 3 questions concerning the concept of the plane. Classification of the different answers has revealed that some of them have their analogues in history, close to points of view expressed by Parmenides, Heron, Leibniz, Peano, Hilbert and others. In addition, our results helped us to realize the existence of difficulties in understanding a simple geometrical concept, which normally remain hidden because of consolidated mathematical conceptions which are tacitly taken for granted.*

Keywords: concept of a surface, Euclidean plane geometry, historical development of the concept of the plane, ontogenesis vs. phylogenesis.

INTRODUCTION

In the late 19th century, David Hilbert placed three geometric concepts as primary at the foundations of his axiomatic system of Geometry; those of a point, a straight line and a plane. He gave no definition, but formulated axioms, which determined the relations among them. Since then, the Mathematics Education community has generally followed some version of Hilbert's axiomatic system, regarding these concepts as primary, simple in comprehension and needing no definition. Didactical questions that arise are a) to what extent students understand in depth these primary concepts, in particular, that of the plane, and b) what are their ideas of this concept.

In her research on the historical development of the concept of the plane, one of the authors (K.Z.) found that there are indications that, even today, non-Mathematics university graduates have difficulties in comprehending this concept. Motivated by this, an empirical study was conducted with a small sample of university graduates (non-mathematicians), who answered questions related to the notion of the plane (descriptive definition, relation to the notion of a surface, and construction). We present elements of this research, which suggest that: (a) (non-mathematics) university graduates face problems in comprehending the plane concept and its relation to that of a surface; (b) often, their views have some analogy to those that appeared in the history of Mathematics.

The idea that the learning of mathematical notions follows the path of their historical development dates back to the 19th century, epigrammatically expressed by the motto "ontogenesis recapitulates phylogenesis" (Fauvel 1991, Radford in Fauvel & van Maanen, 2000, §5.1). In its strict form, this "historical parallelism" has been strongly criticized and abandoned, mainly because the modern conditions under which a person, or a group learns a concept are quite different from those under which mathematicians in the past have conceived, formulated and elaborated the concept in question (Thomaidis 1997). On the other hand, recent studies suggest that, at least in certain cases, there seems to exist a "mild historical parallelism" i.e. the learners have attitudes and formulate views strongly similar to those encountered in history (Harper, 1981, 1987, Moreno & Waldegg 1991, Thomaidis 1997, Thomaidis & Tzanakis preprint, and others). Clearly, to explore the existence of such analogies it is required a detailed historical analysis of particular cases, a careful collection of relevant empirical data from the target contemporary population, and a thorough comparison, keeping in mind the difference between modern conditions of the target population and those met during the historical development of the topic.

The present paper focus on this and attempts to look for such analogies, by comparing the historical analysis with empirical data on the contemporary views of the notion of a plane held by university graduates, whose systematic education in Mathematics stops at the end of secondary education. Section 2 gives a concise presentation of the main efforts to define and construct a plane, from Euclid to Hilbert. Section 3 outlines the main elements of the empirical research (population and questionnaire), whereas, section 4 analyzes the answers. Finally, section 5 summarizes the conclusions obtained, stressing the need and interest of further detailed research on this topic.

A BRIEF ACCOUNT OF THE HISTORICAL DEVELOPMENT OF THE CONCEPT OF THE PLANE IN ELEMENTARY GEOMETRY

Geometry was structured axiomatically for the first time in Euclid's *Elements*, albeit incompletely, according to modern standards. In the *Elements* the concept of the plane is fundamental. Euclid deals with the geometrical relations, first on the plane, and subsequently in space; as a result, a distinction arises between plane and solid geometry. Based on this distinction, two views for the plane result. In solid geometry, the plane is an independent object, like the sphere, or the cylinder, and one examines its relations to other geometrical objects, such as straight lines, spheres, etc. On the other hand, however, the plane is not an object of study in plane geometry. It is the substratum needed for studying the relations of one-dimensional objects, such as straight lines, circles, or other two-dimensional objects (tacitly assumed to be "flat", like polygons, circular disks etc). Because of this distinction, the plane has a peculiar position in the constitution of geometry. Even more importantly, from all two-dimensional geometrical objects (i.e. surfaces), Euclid deals only with the plane.

It was Euclid, who clearly made this distinction between plane and solid geometry in Mathematics. However, it had already been expressed philosophically by Plato, who separated Mathematics into four areas; Arithmetic, Geometry, Solid Geometry (i.e. "Stereometry") and Astronomy. For Plato, Geometry was "the study of the plane".[1] However, it seems that he identified the general notion (*genus* [γένος]) of the surface with the special notion (*eidos* [είδος]) of the plane. According to Proclus, this seems to be true for Aristotle as well. The distinction between these two notions became clear with Euclid[2] (Heath 1956, vol. 1, p 169), but already Parmenides had expressed an

[1]To the question "... the investigation of plane surfaces [την του επιπέδου πραγματείαν], I presume, you took to be geometry?", Socrates answers "... Yes, said I ..." (Plato, *Republic*, 528d, Shorey (trans.) 1935; the translation is inaccurate, since Plato uses the word "plane", not "plane surface").

[2] "The older philosophers did not think to posit the plane (επίπεδον) as a species [είδος] of surface (επιφάνεια) **but look the two terms as equivalent for expressing magnitude in two dimensions.** Thus the divine Plato said that geometry is the study of planes (επίπεδα) and contrasted it with stereometry as if he thought surface and plane were the same thing. Likewise also the inspired Aristotle. But Euclid and his successors make the surface the genus [γένος] and the plane a species of it [είδος], as the straight line is a species of line" (Proclus'

interesting perspective to study geometrical objects, different from Euclid's. He classified geometrical objects into "straight" (i.e., flat), "circular" (i.e., curved), and "mixed", referring simultaneously to one- two- and three-dimensional objects (lines, surfaces and bodies respectively) with two-dimensional objects serving as illustrative examples:

> "....Hence Parmenides says that every figure is straight, circular, or mixed. If, then, you wish to consider straightness in surfaces, **take the plane, which the straight line fits on in all ways**; or if circularity, take the spherical surface; or if the mixture of the two, take the cylindrical, or the conical, or some similar surface. (Proclus' *Commentary on Euclid, Book I*', 117, 17–22, Friedlein 1873, p.117, Morrow (trans.), 1970, p.95; our emphasis).

In other words, Parmenides regarded that "to be straight" is in the nature of the plane. We encounter the same characteristic feature for the straight line and for the right-angled parallelepiped. Although, Euclid did not follow Parmenides' distinction, he did assign to the plane the property of "straightness", reducing it to the straight line: "*A **plane** surface is a surface which lies evenly with the straight lines on itself*" (Thomas 1939, p. 438, our emphasis). The Euclidean definition, however, raises significant problems: firstly, of a semantic type, because the expression "... lies evenly ..." is ambiguous and seems to refer to some notion of uniformity, which has not been previously defined; secondly, Euclid ensures the existence of concepts via their construction (by ruler and compass), which, for primary concepts is given axiomatically (e.g. for the straight line and the circle), whereas, for derivative it is proven (e.g. for the triangle, the median and parallel lines). Even in solid geometry, Euclid ensured the existence of solid bodies via rotations; e.g., that of a cone, by the rotation of a right-angled triangle around one of its perpendicular sides (Stamatis 1957, vol. 4, p. 13). However, Euclid did not ensure the existence of the plane, neither axiomatically, nor via a construction (Heath, 1908-1926, vol. 1, p. 172). In other words, although Euclid constructs both one- and three-dimensional objects (straight line, circle etc.), he does not offer any construction of the plane, the only two-dimensional object he deals with. Although the construction of the plane by ruler and compass would be impossible, since these instruments presuppose the existence of a plane, a construction in three-dimensional space could have been given, for instance, one similar to that of the cone. Thus, after Euclid's *Elements*, it was needed a clearer definition of the plane and a construction that would ensure its existence. For our analysis, we discuss below well-known mathematicians' attempts made along these

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Commentary on Euclid, Book I, 116, 17–117, Friedlein 1873, p.116ff, Morrow (trans.) 1970, p.94; our emphasis).

lines.

Mathematicians and commentators after Euclid tried to give a clearer definition of the plane, notably, Heron of Alexandria (circa 1st century BC - 1st century AD). Essentially, he gave a list of definitions all assigning to the plane the characteristic feature of "being straight":

> "A plane surface is one, which lies evenly with the straight lines on itself, and is extended straight. **When a straight line touches two points** [of this surface], **then the whole** [straight line] **fits in every way in every position** [on this surface], that is, [it is the surface] which fits on a straight line on all its length, and is the least of all surfaces, which have the same ends and all parts of which, by their nature, fit [among them]" (Hero's *Definitiones*, 9, 1, 2-8; our emphasis and translation).

In the 17th century, Pascal seems to conceive the plane in a similar (though not very clear) way, since in his essay *De l' Esprit Géométrique* he writes that ".... plane surfaces are bounded from all their sides by straight lines, which are extended directly from the one [side] to the other" (Pascal 1954, our translation). On the other hand, Leibniz made several attempts to eliminate Euclid's logical imperfections. In *In Euclidis Prota* and *Initia rerum mathematicarum metaphysica*, he referred to the definition and more generally, to the foundation of basic geometrical notions, like the straight line, the plane and the circle. We present only a few of his attempts. In *In Euclidis Prota*, he defines the plane as a section of a body, which behaves everywhere in the same way (Leibniz 1849 – 1863, vol. V, §VII, 6), and draws two figures: a line segment and a curve, with top side A and bottom side B (*ibid.* vol. II, figure 83; see Figure 1). The first figure corresponds to a plane section, since it behaves the same on both sides A and B, but the second corresponds to the section of a non-flat surface, which in general has a concave and a convex side, as Leibniz himself points out. Here, the issue of curvature comes in, which, as we know, is zero for the plane, but nonzero for other concave, or convex surfaces. This specifies in more detail the "straightness" of a plane.

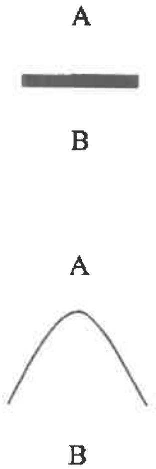

Figure 1

Elsewhere, he criticised Heron's definition that a straight line can fit on a plane, because Leibniz considered that in this definition, there is a "pleonastic subreption (purloining)" (subreptio pleonastica) (*ibid.* vol. V, p. 188); in other words, that this definition contains more than what is required for characterizing a plane. Thus, another, simpler definition of the plane could be given and nonetheless, be able to prove that particular property. In an important letter to Huygens, using the notions of *equivalence* and of a *point*, Leibniz defined the basic geometrical notions differently, without any reference to geometrical figures, as he, himself, points out (Leibniz, 1966, vol. I, pp. 77-83): A plane is a set of points, which are equidistant from two given points, a straight line is a set of points, which are equidistant from three given points, etc (*ibid.* pp.73-88). Evidently, Leibniz considered and defined the geometrical objects in space and provided a figure, depicting a plane perpendicular to the middle of a line segment *AB*. Thus, he essentially gave a construction of the plane.

Since Euclid's time, the construction of the plane was difficult. Probably, by conceiving geometrical objects in three-dimensional space, Leibniz paved the way for this construction to become possible. Well-known mathematicians of the 19th century, like Gauss, W. Bolyai, and his son, J. Bolyai, apparently followed Leibniz' steps. W. Bolyai constructed the plane with the aid of motions in space, particularly, the rotation of a straight line around a second straight line perpendicular to it (W. Bolyai, 1913, p. 65). Gauss defined the plane as the surface containing all straight lines perpendicular to a given straight line at a given point, and he constructed it exactly like Bolyai (Gauss, 1863 – 1929, vol. VIII, p.194). Finally, J. Bolyai constructed the plane through motions, but using the notion of symmetry (though interesting, his definition is not directly related to the present study, so that we will no longer consider it here).

Earlier, the English commentator of the *Elements*, R. Simson, had approached the notion of the plane differently ("*The Elements*" 1756), as the surface with the property that, the straight line through any two of its points lies entirely on it (Simson 1866, p.1). This definition was accepted by the majority of the authors of Geometry books, from the 18th century until about the end of the 19th century, and essentially expresses Hilbert's 6th connexion axiom. Indeed, Heath calls it "Simson's definition" (Heath, op.cit. p. 172), a terminology we will follow, as well. Simson's definition is equivalent to Heron's referred to above, a definition, which, as we saw, was questioned by Leibniz, who thought that it contains redundancies.

In the correspondence with his friends, W. Bolyai (in 1832) and Ch. Gerling (in 1825), Gauss expressed roughly the same objections against Simson's definition. He considered Simson's property too strong to be posed as a definition. He thought that it should be possible to give another, weaker definition, and then prove Simson's property. In his *Archives*, a note was found in which he gives the definition of the plane mentioned above[3] and proved that this object satisfies Simson's property (Gauss, 1863 – 1929, vol. VIII, p.195). With this definition, Gauss was also solving the problem of the existence of the plane (via its construction), a problem left open since Euclid. As we saw, Gauss' friend, W. Bolyai, gave a similar construction.

At the end of the 19th century, significant developments in Geometry took place: Hilbert's *Foundations of Geometry* in 1899 constitutes a complete axiomatization of Euclidean Geometry. A little earlier, Peano's mathematical school was founded. This school made significant contributions to Arithmetic and the axiomatization of Geometry. M. Pieri, an important of its members, presented an axiomatization of Geometry based on three primary concepts; those of a point, a line segment and a motion. He defined a plane in the following unusual way: "Having been given three non-collinear points *A*, *B* and *C*, one will call *plane ABC* - or simply *ABC*- the figure occupied by all the lines that unite the point *A* with various other points of *BC*, or the point *B* with points of *CA*, or the point *C* with those of *AB*" (Marchisotto 1990, p.272). He proved the consistency of his definition, in other words that the three sets of straight lines in his definition, coincide. On the other hand, Hilbert, probably

[3] "A plane is a surface in which lies every straight line *AD* that is perpendicular to the straight line *AB*. Such a plane is generated if *AD* is rotated around the axis *AB*." (Gauss 1863 – 1929, vol. VIII, p. 194, our translation).

influenced by an emerging tendency towards abstraction and axiomatization in Mathematics at that time[4], did not define the plane, but regarded it as a primary concept, like the point and the straight line (Hilbert, 1899). For Hilbert, the axioms determine the relations among primary concepts, whose **only** meaning is that expressed in the axioms, which play the role of definition of primary concepts. In principle, the definition of any derivative concept can be given in terms of these primary concepts. In this context, the existence of any concept is ensured by the logical consistency of the axiomatic system in which it belongs.

Thus we see that until the end of the 19th century, the existence of geometrical objects was proven via Euclidean, or, let we call it, of Leibnizian type constructions; it is existence of an **empirical** nature. As we have seen, for the plane such a construction becomes possible by considering geometrical objects in three-dimensional space. On the other hand, however, after Hilbert, the existence of an object is ensured by the non-contradictory nature of the axiomatic system in which it belongs; it is existence of a **logical** nature.

Hilbert's conception was accepted, not only by the majority of mathematicians, who have been dealing with the axiomatization of Geometry after him, but also by the Mathematics Education community. As a consequence, in school and university textbooks the plane is not defined, since it is considered as primary.

In summary, the unclear definition and the absence of construction of the plane in Euclid's *Elements*, led many mathematicians, to provide definitions that assign to the plane its characteristic property of "being straight", and/or, to construct it. Hilbert's approach led to a very different conception, which essentially stopped all these attempts. The question is, whether this conception, acceptable by modern mathematical standards, is that possessed by non-specialists in Mathematics (e.g. those, who have at most a secondary school mathematics education); and additionally, to what extent do they understand and how do they perceive the notion of the plane.

An Empirical Investigation Of Contemporary Views On The Notion Of A Plane

To look for the answer to the above questions, we composed a questionnaire with three questions.

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[4] We can trace such a tendency in the work of Cayley, Pasch, Peano, Pieri and others (Dieudonné 1978, §XI.II).

1. Describe what a plane is.

2. In your opinion, what are the differences between a "plane" and a "surface"?

3. Construct a plane.

The first question would help to understand the ways the notion of a plane is being conceived today. E.g. whether the subjects in this study consider it as a primary concept, as they are taught in Geometry, or they try to assign some characteristic property to it and what this property is. It is worth noting that, when asked questions 1 and 3, Mathematics graduates answered that the plane is a primary concept, hence, there is no question to describe it (via a definition), or construct it.

We posed the second question because, in the initial design of our research, we noticed that non-Mathematics graduates had a difficulty to perceive a surface as a two-dimensional object independent of the three-dimensional object, of which it may form part. In contrast, nothing similar was noticed for the plane, for which it seems that they had clearer perceptions.[5] These perceptions bare some resemblance to those expressed by Aristotle, who occasionally considers the plane as a "genus" (identifying it with the surface) in the same way that this is true for the "geni" line and solid[6], and in other cases he perceives a surface (as synonymous to the plane, and) as the boundary of a solid (Heath, 1956, vol. 1, p. 170)[7].

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[5]Accepted answers should be based on the fact that a surface is a more general concept, while a plane is a special case of a surface; more specifically, a "flat" (non-curved) surface. Obviously, no technical interpretation of the term "flat" was expected from non-mathematicians.

[6]See footnote 2 and Heath 1956, vol.I p.169. Also, "A magnitude if divisible one way is a line, if **two ways a surface** [επίπεδον], and if three a body. Beyond these there is no other magnitude, because the three dimensions are all that there are, and that which is divisible in three directions is divisible in all" (Aristotle *On the Heavens*, 268a, 7-10, Barnes (trans.) 1984, vol.I, p.447; our emphasis - the translation is inaccurate, since Aristotle uses the word "plane", not "surface"). Also, "Thus that which is quantitatively and *qua* quantitative wholly indivisible and has no position is called a unit; and that which is wholly indivisible and has position, a point; that which is divisible in one sense, a line; **in two senses, a plane**; and that which is divisible in all three senses, a body" (Aristotle, *Metaphysics*, 1016b, 24-28, Tredennick (trans.), 1933; our emphasis).

[7] "But points and lines and **surfaces** cannot be in process of becoming nor of perishing, though they at one time exist and at another do not. For when bodies come into contact or are separated, their boundaries instantaneously become one at one time – when they touch, and two at another time – when they are separated .... And evidently the same is true of **points and lines** and **planes**; for the same argument applies, as they are all alike either **limits** or divisions." (Aristotle, *Metaphysics*, 1002a 32 – 1002b 3, 1002b 9-11, Barnes (trans.) 1984, vol.II p.1583;

The third question stems naturally from the history of the concept of the plane, which, as we saw, was a continuous attempt to prove its existence, and particularly to construct it. With this question, we would like to examine: (i) whether the subjects can give a (possibly incorrect) construction, or whether they think that the plane cannot be constructed, and (ii) whether they think that it is presupposed for every other construction. In other words, that it is a primary notion, at least for constructing geometrical objects. At the same time, from questions 1 and 3 we could detect possible analogies between contemporary views and those formulated by mathematicians in the past prior to Hilbert's axiomatization of Geometry.

The sample consisted of non-mathematicians, since most mathematicians who have been asked, gave the standard answer, that the plane is a primary concept, and wondered why they were asked to construct it. We do not refer to this group here. Our analysis concerns other graduates; sociologists, elementary school teachers, German and English teachers, psychologists, lawyers and doctors. In total, 51 people were given the questionnaire. Since for those who know Euclidean Geometry, geometrical constructions should be done using only ruler and compass, a clarification to the third question was given when the questionnaire was handed; we explained that the construction of a straight line can be done with a ruler, that of a circle with the compass, but that of a cylinder can be done with motions in space, such as the rotation of a rectangle around one of its sides.

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our emphasis). Also, "Among definitions of this kind are those of the point, the line and the plane; for all these demonstrate the prior by means of the posterior – the point being called the **limit** of the line, the line that of the plane, and **the plane that of the solid.**" (Aristotle, *Topica* 141b 20-23, Forster (trans.), 1960; our emphasis). "But if we suppose lines or what comes after these (I mean the primary plane figures [επιφανείας τας πρώτας]) to be principles, these at least **are not separate substances, but sections and divisions** – the former of surfaces, **the latter of solids** ….; and further they are limits of these same things; and **all are in other things and none is separable.**" (Aristotle, *Metaphysics*, 1060b 12-17, Barnes (trans.) 1984, vol.II pp.1675-1676; our emphasis- the translation is inaccurate, since Aristotle uses the word "surfaces", not "plane figures"). "There are some, who, because the point is the **limit and extreme** of the line, the line of the plane, and the **plane** of the **solid**, think there must be real things of this sort. We must therefore examine this argument too, and see whether it is not remarkably weak. For extremes are not substances, but rather all these things are mere limits." (Aristotle, *Metaphysics*, 1090b 5-9, Barnes (trans.) 1984, vol.II, p.1723; our emphasis). Concerning the last quotation, it seems that Pascal also conceived a plane surface as a finite 2D-domain bounded by straight lines (see the quotation from Pascal 1954 in section 2 above).

Presentation And Discussion Of The Answers

In what follows, we group the answers, based on similarities among them. The title of each group expresses the common element of the corresponding answers, which the subjects consider as the basic characteristic that describes a plane. In certain cases, the same subject gave several descriptions, which are placed in different groups. They are codified with the number of the particular subject, followed by a letter (a, b, c, etc.), which indicates the different answer to this question. The relative frequencies are calculated with respect to the total number of answers given to that question. We give characteristic representatives of each group.

ANSWERS TO THE FIRST QUESTION: <u>Describe what a plane is</u>

The total number of answers is N = 63. In one case (Y20) the second answer is formulated as a result of the first. In this case the subject's answer appears in two different groups.

(a) *A surface on which a straight line **fits exactly*** (n = 10, 15.9%). This is the largest group of answers, some of which directly refer to Heron's description:

Y15. A plane is a surface in which, a straight line fits (is tangent) completely, no matter in which direction we rotate it on the surface.

Y20*. A plane is a surface on which a ruler fits no matter where and how it is placed. Hence, it contains every line, which has two points in common with that surface.

(b) *A surface that **contains each straight line through any two of its points*** (n = 3, 4.8%). This is a small group of answers, but with close similarity to Simson's rule:

Y3a. A plane is a surface, which **contains completely the straight line** passing through any two of its [the surface's] points.

Y16a. A plane is a surface for which, if any two of its points are connected by a straight line, then all points of this line belong to the surface.

Y20*. A plane is a surface on which a ruler fits no matter where and how it is placed. Hence, it contains every line, which has two points in common with that surface.

(c) *A surface defined by **three points** and the lines passing through them* (n = 8, 12.7%). These are answers, which express in some form Hilbert's 5th axiom of connection. At the same time, however, some of them have a close similarity to Pieri's definition, discussed earlier (Y13 and to a smaller extent Y31).

Y13. Three arbitrary points in space, which do not lie on the same straight line, define a plane. To the plane belong all points of the lines which intersect two of the lines determined by the three given points, or, all points of the lines passing through one of the three given points and intersecting the line determined by the other two.

Y16b. A plane is determined completely by three points, but which do not lie on the same straight line...

Y31. Three points in space determine a plane. The plane is an infinite set of points with no limits, which can be considered as the figure, and its extension, which is formed if we connect three non-collinear points.

(d) *A set of points* ***equidistant from two given points*** (n = 1, 1.6%). A single answer, essentially equivalent to Leibniz' formulation.

Y3b. The set of points, lying at the same distance from two given points.

(e) *A* ***smooth straight*** *surface* (n = 8, 12.7%). This is a group of perceptions related to the fact that the plane illustrates the meaning of "straightness".

Y17. A smooth surface, without cavities and prominences, without depth.

Y42. The plane is a straight surface.

Y47. A plane is an object (a surface) that does not have corners, that does not have curves and all points of this object are along a straight line.

(f) *Answers referring to concrete* ***physical situations*** (n = 4, 6.3%). Some subjects do not go beyond the level of description in terms of concrete physical situations:

Y28. The surface of calm water.

Y48a. I could not describe it. I would point out the surface of a table.

Y50b. A plane is that surface which, while being in a horizontal position, a sphere has a stable balance at any of its points.

Approximately 1/3 (33.3%) of the answers is either manifestly wrong, unclear, or incomplete, or is based on logically circular sentences.

(g) *Answers either* ***unclear, logically circular****, or meaningless* (n = 7, 11.1%). Two of them seem to be logically circular (Y10, Y14), probably indicating difficulties to define a primitive notion. The other answers are unclear or meaningless.

Y10. A plane is a surface such that, if we draw a straight line on it, all its points will lie on the surface.

Y14: A plane is a surface defined by points that lie between straight lines, which are intersecting each other and determine the plane.

Y51a. A plane is a surface without depth.

(h) *Incorrect answers* (n = 10, 15.9%)

Y24. It is a surface such that, all lines passing through its points, lies on it.

Y25. A part (piece) of a geometrical body.

Y39. A plane is a geometrical figure, which has a base and a height.

(i) *Incomplete answers* (n = 4, 6.3%).

Y6. A plane is a surface in which two arbitrary points determine a straight line.

Y51b. A plane is a geometrical surface that has two dimensions.

(j) *Other answers* (n = 5, 7.9%). Finally, a number of answers are of a rather peculiar nature. One of them (Y16c) seems to refer to Gauss' and W. Bolyai's description, especially if combined with the answer which this subject gives to the 3rd question; another one (Y43) reminds Leibniz' description corresponding to figure 1:

Y16c. A plane is completely determined ... by a point, which lies in the plane and a straight line perpendicular to the plane.

Y43. A plane is something, which gives us the same impression from both its upper and lower side.

(k) *No answer*: (n = 3, 4.8%).

ANSWERS TO THE SECOND QUESTION: In your opinion, what are the differences between a "plane" and a "surface"?

The total number is N = 58. It is interesting to observe that for a significant part of the sample, the surface is perceived as the boundary of a three-dimensional finite body and consequently, it requires the existence of three dimensions in order to be defined. In a few, manifestly incorrect formulations, a "surface" seems to coincide with the area concept, a fact which may be related to the use of the Greek word for "surface" in everyday language; in others a "surface" indicates the outer surface, or a face of a solid, a terminology which is also used in everyday life. Finally, certain answers invoke Simson's property to differentiate a plane from the more general concept of a surface.

(a) *The* ***"surface"*** *as the* ***general*** *notion. The plane as a special case of a surface* (n = 13, 22.4%). These are answers showing a correct conception of the difference between the two notions, despite the use of peculiar, or not strictly correct expressions in some cases.

Y50. A surface has an arbitrary shape (wave-like, cylindrical). The plane is the "shortest" shape determined by three points in space, as the straight line is the shortest path between two points.

Y51a. A surface might not be plane. The plane is a kind of surface.

(b) ***Simson's property*** *as the key property determining the difference between a "surface" and a "plane"* (n = 4, 6.9%). These are correct answers, which essentially define a plane as a surface satisfying Simson's property.

Y12b. For plane surface, the straight line determined by any two of its points belongs to the plane surface. This does not hold for all surfaces.

(c) *A* ***"surface"*** *as the* ***boundary*** *of a three-dimensional* ***body*** and/or something limited, as opposed to the "plane" which is unlimited (n = 12, 20.7%). For 1/5 of the subjects a surface cannot be considered as an independent object, but only in relation with a solid. They conceive it as the surface layer, or boundary of a solid and only in contrast to a plane, which is understood as an independent object (a conception close to similar views expressed by Aristotle; section 3 and footnote 7). In some cases, this view is expressed by appeal to the concept of measurement: Since any surface is a surface of a solid body, it can be measured, whereas, a plane, which is an unbounded, independent object, cannot be measured.

Y5. A surface of a body is the set of its extremities, which separate it from space around it.

Y24. A surface has boundaries, whereas, a plane is defined at infinity.

Y27a. A plane is extended without limit, a surface is finite.

Y43a. A surface is a surface of a solid body. It always refers to some solid body. The plane exists independently of a solid body.

(d) *A combination of cases (c) and (a) or (b)* (n = 2, 3.4%). These are answers illustrating a conception of the plane, correct as far as the absence of curvature is concerned, but nonetheless, identifying a surface with the boundary of a body.

Y3. A surface: the outer in length and width extension of an object, the upper part of a certain object. A plane: an unlimited surface that contains all points of a straight line, which connects two of its points.

Y17. A surface is the outer part of a body, the extension in length and width, without depth, which is not exclusively plane.

(e) *"Plane" = a plane solid body, "Surface" = a face of a solid body* (n = 10, 17.2%). This is a large group of answers, in which a "plane" is identified with a plane solid body (i.e. a polyhedron), and a "surface" with its lateral surface (see Y34), or, with one of its faces. They possibly reflect the everyday use of the term "surface" as the lateral surface of a body (that is, in topological terms, its boundary as a subset of $\mathbf{R}^3$), except two almost identical answers (see Y14 below), where, by a "plane" is understood a plane face, whereas, a "surface" indicates the lateral surface.

Y34. The plane is related to a whole figure, while a surface is related only to the upper extremity of a figure, that one, which "can be seen".

Y14. A plane has only one surface, whereas, a surface can have many planes.

(g) *Unclear answers* (n = 4, 6.9%).

Y11. A plane contains straight lines in all directions, while this is not necessary for a surface.

(h) *Manifestly incorrect answers* (n = 9, 15.5%). This is a largely inhomogeneous class of answers. Most likely, Y32 and Y23 reflect the fact that in everyday language a "surface" is used as synonymous to the "area". Y9 seems to suggest what is explicitly stated in the answers of type (c).

Y32. The plane defines the straight line, while the surface defines area.

Y23. The surface is a measurable quantity, while the plane is not.

Y9. To define a plane, two dimensions of space suffice, whereas, to define a surface all three dimensions are necessary.

(j) *Other answers* (n = 2, 3.4%). These are isolated answers. The first presents similarities with the answers of type (a), and the second with those of type (c), at the same time containing some clearly incorrect conceptions.

Y36. The difference consists of the fact that the first is a property of the second, on the basis of the expression "plane surface".

Y49. Mathematically, a plane can be defined by three points. A plane is a geometrical figure and can be defined, whereas, in contrast, a surface is not a geometrical figure and determines the outer layer of any object.

(k) *No answer* (n = 2, 3.4%).

ANSWERS TO THE THIRD QUESTION: Construct a plane

There were 56 answers to the third questions. Y20 gave four different constructions and for Y21 the answer consists of two constructions belonging to different classes. We note the existence of great difficulties, which, in certain cases, lead to surprise, or puzzlement. Many subjects did not answer (21.4%), answered incorrectly (10.7%), or gave unclear, confused, or logically circular answers (17.9%).

(a) *Construction of, or* ***drawing, a straight line figure*** *either (i) with an explicit note that the plane is obtained as an extension of the figure, or (ii) without such a note* (n = 3, 5.4%). It is not entirely clear, particularly in case (ii), whether the subjects understand the plane as infinitely extended, especially, if we take into account their answers to the other two questions.

Y2. [A plane] is an unlimited space and is constructed symbolically with a parallelogram, or, a triangle (considered as a part of the plane).

Y4. [He/she draws a circle with radius $a = 2$ cm. and writes]: Construction of a circle.

Y33. With the aid of a ruler we draw straight lines, which form a triangle, or a rectangle.

(b) *Construction of, or* ***drawing, a straight line figure*** *without any* ***comment*** (n = 5, 8.9%). Here again it is unclear whether the subjects understand the plane as infinitely extended, or simply as the adjective in the expression "plane surface"; see their answers to the 1st and the 2nd question, especially that of Y14 (possibly except Y24, if we consider his/her answer to the 2nd question, category (c)).

Y14. [He/she draws a rectangle *ABCD*]

Y24. [He/she draws a non-rectangular parallelogram with dashed lines].

(c) *Construction with the aid of three* ***points*** (n = 6, 10.7%). These are answers expressing Hilbert's 5th axiom of connexion by a construction. Y20a is closely related to Heron's definition.

Y19. [He/she draws a non-rectangular parallelogram with three points *A*, *B*, *C* belonging to it and adds]: From three non-collinear points passes one and only one plane.

Y20a. With three non-collinear points. In any case, the plane constructed is the surface, which includes the aforementioned in the above cases[8] and on which a ruler fits, no matter how the ruler is placed.

(d) *Construction with the aid of* ***two intersecting straight lines*** (n = 3, 5.4%). These are correct answers.

Y20d. With two intersecting lines … In any case, the plane constructed is the surface,

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[8] This subject provided 4 constructions, which are accompanied by the same comment.

which includes the aforementioned in the above cases[8] and on which a ruler fits, no matter how the ruler is placed.

Y41. Two intersecting lines construct a plane.

(e) *Construction with the aid of* ***a straight line and a point*** *not belonging to the line* (n = 1, 1.8%).

Y20b. With a straight line and a point not belonging to it ... In any case, the plane constructed is the surface, which includes the aforementioned in the above cases[8] and on which a ruler fits, no matter how the ruler is placed.

(f) *Answers referring to concrete* ***physical situations*** (n = 3, 5.4%)

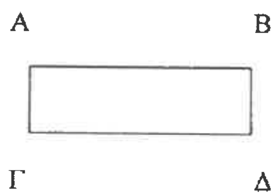
Y25. The paper I used defines a plane, if extended infinitely.

Y34. We could construct a wooden square, which would be a plane.

(g) ***Unclear*** *and confused, or* ***logically circular*** *answers* (n = 10, 17.9%). Two of them (one is Y27a) are logically circular. The rest are particularly unclear and some of them (e.g. Y21) seem to show that the subjects conceive the "plane" as a "plane figure".

Y27a. By considering two parallel straight lines ..... I define a plane, which contains them.

Y21. A plane surface is a surface having equal distances from the straight lines, which it consists of (the straight lines which constitute it, intersect each other, forming points which, taken together, create the surface). In a relation, straight lines and points complement and create the whole, specifically, the flat surface which, if we analyse it, we will be led to the smaller pieces which constitute it, to straight lines which intersect each other. According [to the above]



(h) *Incorrect answers* (n = 6, 10.7%). These express manifestly incorrect conceptions, and all but one (Y32) seem to show that the subjects understand the plane either as a "plane solid body" or a "plane figure".

Y32. It seems that a ruler is sufficient to construct a plane by drawing a straight line.

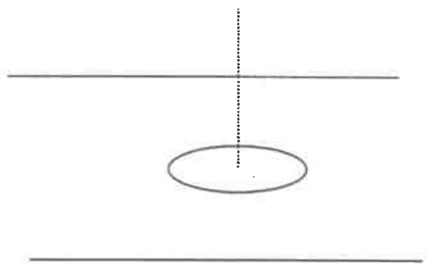
Y30. We mark 4 non-collinear points on a piece of paper and draw the lines that pass through them. In this way a plane results.

(i) *The **construction is impossible*** (n = 1, 1.8%).

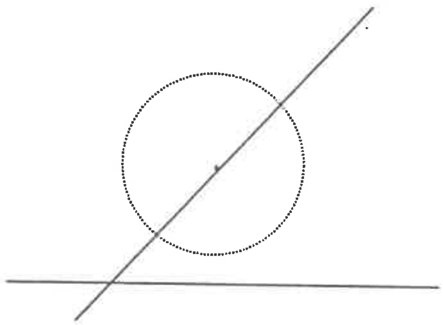
Y12. We cannot construct a plane, since a plane does not possess width. We get an image of a plane when we form a non-rectangular parallelogram.

(j) *Other answers* (n = 6, 10.7%). These are isolated answers, some of which, at least indirectly, present an analogy with ideas that appeared during the historical development of the concept of a plane. Y16 exhibit similarities to W. Bolyai's and Gauss' construction; Y49b is unclear, but could correspond to a similar construction; Y13 describes Pieri's construction, albeit expressed in a peculiar way! Others refer to the development of solids, having a boundary consisting of flat surfaces (e.g. Y47).

Y16. [He/she draws two parallel straight lines, a perpendicular one in perspective and a circle centred at the intersection of the perpendicular line and the plane defined by the parallel lines]



Y13. I take two arbitrary points. I draw the straight line that they define. I take a third point not belonging to the line and draw any straight line passing from this point and intersecting the line through the first two points. I rotate the second straight line around any of its points (different from the point of intersection of the two lines) completely by 360°, so that it intersects the first line (the points of the parallel to the initial line are left out). Once again, I rotate the line around a different point and thus all points of the plane are defined.



Y47. [Construction with straight lines on a piece of paper]. I take a cylinder, I open it and I obtain a plane.

Y49b. From a point in space, I draw two straight lines and taking one of them as an axis, I rotate the other one perpendicularly.

(k) *No answer* (n = 12, 21.4%): This is the largest class.

In the table below we summarize the various types of answers and their relative frequencies.

TABLE: CLASSIFICATION OF THE ANSWERS BY QUESTION

No of class	1st question (Q1)	2nd question (Q2)	3rd question (Q3)	Relevant historical views
(a)	*A surface on which a straight **line fits exactly*** (15.9%)	*The **"surface"** as a **more general notion**. The plane as a special case of a surface* (22.4%)	*Construction of, or **drawing, a straight-line figure** and providing additional comments* (5.4%)	**Q1**: Heron, c.1st cent BC-1st cent.AD
(b)	*A surface that **contains each straight line** through **any two of its points*** (4.8%)	***Simson's property** as the key property determining the difference between a "surface" and a "plane"* (6.9%)	*Construction of, or **drawing, a straight-line figure** without any **comment*** (8.9%)	**Q1, Q2**: Simson, 18th century
(c)	*A surface defined by **three points** and the lines passing through them* (12.7 %)	***A "surface" as the boundary of a three-dimensional body** and/or as something limited, as opposed to the "plane" which is unlimited* (20.7%)	*Construction with the aid of **three points*** (10.7%)	**Q1, Q3**: Hilbert 5th axiom of connection , 1899 **Q2**: Aristotle 4th cent. BC
(d)	*A set of equidistant points **from two given points*** (1.6%)	*A combination of cases (c) and (a) or (b)* (3.4%)	*Construction with the aid of **two intersecting straight lines*** (5.4%)	**Q1**: Leibniz, 17th century
(e)	***A smooth straight** surface* (12.7%)	*"Plane" = a plane solid body, "Surface" = a face of a solid body* (17.2%)	*Construction with the aid of **a straight line and a point** not belonging to the line* (1.8%)	**Q1**: Parmenides, 5th century BC
(f)	*Reference to concrete **physical situations*** (6.3%)		*Reference to concrete **physical situations*** (5.4%)	
(g)	***Unclear, logically circular, or** meaningless answers* (11.5%)	*Unclear answers* (6.9%)	***Unclear** and confused, **or logically circular** answers* (17.9%)	
(h)	*Incorrect answers* (15.9%)	*Incorrect answers* (15.5%)	*Incorrect answers* (10.7%)	
(i)	*Incomplete answers* (6.3%)		*The **construction** is **impossible*** (1.8%)	
(j)	*Other answers* (7.9%)	*Other answers* (3.4%)	*Other answers* (10.7%)	**Q1**: Gauss 19th cent., or Leibniz 17th cent. **Q3**: Gauss, or Pieri, 19th cent
(k)	*No answer* (4.8%)	*No answer* (3.4%)	*No answer* (21.4%)	

CONCLUDING REMARKS

To answer the first question, the subjects attempted to attribute to the plane its characteristic property of "being straight" (types (a), (b), (e), 33.4% in total). Specifically, type (e) is similar to Parmenides' view (Section 2), who thought that it is in the nature of a plane to "be straight". This point of view is encountered in all definitions of the plane from Euclid and Heron, to Simson, which reduce it to the notion of a straight line. Answers of type (a) and (b) express points of views similar to those of Heron and Simson, respectively. The isolated answers Y3b (type (d)) and Y43 (type (j)) are also noteworthy; they directly refer to Leibniz' views. Y16c (type (j)) is related to Gauss' and W. Bolyai's approach. Type (c) (12.7%) is directly related to Hilbert's axioms. However, we should notice that answer Y13 in this category bears a strong similarity to Pieri's definition of a plane (Section 2). Finally, about 1/3 of the answers (33.3%) are characterised by vagueness, incompleteness, or clearly incorrect conceptions and logical errors, while 6.3% does not go beyond a description in terms of concrete physical situations.

In spite of the fact that a significant percentage of the subjects could describe what a plane is, the answers to the second question reveal important difficulties in identifying the difference between a plane and a surface. For 41.3% (types (c), (d), (e)) the surface cannot be understood as an independent object, but only in relation to a solid; it is conceived as the outermost layer of a solid and only, in contrast to the plane, which is understood as an independent object, in analogy with a similar point of view of Aristotle. Specifically, in category (e) the confusion of a plane with a plane solid body, is evident, whereas, a surface is understood as the outermost layer of such a body. Similarly, in certain clearly incorrect answers (Y9 and Y23 of type (h)) a surface expresses area, without being clear if a plane solid is understood as a plane (these subject did not answer the first question). Finally, 29.3% (types (a) and (b)) have a correct view of the difference between the two concepts, with category (b) being particularly interesting, since it essentially uses Simson's property as the key difference. Additionally, we notice that the four subjects in this group are consistent enough, since they answer the first question in an analogous way (they are of type (c), (a), or (b)).

The results are different for the third question. 25% of the answers are correctly based on the use of points and/or straight lines (types (c), (d), (e) and four of type (j)), with Y16, Y49b and Y13 being particularly interesting, since they refer, respectively, to the constructions by Gauss – W. Bolyai and Pieri! Types (a), (b) and two of (j) (total percentage 17.9%) are intermediate cases, in the sense that they construct the plane by constructing a plane figure. However, it is unclear if this is what they mean, or they implicitly mean its infinite extension (Y2 of type (a) being excepted). Finally, 29.5% express unclear, logically circular, or incorrect conceptions (types (g) and (h)), 21.4% does not answer, 5.4% invokes the use of natural bodies (type (f)), while one subject (1.8%) claims that the construction is impossible (type (i))! Thus, approximately 58% cannot provide a construction, because of confused, or incorrect conceptions, possible weakness to think abstractly, or other reasons, which remained hidden in the framework of the present study.

No attempt is made to correlate systematically the answers of each subject to different questions, or, report on Mathematics graduates' answers to these questions. However, the present analysis suggests that people with university (but not strictly mathematical) education formulate definitions, constructions and views of the concept of a plane, similar to those that appeared in the historical development of this concept. This is a phenomenon of "historical parallelism" which is worth studying in more detail.

On the other hand, the historical analysis showed that Hilbert's view, which considers the plane as a primary concept, is the outcome of a long series of attempts to get a detailed description and understanding of this notion. Consequently, it is not obvious didactically that it can be taken as the starting point in the teaching of (theoretical) Geometry. Presumably, the many incorrect views that appeared in this study are related to the subjects' school mathematics education, and specifically, to the lack of any attempt of a detailed elaboration of the basic characteristic properties of the plane. Hilbert's approach to this notion may have solved several mathematical problems, but it may have produced a new problem of a didactical and ontological nature; namely, the impossibility to understand deeply the nature of the concept of a plane. This is an interesting subject for further research.

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Problem Solving, Collaborative Learning and History of Mathematics: Experiences in Training In-Service Teachers

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Abstract. *This paper presents and discusses the author's experiences in continuing education of junior high school mathematics teachers. Twelve in-service teachers participated in a 48-hour workshop, consisting of twelve weekly sessions of four hours each. Teachers were required to collaboratively solve a series of classical problems from elementary arithmetic, algebra and calculus taken from history, in order to identify possible obstacles encountered by their students in comprehending similar concepts. Sessions were divided into four stages: (a) individual work, (b) collaborative work in small groups of three teachers each, (c) collaborative work within the whole group, (d) metacognitive individual work.*

Keywords: Problem solving, collaborative learning, teachers' education, history of mathematics

HISTORIC-EPISTEMOLOGICAL APPROACH IN TEACHER EDUCATION

It is widely accepted (Brickhouse, 1990; Chen, Taylor, Aldridge, 1997; Hashweb, 1996; Laplante, 1997; Lederman, 1992; Pomeroy, 1993; Tobin, McRobbie, 1997; Yerrick, Parke, Nugent, 1997) that science teaching methods are influenced, consciously or unconsciously, by the teacher's own conception of scientific knowledge construction, evolution and nature.

Regarding mathematics education, Carrillo and Contreras (1995), Pajares (1992), and Thompson (1984, 1992) notes that the consistency observed between teachers' conceptions and the manner in which they present the contents reflects how their visions, beliefs and preferences influence their classroom practice. Along the same line, Moreno and Waldegg (1993) argue that mathematics teachers have their own ideas of what the discipline and its knowledge are, which exert impact on the teaching and learning processes and that "... Frequently, these opinions have been indirectly acquired or inherited during the person's own training; but, equally often, they respond to trends or international fashion which sometimes are incompatible with the former." (Moreno, Waldegg 1993: 654). The history of mathematics provides an ideal means of testing epistemological theories – personal or shared – and confronts them with teaching ideas. Analysis of different notations, application fields and modes of operation and validation of concepts at different moments in history, favors reflection upon one's own conceptions and allows for a better understanding of the processes by which concepts are acquired and mastered.

A quarter of a century ago, the general tendency in education was to present mathematics according to the logical organization of the discipline. This organization emphasizes the foundations of mathematics and its axiomatic-deductive systematization. Many of the teachers that are currently active were trained in this tradition. However, history shows us that in mathematics, as in all sciences, foundation problems are second level problems that arise only when one has fully engaged with the concepts. Thus, it is clear that foundation problems are not a suitable basis for mathematics teaching, since they only acquire meaning when practical functional difficulties arise that are impossible to overcome.

The workshop that we present and discuss in this paper was conceived as a way of provoking a metacognitive exercise for the teacher about the operations that are realized with mathematical objects, rather than about mathematics foundation problems. The workshop included didactic techniques of collaborative work and problem solving. The teacher was confronted with problematic situations whose solutions contained theoretical elements familiar to the teacher, but were not part of his or her routine application exercises.

THE WORKSHOP PARTICIPANTS

The workshop took place in the second semester of 2002. Twelve junior high school teachers (in charge of students aged 11 to 14) with more than 15 years of teaching experience participated. The participants were also enrolled in a master programme on education of mathematics. They all had a similar background, having graduated from the *Escuela Normal* – the Mexican national institution in charge of basic training for teachers – with a major on mathematics education. Their ages ranged from 37 to 45.

The curriculum of the *Escuela Normal* has been recently modified. However, when the participant attended the institution, training in mathematics was weak and in many cases restricted only to the topics that the future teachers were to cover in their courses. Hence, in many schools remedial programmes were implemented to reinforce the mathematical knowledge of the teachers that graduated from the *Escuela Normal*.

Didactic Design

The collaborative learning model used in the workshop is a modified version of Slavin's jigsaw puzzle model (1978), Jigsaw-2. This model facilitates collaboration by dividing the teamwork in equal and interdependent parts. Participants in the workshop become "experts" on a particular aspect of the main topic. They are also in charge of reporting the collected information to their team, or base group, such that all its members benefit from the expert's knowledge. In other words, each participant is responsible for learning something and for teaching it to the members of his or her team. The organization of such a group is depicted in Figure 1.

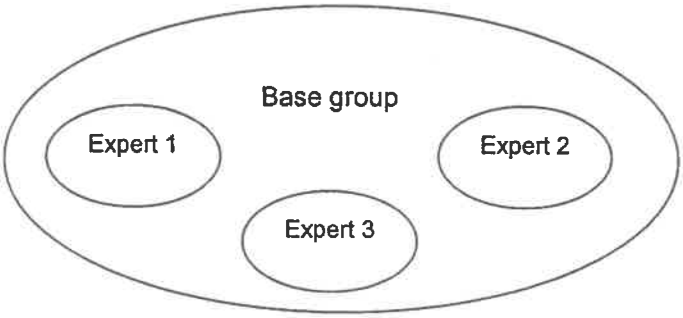

Figure 1. Organization of a collaborative group following the Jigsaw-2 model

Figure 2 summarizes the interactions between the experts of the different base groups, who work on the same subtopic and exchange information in order to achieve a better understanding of it.

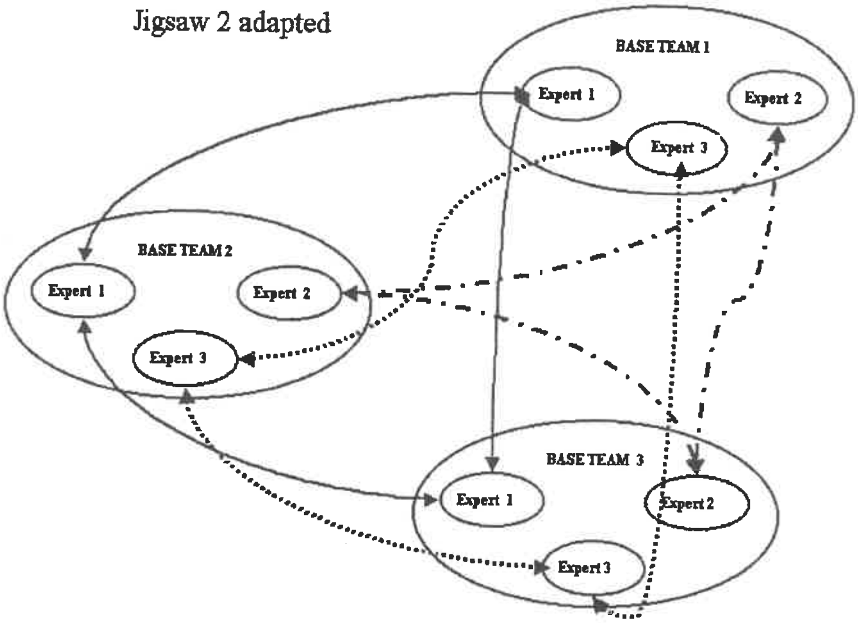

Figure 2: Jigsaw 2

The design of the workshop's content was based on the Nuffield Foundation's text *The History of Mathematics* (1994). We selected the following topics:

1.Babylonian mathematics

a.Numbering systems

b.Equation solving

2.Greek Mathematics

a.Constructions with compass and ruler

b.The Elements

c.Geometric algebra

3.Arabic mathematics

a.Equation solving

4.Descartes: Analytic geometry

a.Algebraic solutions to geometric problems

5.Beginnings of calculus

a.Infinite processes. Archimedes

b.The tangent problem

c.Quadrature problems

6.Negative numbers

Each of the eleven topics was studied in one session. The twelfth session was devoted to a collective reflection upon the process.

Sessions took place on a Saturday, when school was off for the day. Three problems on a particular topic were selected for each session from Nuffield's text. Each problem had to be solved independently by one of the three teachers that formed a base group, and it was later discussed with the other team members and with the whole group. Activities were given a fixed amount of time and were performed in a predetermined sequence. They had to be fully completed within the four hours of the session. The tasks that teachers had to perform in each session are summarized in Table 1.

Sequence	Task	Work modality	Time (minutes)
0	Read information about the historical context	Individual	30
1	Solve the problem, including counseling and interaction with other "experts"	Individual and collaborative between experts	45
2	Explain the problem and its solution to base team members (each team member is responsible for their colleague's learning)	Collaborative, base team	30
3	Analyze the solutions proposed by all team members and decide if they are adequate.	Collaborative, base team	30
4	Analyze the concepts involved in the problem and identify the encountered difficulties.	Collaborative, base team	30
5	Explain the problems and their solutions to the whole group and compare the different answers. Discuss.	Collaborative in plenary session	45
6	Write a diary entry with personal thoughts on the session and possible usage of the experience gained in lectures for students.	Individual	30

Table 1

The theoretical basis for our proposal can be found in constructivist experiences, such as Fosnot's (1996), which proposes that:

…teacher education needs to begin with these traditional beliefs and subsequently challenge them from activity, reflection, and discourse in both

> coursework and field work [...] participants need experiences as learners that confront traditional views of teaching and learning in order to enable them to construct a pedagogy that stands in contrast to older, more traditional held views (Fosnot 1996: 206)

Diaries were analyzed weekly by the facilitators and designers of the workshop, and contrasted with notes taken during the session.

Problem Solving

In selecting the problems to be treated in the workshop, we targeted the following characteristics:

1. Problems should use unconventional notation (e.g. Babylonian fractions)
2. Problems should entail unconventional algorithms (e.g., Heron's algorithm to compute the square root)
3. Problems should be stated in an unconventional fashion (e.g., Arabic inheritance problems)

After reading about the historic context, the teacher had to solve the problem individually and reach a well-founded solution. At this first stage, the teacher could interact with teachers in other base groups whose assigned problem was the same. In the second and third stages, the base team explained and discussed the solutions given to the three parts of the problem by the experts. The fourth stage was devoted to a scholastic reflection where teachers searched for similarities between the analyzed concepts and those that they would need to teach, trying to match their own difficulties to those that their students would encounter. In the fifth stage, teachers discussed the problems (all base teams had solved the same problems) and the different solutions, so that in the sixth stage they could reach their individual conclusions and enter them in their diary.

The Teachers' Reactions

The individual resolution of problems proved to be, in general, a difficult task for the teachers, since they found themselves in conditions similar to their own students' who confront a novel situation. These difficulties were part of the didactic design of the workshop – teachers were required to face new notations, new algorithms and unfamiliar problem statements. These difficulties reproduced, as the teachers acknowledged, those that their students have to face. Many of the teachers' reactions were similar to students' typical strategies: for example, searching for keywords in the problem statement in order to "guess" the operation to use; making use of "external representations", such as icons or finger-counting, as aids in computations; appealing to "trial and error" or trying to "translate" the algorithm in terms of well-known algorithms. These reactions helped the teachers realize that they were in the same "apprentice" position as their students and made them more understanding of the difficulties that their pupils confront.

One teacher commented, for example, that she finally understood the difficulties that children face when they are required to memorize multiplication tables: she had felt the same pressure when asked to work with sexagesimal tables, where symbols do not represent the values that she is familiar with.

Similarly, the workshop allowed the teachers to identify some of their beliefs about mathematics learning and to contrast them with their own learning processes. The impressions of one of the participant, recorded in her diary, illustrate this epistemological reflection on this mathematical activity:

> *Margarita.* - I always thought that a pupil's goal in the classroom should be to produce correct answers, and that my duty was to help the student achieve this. Today I realize that the activities the students need to perform in order to obtain the answer are more important than the answer itself, and that mistakes teach more than correct answers to both the teacher and the student...

Or the teachers' thoughts about didactic techniques:

> *Rubén.* - I now know that it is very important for the student to feel that he or she is part of the process, and not only a 'processor' of facts and rules that must produce a correct answer for the teacher.

> *Francisco.* - This week I learned the potential of collaborative learning (which I only knew from books). Group work facilitates learning because defending our ideas forces us to search for arguments that can convince us we are right (or wrong).

However, not all the results of the workshop were as expected. Some teachers were not able to transfer their role as a student to their role as a teacher, and tried to apply textually, in their classroom, the problems that they had solved, yielding inappropriate results (e.g. teaching numbering systems departing from non decimal bases). In some cases, the teachers did not achieve, within the four hours of the session, a consensus on the "correct" solutions to a problem, which required the facilitator's intervention. There were also flaws in the design of the workshop. For instance, not all problems aroused interest among the participants, or not all parts of the problem had the same degree of difficulty.

CONCLUDING REMARKS

Our experiences in using history in the didactic design of a workshop for training in-service teachers have convinced us of the benefits of this approach. Even if the results we obtained cannot be generalized, they allow us to set the foundation for new ways of dealing with the problem of teachers' training – not only for active teachers, but also for teachers in their initial stage of education.

The combination of didactic techniques of collaborative work and problem solving proved to be motivating for most of the teachers and kept them interested in the task during the four hour sessions – which, in several occasions, were extended to 30 or 45 minutes more.

Comments from the teachers showed a clear comprehension of the problem's conceptual elements and of the cognitive problems of their students. The teachers were made conscious of the teaching and learning processes, which they had not been made aware before.

Naturally, a follow-up stage needs to comprise the observation of classroom practices to verify if there is any effect on the teaching of the teachers who have attended the workshop.

There is a major difficulty in the didactic design of the workshops: it is necessary to build a database of problems taken from history, that can be divided into a number of parts equal to the number of team members so that they all work in a coordinated fashion on sub-problems with equal level of difficulty. Contributing to such a database will be a task for mathematics historians.

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Learning-and-Teaching Project in the History of Mathematics for Pre-service Teachers: Educational and Multicultural Enrichment of their Academic Curriculum

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Abstract: *We present structured academic activities combining learning and teaching of mathematics with the history of mathematics built up as a part of our students' education program. The learning part is implemented in the yearlong seminar related to the history of mathematics. The teaching part is the graduate project, during which each topic is elaborated into a teaching module at a school level for an appropriate grade. The students at the Achva College come from a variety of cultural and ethnic groups (we mean here "cultural" in the broadest sense of the word specified in the paper). It appears that our approach helps to overcome numerous prejudices related to the mathematics both as a part of human culture and as a field of personal academic and professional life. Focusing on eminent mathematicians - e.g. Lobachevsky, Pascal, Fibonacci, Hilbert, Archimedes, Al-Hwaresmi, Russell, Omar Al-Qhayam, as well as on non-mathematicians who did mathematics, like Leonardo da Vinci, Napoleon, Albrecht Dürer, and on social frameworks of some mathematical developments, like the Renaissance or the French revolution, - helps to bring a student at both levels we are involved with: in the teachers' education program and at school – to start forming a more or less adequate idea about the essence of mathematics, its relations to other fields of knowledge and to social and cultural phenomena. In analyzing the multicultural aspects of mathematical education, we relate to Banks' (1989) parameters characterizing them.*

Keywords: eminent mathematicians at school, history of mathematics in teachers' education programs, multicultural value of math education, overcome prejudices concerning mathematics, teaching history of mathematics.

"*Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country.*" D. Hilbert

INTRODUCTION

What we present here is the sequence of structured activities combining learning and teaching of mathematics with the history of mathematics. The activities are built up as a part of our students' education program, and we present the rationale behind these activities, some of the outcomes and our conclusions and observations.

In the whole, we watched our students on their way through two consecutive stages: the course in the history of mathematics during the second year of the program, and the complex seminar-and-project unit during the third year. This complex unit combined the seminar in which an academic review was accomplished as a seminar work, with a teaching module based on the seminar topic, planned and taught at an appropriate school-teaching level, as an implementation of all the academic and professional knowledge acquired by a student up to this point.

The combined unit in history of mathematics focused on outstanding personalities related to mathematics:

- Eminent mathematicians and their role in the development of mathematics: Lobachevsky, Pascal, Fibonacci, Hilbert, Archimedes, Al-Hwaresmi, Russell, Omar Al-Qhayam, Fibonacci, etc.

- Mathematics by non-mathematicians: Leonardo da Vinci, Napoleon, Albrecht Dürer.

- Mathematics during and after the French revolution: Monge, Fourier, Carnot; unified measure system.

The cultural background of the Achva population. The students at the Achva College come from a variety of cultural and ethnic groups (we mean here "cultural" in the broadest sense of the word, which will be specified later). This leads to interactions between various groups of students whose previous activity did not bring them together, at least to an extent that demanded close and prolonged contact. For example, on the one hand there are students who come from religious or traditional Jewish families, and on the other hand those who have obtained a purely secular liberal upbringing. Other examples are those of interaction between Moslem, or Christian students of non-Jewish origin, and Jewish students; between new (or relatively new) immigrants and Israeli-born students; between freshmen starting their higher education, and experienced teachers who complete their education within an in-service program, etc. Students coming from these different groups sometimes have little knowledge (if any) of each other's lifestyles, history, learning habits. This is usually the starting point of their closer acquaintance during the four years of the teacher- educating course as future math teachers[1].

MULTICULTURAL MATHEMATICS EDUCATION - MAIN CONCEPTS AND GOALS

To begin with, we would like to cite here Collins' (1992) attempt of definition of multicultural mathematics. In doing so, we emphasize once more the broad sense in which we relate to the very notion of **culture** in every aspect being referred to in this paper: "**Multicultural mathematics** is best defined by what it is not; it is not ethno-mathematics, nor simply Afrocentrism or Eurocentrism". We start by relating to Banks' (1989) five parameters of **multicultural mathematics education:**

- **Content integration,**
- **Knowledge construction,**
- **Prejudice reduction,**
- **Equitable pedagogy,** and
- **Empowering school culture and social structure.**

Realization of each of these parameters presents a challenge for an instructor, and we shall briefly refer to each of them:

Content integration: We regard this as an ability to use several sources of disciplinary knowledge, and to bring these sources together to form a new piece of information. Moreover, we expect a student to be able to see the interrelation between *seemingly different disciplinary contents*, which is an essential part of the inter-disciplinary approach undertaken in the seminar, as it is to be described herein. When one of the two disciplinary components is of seemingly outer-mathematical nature, like history, the content integration is even less obvious and presents a real challenge.

Knowledge construction: An ability to read and understand literature intended for mathematic teachers, struggling through new ideas and concepts, and to transform this into a piece of pedagogical knowledge acquired by the teacher to the

[1]A more or less similar cultural distribution is valid for the academic staff of the college, as well. To be more specific, speaking of the staff of the Math Department, one finds among its members representatives of all ethnic and religious groups mentioned above. But, referring to the culture in a broader sense, one cannot neglect relevant habits and concepts accumulated by members of the teaching staff during years of academic activity, as an additional important cultural factor. Among those, reading professional literature, the need to maintain professional and personal contacts within one's field of activity on the basis of mutual respect and dignity, etc. This component is less dependent on the background of an individual, and is very important in the multicultural interaction related to the academic activities of the students.

extent that enables him or her to teach it.

Prejudice reduction: We refer to various kinds of prejudice. One of these originates from ignorance concerning a cultural basis of another country, nation or epoch. Another one relates to one's own professional abilities: students tend to underestimate their academic competence, out of fear of confronting a challenge that may seem too difficult before they have succeeded in attaining it for the first time. This is typical of students, whose academic-cultural background has not yet provided them with an appropriate experience at least once, for example, for freshmen or sophomores participating in advanced courses.

Equitable pedagogy: We regard this as attainability of professional knowledge for every mathematics teacher, regardless of the fact whether this knowledge has been obtained at a college, or by one's own effort. Moreover, a teacher should feel him- or herself able and competent enough to judge what may be an interesting, non-routine way of presenting his field to his pupils.

Empowering school culture and social structure. We expect this to be provided by the ability of a teacher (in particular, by a math teacher), to work in harmony with his colleagues in order to maintain cooperation within the school staff in both disciplinary and pedagogic contexts.

We shall focus on attainability of these aims in the framework of the combined teaching-learning unit on mathematics and history of mathematics described herein.

RATIONALE AND AIMS

The rationale behind these contents, and behind focusing on the teaching of the history of mathematics on the whole, was related to several issues arising in the math teacher education in general, and in the teacher education in Israel, in particular:

- The great diversity of cultures presented in any Israeli educational institution, and at the Achva College in particular, hence, the enhancement of the role of mathematics as a common language for all.
- The complete, or almost complete lack of acquaintance of students with outstanding personalities in mathematics prior to their studies in the college, including those originating from their own cultures. E.g., most former Soviet Union students have never before heard of Lobachevsky, and most Arab students have never heard of Al Hwaresmi. It occurs more than once during their college years, that students come to know facts of their own culture, of which they have been unaware before.

- The perception of mathematics as an abstract distant field, whose laws are detached from any historical, physical and social reality, and of mathematicians as queer strangers, who arrive at their somewhat peculiar discoveries for some obscure reasons and aims.
- The perception of mathematics as a field of activity not altogether human: one may be involved with it for practical reasons, i.e. to earn money by teaching it, or because some utilitarian computations are needed, but one is not expected to connect one's pleasure or leisure to it.
- The very narrow perception of mathematics as a field related only to numbers, or figures.

In order to deal with these beliefs, we strive to pour some light on a wide circumstantial background of mathematics in its evolution in time and space, and on its creators as human beings, who have lived in different epochs and countries, and who had discovered, or developed pieces of knowledge that have outlived times and spaces, serve us and will serve the future generations as well. Moreover, we find it important that our students gain confidence in both possibility and necessity to transfer this new knowledge to their pupils at school.

Hence, the structure described herein of the studies in this area.

The Structure And Content Of The Complex Learning-Teaching Unit

The **learning** part of the unit is implemented in the seminar in mathematics. In general, the seminar in mathematics at the math-teaching department of the Achva College is an academic course on an interdisciplinary basis centered in mathematics. What we present here is the yearlong seminar that was related to the history of mathematics.

The **teaching** part of the complex unit is implemented in the graduate project. In the course of the graduate project linked to the seminar, each topic is elaborated into a teaching module at a school level for an appropriate grade.

The seminar on the history of mathematics was an opportunity for each student to get to know more closely the mathematician chosen for his or her seminar work and project, his country, epoch, and some of his scientific achievements within the reach of the student, accounting for his or her mathematical background.[2] The

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[2]The students are always surprised to find out that even most advanced mathematicians still produced some items of mathematical knowledge within their reach.

teaching module built on this basis rendered the theoretical knowledge acquired by the student to be relevant for his or her future professional activity. We find that in the primary school, the teaching modules in the form of learning centers are preferable both for the pupils and for the teachers, as far as this type of content is concerned. The reason for this preference is mostly didactic and functional: the learning centers once presented in the classroom are permanently available for the pupils' activity, and every pupil can learn from them on his or her own, with the guidance of the teacher, if needed. If a learning center is adequately constructed, it contains learning materials within abilities of most pupils of at least two adjacent grades, including a rich assortment of information and assignments in various fields at an appropriate level. It can be planned in a way not too strictly hierarchical in the knowledge construction and thus does not necessarily imply a consequential learning procedure. Furthermore, it includes assessment tasks that enable a teacher to follow-up each pupil's successes and difficulties.

Each learning center contains information on the history of a mathematician's epoch and country, his biography in a most possibly interesting presentation, with pictures and stories of his life, and some of his mathematical discoveries, emphasizing those comprehensible at the ages of primary school.

As to the secondary school projects, we found it appropriate to focus on the content of the project, enabling the future teachers to build it as an extensive enrichment module, reaching other fields of human culture and history and relating them to mathematics.

We present here a selection of topics, to provide a basic idea of the seminar and project activity, and examples of layouts of the mathematical contents of two selected projects: one for the primary school and one for the secondary school.

Table 1: A selection of topics of the seminar and project activity for primary and secondary school levels

Mathematician	**Main achievements as presented in the theoretical seminar work**	**The school level for which the project was elaborated**	**Mathematical issues reflected in the project and taught at school**
Hilbert	Presentation of the 23 unsolved problems of mathematics – among them *Fermat's Last Theorem* and the *Four Color problem*, as those whose assertions are most comprehensible for the students	Primary school	-Graphs and map coloring - Goldbach conjecture
	-Modern axiomatics of geometry -Using graphs to study the map-coloring problem	Secondary school	.- Existence of unsolved or unsolvable problems - Fermat's last theorem - Graphs and map coloring - Goldbach conjecture
Euclid of Alexandria	- First axiomatics of geometry - Algorithm of finding the GCD - Method for finding perfect numbers - Infinity of primes proof	Primary school	-Algorithm of finding the GCD -Perfect numbers
Al-Hwaresmi	- Introducing the Indian numbers representation - Algorithms for decimal computations. - Geometry-based solution of first-order equations - Astronomy studies	Primary school	-Geometry-based solution of first-order equations
Archimedes	- Archimedean solids - semi-perfect polyhedra - Evaluating and representing big numbers - Archimedes' spiral - Geometric constructions using compass and ruler	Primary school	- Archimedean solids - semi-perfect polyhedra - Evaluating and representing big numbers - Archimedes' spiral - Geometric constructions using compass and ruler
	- Archimedes' mobile – solving equation by equilibrium		- Archimedes' mobile – solving equation by equilibrium
Sir Isaac Newton	- Calculus of infinitesimals - Minimum – Maximum problems - Areas and volumes computation - Gravitation law - Conics	Primary school	- Areas computation - Gravitation law
Pythagoras	- Pythagorean theorem - Pythagorean triples - Pythagorean solids - perfect polyhedra	Secondary school;	- Pythagorean theorem - a number of proofs - Pythagorean triples -Pythagorean solids - perfect polyhedra

Table 2: Examples of layouts of the mathematical contents of a project for the primary school

Mathematical topic developed by Archimedes	**Didactic activities in the learning centre at the primary school**
1. Evaluating and representing big numbers	"Popcorn" activity – trying to estimate the number of popcorn grains in a given volume
2. Archimedean spiral	Cutting the paper spirals
3. Geometric constructions using compass and ruler	Actual construction of geometric figures using the compass and ruler

The mathematical focus of the enrichment project related to Leonardo Da Vinci for the *secondary school* was to the geometric cuttings of polygons and computation of areas of curvilinear figures in the spirit of Willis's book (1995).

Its brief layout is as follows:

- Cutting and rearranging a rectangle into a square;
- Cutting and rearranging any triangle into a rectangle;
- Cutting and rearranging any polygon into any polygon.
- What is preserved under these transfigurations?
- How can we use the same idea for non-polygons?
- Leonardo's cuttings: why "no pi"? Impossibility of squaring a circular region: a brief history of the problem. Does it doom squaring of any part of a circular region impossible?
- Who was Leonardo da Vinci? A review of his epoch, life and activities as an artist and as an engineer, scientist, poet, musician. A notion of a "Man of Renaissance".
- Activities on the bases of the ideas from the book, analyzing and emphasizing the preservation of the area.

A special project on the French Revolution. A somewhat special example of the complex collaboration between students, is a project on mathematics during and after the French Revolution, that was accomplished jointly by four third-year students of the secondary school teacher-education program. The topics for this project were elaborated by the students separately, and were dedicated to Monge, Fourier, Carnot and to the unified measure system. The joint part of the students' work was centered on the social and cultural changes during and after the Revolution, and revealed the impact of social processes on the development of sciences – both applied and pure. The fact that some of the scientific accomplishments (mainly those by Carnot) were in physics fit into the general inter-disciplinary nature of the seminar.

CARRYING OUT: FEARS, DIFFICULTIES AND SUCCESSES

We shall relate separately to each part of the complex unit.

The seminar work. The first part for a student to confront with is the theoretical – learning – part, which is the seminar.

An essential feature of this part is the complete, or almost complete lack of previous knowledge by students in the suggested topics. Thus, a student is expected to learn a new subject from several sources, to integrate the information obtained, and to compose a self-contained written and oral presentation on the issue. This is done in the form of tutored self – education. The students find various sources to learn from: books, papers, web sites, etc., at various scientific levels, some of these fit their academic abilities, and some others are above them. They have to struggle through these sources, to understand their contents, and to be able to "filter" and synthesize them, keeping in mind two ultimate goals of their toils: an oral and written presentation of the theoretical part, and the school teaching module to be built up on the basis of the theoretical knowledge acquired. Historical and cultural aspects add to the interest of a student in his or her work, helping him or her feel closer to the personality of the chosen mathematician, but they are to be learned and integrated as well, which is far from being easy.

The seminar instructor's part at this stage is to help the students out of an almost inevitable confusion, to teach them to read and understand texts, which may be above their academic training, in a way that enables one to understand the general idea without getting bogged down by the technical details, and to compose an integral unit out of pieces of mosaics picked out of those sources. We assert that the most important output of this part is to provide one with a primary experience of replenishing one's professional stock of knowledge by oneself and with a deeper insight into various fields of knowledge through mathematical, historical and scientific literacy.

One of the inevitable problems is reading scientific texts in English, which is a foreign language learned at school and in the college. The assortment of appropriate materials in Hebrew is usually not sufficient. Besides, we induce the students to go through this experience in order to teach them to read professional texts in English and to overcome their anxiety about the issue.

The graduate project. In the framework of the graduate project, the teaching module is elaborated on the basis of the seminar and taught during the continuous teaching-practice session at school.

In the process of elaborating it, the students also go through several stages.

At first, they formulate the aims of teaching the part of scientific heritage of the mathematician they have come to know. In doing so, they keep in mind several aspects:

From the educational point of view:

- Enrichment of the pupils' general education through more or less close acquaintance with the personality of the chosen mathematician, his life and time;
- Showing mathematics as omnipresent and equally challenging in different times and cultures;
- Broadening the pupils' perspective of the range of themes included in mathematics;
- Drawing links between mathematical discoveries of different times and epochs (e.g. geometry from Euclid to Hilbert);
- Embedding of mathematical and, in general, scientific, research in historic and social processes and phenomena.

From the pedagogical and didactic point of view:

- Development of pupils' basic skills of self-instruction using the materials worked out for that purpose by the students;
- Integration between pieces of knowledge coming from various sources (e.g. historical maps and atlases, information cards, web sides), and from two different fields of human knowledge represented at school within different lessons.
- Development of various forms of social learning processes: small groups, individual learning, etc.
- Utilizing one's spare time to learn something new on one's own.

From the cultural and multicultural point of view:

- Human culture and history: exposition of the school-pupils to the variety of cultures mostly unknown to them, in the context of mathematical creation; emphasizing dignity and mutual respect of various cultures and nations.

- Culture of learning: creation of learning habits and skills; arousing of curiosity, expanding of mind horizons; using special learning environment in mathematics and math history for learning and multicultural collaboration between pupils.

- Academic and general culture: presenting mathematics as a field of human historical heritage, as an integral part product of social creative processes.

The next stage is to identify the mathematical achievements of the chosen personality that can be made perceivable for the children of the given age, accounting for their mathematical background, and to sketch the mathematical contents of the learning unit.

The contents thus outlined are then planned and divided into information cards, or lesson plans - depending on the chosen teaching framework, - and assignments and tests of the teaching module. In doing this, the students try to foresee the possible difficulties and the ways to cope with them. They also have in mind a variety of activity types to fit preferences and learning abilities of various learners of the class.

ASSESSMENT

The assessment of the whole unit consists of two main components: the formative assessment during the process of elaboration of the unit, and the summative assessment of the output products. Each component in turn refers separately to the theoretical part, which is the seminar work, and to the didactic implementation, which is the teaching project.

The formative assessment of the theoretical part consists of constructive recommendations on several "iterations" of the work done. The "iteration" stages required from every student are:

- *Primary selection of sources: books, papers, essays, web sites etc.* The student is supposed to perform primary superficial reading to get an initial idea of the variety of sources, of their level and style, of the information contained in them and so on. In doing this part of work, he or she either gets trapped, or gets lost in vast and unfamiliar texts, terminology, names, epochs and notions. At this point the students consult the instructor as to the reasonable selection of sources and issues in the sources, which guides the contents of the future work. The instructor's part at this stage is to guide a student so that he or she cope with sources of the level attainable to him or her, and to restrict the amount of new information to some reasonable amount appropriate for the requirements of students' work.

- *The tentative structure of the future work*, its subdivision and main sections. This stage helps a student to structure the future composition and to see it in the whole.

- *The brief oral outline of the work before the group*. This is done so that everybody can get an impression of other's work, and provides an input of everybody to everybody concerning the clearness of the main ideas and the configuration of the work, as it seems to shape. The necessity to present orally the outline of the work before the group and the instructor disciplines the student and is an essential stage of work.

- *Presentation of the draft of one chapter chosen by a student as a most representative one, for the judgment of the instructor*; no marks are given, and the detailed notes and commentaries of the instructor on this chapter reflect the requirements of the final written work. The final work is not returned to the student and not corrected.

The summative assessment of the theoretical part is based on several components:

- The academic quality of the written essay, which is essentially a scientific review on the chosen topic. This implies educated integration of historical and mathematical pieces of information, with emphasis on those understandable for the student; logical structure of the text; clear and consistent layout of ideas; accurate and adequate terminology usage; vast and relevant bibliography; the student is supposed to outline in detail some mathematical ideas of the mathematician he or she has studied: problems solved or posed, theorems proven, notions introduced etc.

- The quality of the oral presentation, which is a 40-45 minutes lecture based on the seminar work.

The relative weights of these two parts are: 60% for the written work, and 40% for the oral presentation.

The formative assessment of the project is related to the working stages described above. The summative assessment of the project refers to the following components:

- *Pedagogical skills*: understanding and implementation of the pedagogical principles and concepts involved in the project; usage of theoretical knowledge and relevant concepts acquired in the course of pedagogy;

- *Strategic knowledge*: adequate sources' usage; consciousness of one's own strong sides and difficulties as they appeared while carrying out the project; ability to improve one's performances upon reflection on them;

- *Communication*: awareness of meanings and messages conveyed both by learners and instructors; ability to arrive at mature conclusions and adequately formulate them; style and level of written and oral speech.

Reaction Of Students

After the session of oral presentation, which takes a daylong marathon of 40-45 minutes lectures, the students are requested to fill in a questionnaire anonymously assessing the course and the final presentation. An essential part of the questionnaire is the verbal part in which they are encouraged to express explicitly their impressions and opinions. Here are some of these, translated into English:

"I never knew there was so much to learn in fields related to mathematics!"

"I never expected to have been able to present such a lecture, and all this on my own!"

"When I chose a topic, I was very pessimistic as to my chances to succeed. The only thing that supported me was that everybody was equally pessimistic, and that I knew the previous groups had successfully overcome the same difficulties".

"This presentation was a real mathematical festival".

These expressions of acquired self-esteem and self-confidence are best appreciated keeping in mind what Shoenfeld has formulated as "**Belief 3** Only geniuses are capable of discovering or creating mathematics" (Shoenfeld 1985), which had been almost all students' conviction prior to their experience.

Consequences For The Students As Future Teachers

The history of mathematics is one of possible topics of the inter-disciplinary seminar-and-project complex unit. The central topic of the unit is being changed every year, and varies between several fields. The unit presented herein, was the second endeavor in history, and we have learned from our graduates, who have participated in the first one that they utilize the knowledge acquired in it in their teaching practice.

The department has worked out an electronic library of projects that are the product of such courses, especially of their didactic components, so that they are accessible for all our graduates to be used at school.

Some of the consequences of these activities for the future teachers are as follows:

- The fact that these and other names have become whole-world intellectual property inspired a positive interaction within the learning group. Acquaintance with original sources of elementary mathematics: Greece, Ancient East, Asia, medieval mathematics, - enhanced this effect.

- Simultaneous presentation of mathematical, artistic and technological achievements of artists like Leonardo da Vinci and Albrecht Dürer made it necessary to get more or less closely acquainted with Italian and German Renaissance as epochs in world culture and science, and elucidated the notion of the "Renaissance man".

- The seminar highlighted the omnipresence of mathematics and presented mathematics as having been contributed by all cultures, nations and in all epochs. In this connection, we cite here McFadden et al. (1997, p.1): "Multicultural education strives to value and respect the uniqueness of persons within a common human community. It expresses the democratic ideals of equality, of unity within diversity....", and C. Zaslavsky (1994, pp.3-4): "Teachers can enrich the mathematics curriculum by taking examples from our own diverse society and other societies. A multicultural perspective can also be adopted through the study of the history of mathematics." This last citation is more than relevant for the highly multicultural Israeli society.

- The revelation of interrelations between the development of numerous fields of mathematics and historical and social processes (e.g. the French Revolution) changed the attitude towards mathematics among the future teachers, and as a result, to their own future profession as one closely related to the society and to the human culture.

- The students become more aware of a variety of possible alternatives to a teaching – learning routine, both in contents and in the form and methods, and have an opportunity to experience the whole process of planning and implementing one of these alternative forms relevant to the contents of the lesson and to its purposes.

The students have overcome some of deep prejudices they had owed prior to this experience. We shall refer here explicitly to only one kind of prejudice – that which originates from under-estimation of one's own professional abilities: students tend to misjudge their academic competence, out of fear of confronting a challenge that may seem too difficult before they have succeeded in attaining it for the first time. Speaking of students' professional prejudices concerning their academic abilities, one has to emphasize that not only each individual relates to himself, but also to the group as a whole, being skeptical as to the adequacy of the 3rd year students' level to meet performance expectations. After oral presentations by all participants have taken place, they are much more optimistic about their abilities and gain greater self-esteem: "We can do it!" "It's both useful and enjoyable!". On the whole, this experience encourages students to read and learn mathematics beyond a minimum required for graduation, by giving them faith in their capabilities. This is achieved as a result of developing those academic habits and views that are part of the professional and general culture to be accumulated throughout our graduates' professional lives.

The main value of the course as we see it, apart from attaining its purely academic goal, is simultaneous integration in several dimensions: integration between seemingly unrelated fields of knowledge, between various ways of learning, along with cultural and social integration, which emerged as the learning process evolved. This last helped the students to overcome the difficulties of the course and encouraged them to overcome prejudices of various origins.

Another valuable outcome is the experience in working out theoretical knowledge acquired by self-education, into a school teaching module. Keeping in mind that our students are future teachers, we encourage them to believe in abilities of their pupils not less than we have justly believed in theirs, and to bring them interesting and challenging materials within their reach. Moreover, historical and cultural context being transferred from the college to the classroom during the math lesson enriches the learning routine. We know our graduates to successfully practice this enhancement of the learning routine.

Discussion Of The Multicultural Aspect

The value of our experience described above, is amply supported by numerous opinions of teacher educators who have been involved in this issue. See, e.g., Collins (1992): "Pre-service education for teachers is the best opportunity to introduce a multicultural perspective on mathematics. Multicultural mathematics education should be taught in in-service programs as well. Regardless of the depth of instruction necessary to acquaint teachers with multicultural mathematics, the important ingredient is a commitment to inclusion". Students at the seminar-and-project unit come both from pre-service and from in-service systems, since in Israel there are many teachers who complete their academic degree during their in-service training. It is worth mentioning that the multicultural approach has not been the major goal of the seminar, but rather a "byproduct", which proved its own educational and cultural value

Last but not least to be emphasized again is the development of academic and learning culture as an integral component of the teachers' education process. This includes various learning abilities enhanced by the experience of the work over the unit, including self-instruction abilities that seem to us almost vital for teacher to continuously develop during his or her carrier; this also includes re-estimation of learners' abilities at all levels, towards higher esteem and self-esteem. This re-estimation leads to higher level teaching and learning procedures planned and implemented by the teachers. The learning and academic culture of a school math teacher is also an almost crucial component in developing the higher-level learning procedures environments for the future pupils.

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History of Mathematics in Digital Kinematic Mechanism Collection

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Abstract: *Kinematic mechanisms were a technology used in teaching during the 19th century. They were designed based on mathematical ideas and used for teaching and the invention of numerous practical machines. Cornell University has digitized a famous 19th century kinematic mechanism collection not only for its archival value, but also because there is a growing interest in modern machine design, for exploring kinematic properties of mechanisms, and the applications of mathematical ideas behind them. One part of this project (KMODDL project) is to unpack the mathematics imbedded in the mechanisms and to develop educational materials for use in mathematics classrooms at all levels. This material also helps to demonstrate interdisciplinary connections between mathematics, mechanics, and engineering from a historical perspective. We are doing research on what mathematical ideas lie behind them. In some cases we can find references to the ideas, but for some models we have not been able to find any information about what the ideas were that precipitated the model. We are continuing work on the KMODDL project now, developing more tutorials for use by teachers and their students. Also, in conjunction with the Museum of Science in Boston we are designing didactical exhibits (for secondary school students) that will involve the possibility of 3-d printing of the models.*

Keywords: applications, constant-width curves, digital collection tutorials, history, inversions, involutes, kinematic mechanisms, Reuleaux triangle.

INTRODUCTION

Usually when we talk about using technology in teaching we think about it as something that has been developed in the 20th century. Kinematic mechanisms were a technology used in teaching during the 19th century. They were designed based on mathematical ideas and used for teaching and the invention of numerous practical machines. Later they were forgotten. During the 20^{th} century, machine design was mostly concentrating on the invention and use of new materials and the introduction of electronics, but this direction seems to be getting close to its limits. Now, 21^{st} century machine designers are going back to rediscovering mechanical principles, which are closely connected with mathematics. Cornell University has digitized a famous 19^{th} century kinematic mechanism collection not only for its archival value, but also because there are growing interests in modern machine design, for exploring kinematic properties of mechanisms, and the applications of mathematical ideas behind them. One part of this project is to unpack the mathematics imbedded in the mechanisms and to develop educational material for use in mathematics classrooms at all levels. Using this knowledge in classrooms gives a different perspective on applications of geometry and raises student interest about some particular problems. This material also helps to demonstrate interdisciplinary connections between mathematics, mechanics, and engineering from a historical perspective.

KMODDL DIGITAL LIBRARY PROJECT

Our project *The Kinematic Mechanisms for Design Digital Library* (K-MODDL) (see [9]) represents Cornell's collection of mechanisms as photographs, interactive movies, interactive simulations of their movements, and also downloadable files for printing 3-D working models. Just as importantly, the project restores the mechanisms to their intended classroom use as virtual teaching models of geometric and kinematic principles and gives a chance to explore the history of mathematical and mechanical ideas behind the models. K-MODDL is a freely accessible, Web-based resource. The core of the K-MODDL collection is more than 250 mechanism developed by Franz Reuleaux (1829-1905) (Figure 1) specifically for use in the classroom.

In our work on developing tutorials we found many interesting facts that allow bringing together mathematical knowledge and its application to engineering (for example, the story about James Watt searching for solutions of the straight-line problem, or about the use of Geneva wheel mechanism in early movie projectors). At the same time we experienced certain difficulties trying to balance the differing cultures of the librarians (who saw the project as building a digital library) and of the academic content people (who saw the project as producing material useful for education). The original vision from the library side was to have all developed materials in a unified format, which became a problem in some cases. For example, we experienced difficulties in posting mathematical formulas on the web within the unified format because there is still no generally accepted way to do so. We used MathType, but it produced results that are not viewable in some browsers. Entries were designed according to library standards, but they were not attractive to students, as was shown also in some of the preliminary evaluations of our project [17] Therefore still some of our materials are not in final polished form. We are also looking for feedback from educators after they have used our materials in classrooms with their students.

FRANK REULEAUX, MECHANICAL ENGINEER WITH MATHEMATICAL THINKING

Figure 1: Franz Reuleaux

Reuleaux laid the foundation for a systematic study of machines by defining clearly their basic building blocks (mechanisms), and developing a system for classifying known mechanism types. Reuleaux created an abstract symbol representation of machines, which are built from a combination of mechanisms. Each mechanism was described by a collection of symbols, which formed a 'word'. A complete assembly of mechanisms into a machine is then a 'sentence' of these 'words' in this symbol language (see [10]). In the 20th century the same approach in codifying problems was used in information theory and theoretical computer science. The models in the Cornell Reuleaux Collection are classified according to the alphanumeric schema employed in the catalog of the manufacturer, Gustav Voigt. The letter in a model's ID (e.g., B14 or S35) refers to a class of mechanisms; the number is a specific instance of the class. This classification scheme is a simplified version of the taxonomy of machine elements elaborated in Reuleaux's work. For example, Reuleaux triangle (see section 4 below) can be found in B-series, but it is also in L-series as a cam. Straight-line mechanisms (see section 6 below) are in S-series.

Reuleaux believed in the use of demonstration models to express mathematical and kinematic ideas. Following some ideas of F. Redtenbacher(1809-1863) and R. Willis(1800-1875) and adding his own, Reuleaux built a large collection of 800 mechanism models in Berlin and marketed 350 of them to universities around the world. Collections of Reuleaux mechanisms were widely used in Europe, especially in Germany, from the 19th century up until the Second World War. Most of the mechanisms in Europe were lost in the destruction of 1941-45. Cornell's Kinematics Model Collection seems to be the largest remaining collection of Reuleaux's model-mechanisms. The first president of Cornell, A. D. White, acquired the models in 1882. This collection has now been digitized (see models [9]) as part of the KMODDL project.

CURVES WITH CONSTANT WIDTH

Franz Reuleaux incorporated mathematics into the design and invention of machines in his work *Kinematics of Machinery* [11]. Reuleaux had brilliant ideas about the synthesis, optimization, and aesthetics of machine design. To mathematicians he is best known for the "Reuleaux Triangle", which is the simplest noncircular curve with constant width. The "width" of a closed convex curve is defined as the minimum distance between parallel lines bounding it. Curves of constant width have the same "width" regardless of their orientation between the parallel lines.

A Reuleaux triangle can be constructed starting with an equilateral triangle of side s and then replacing each side by a circular arc with the other two original sides as radii. (See Figure 2)

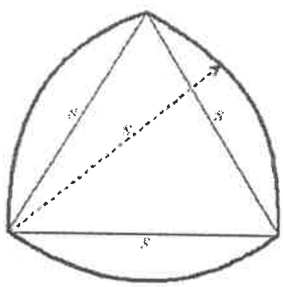

Figure 2: Construction of Reuleaux triangle.

Other symmetrical curves with constant width result if you start with a regular pentagon (or any regular polygon with an odd number of sides) and follow similar procedures. This construction is used in the design of some British coins – such coins will roll similar to round ones.

It is mentioned in the literature that the first mathematician to discuss curves with constant width was Leonard Euler [5], but some curved triangles can be seen in Leonardo da Vinci's 15th-century *Codex Madrid* [9]. In addition the 13th-century Notre Dame cathedral in Bruges, Belgium, has several windows clearly in the shape of a Reuleaux triangle (Figure 3).

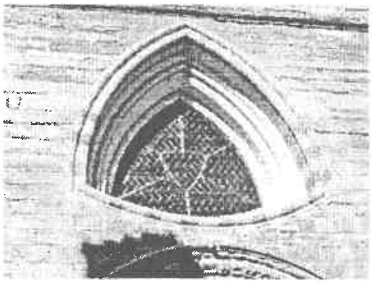

Figure 3: Window of Notre Dame Cathedrale, Bruges, Belgium.

Another early reference to Reuleaux triangles is [1], which mentions that the perimeter of a curve of constant width equals π times its width, and this is often called the Barbier Theorem. Reuleaux in his *Kinematics* gave the first complete analysis of such triangles, and he also noticed that such curves could be generated from any regular polygon with an odd number of sides. A modern application of the Reuleaux triangle can be seen in the Wankel engine (Figure 4).

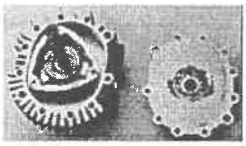

Figure 4: Reuleaux triangle in Wankel engine

In our KMODDL collection we can see several applications of these curves in the design of machines. We have developed a tutorial for high school students and their teachers for exploring properties and applications of Reuleaux triangles, involving the KMODDL digital resources. In this tutorial we have collected problems, references, and links about different properties of Reuleaux triangles and other curves of constant width. See [15]

Involutes In Machines

There are many other mathematical ideas connected with kinematic mechanisms. Most of the mechanisms in the KMODDL collection produce change in motion: circular to trigonometric (slider crank), circular to elliptical (double slider crank), circular to straight-line motion (straight-line mechanisms) – analysis of these motions involves calculus, triangle geometry (both planar and spherical), and inversive geometry (or geometry with inversions, see [13]). Gear mechanisms in the kinematic mechanism collection use the geometric curves, involutes and cycloids (see Figure 5)

Figure 5: Involute gears from KMODDL

The first mathematics of gear teeth in print is to be found in Girolamo Cardano's (1501-1576) work of 1557, where he gave empirical rules [2]. In 1694 Philip de la Hire (1640-1718) recommended to use involute curve for gearing, but in practice it was used only about 150 years later. In 1733 Charles Camus (1699-1768) expanded work of la Hire, in 1754 Leonard Euler (1707-1783) worked on design principles of involute gearing [3]. We have developed a tutorial [12] about involutes, which allows learners to see mathematical principles imbedded in some mechanisms (like the pump in Figure 6).

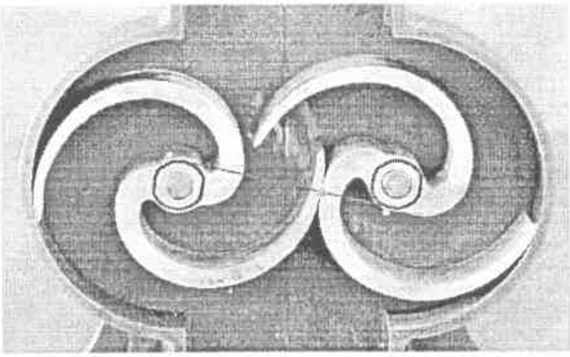

Figure 6: Involute in pump.

Imagine that one of the circles in Figure 6 is a spool that has a thread wrapped around. Now imagine unwinding the thread, keeping the spool fixed, and keeping the thread pulled taut. The end of the thread traces the outer edge of the spiral arm. This curve is called the *involute of a circle*. Instead of keeping the spool fixed and unwinding the thread, we could rotate the spool and pull the thread taut in the same direction. Imagine that there is a thread rolled around the left spool and then pulled taut and wrapped around the right spool as indicated in the Figure 6. Notice a dot on the thread at the place that two arms touch each other. Instead of unwrapping the thread we can turn the two spools at the same rate {PRIVATE}{PRIVATE} always keeping the thread taut. Since the spiral arms are in the shape of an involute, the dot will follow the outer edge of both spiral arms. Thus, as the spools rotate, the spiral arms will always stay in contact. R. Willis first showed this property in 1841. In [12] the above-described motion can be seen in an interactive movie. W. Payton used this mechanism in a water meter. It was exhibited at the Second World Exhibition in Paris in 1867 and patented in 1868.

In the KMODDL collection, models that in the R-series represent different spherical curves, generated by rolling cones and plates. Later these mathematical models were of great interest in automotive industry where they found applications as bevel gear pairs. These gears are useful when direction of the motion has to be changed. Several of these Reuleaux models (see, for example, Figure 7) were included in Walter Dyck's *Katalog mathematischer und mathematisch-physikalischer Modelle, Apparate und Instrumente* [4].

Figure 7: Reuleaux model of spherical cycloid

They are not actually mechanisms but rather dynamical representations of spherical cycloids – now we use computer graphics to represent these 3-D curves on a flat screen. Walter Dyck (1856-1934) was appointed Director of the Munich Polytechnikum in 1900 and under his inspired leadership the institution rose to university status becoming the Technische Hochschule of Munich. He served as rector of the Technische Hochschule for two terms. There was another important project in which Dyck played an important role. This was the creation of the Deutsches Museum of Natural Science and Technology in conjunction with Oskar von Miller, an electrical engineer, and Carl von Linde, professor of machine design. They proposed that a museum be built in Munich that would both preserve technological artifacts and let visitors learn about the scientific principles through interactive displays. The Deutsches Museum was the first of its kind and its ideas were soon copied by other science museums around the world. Not only was Dyck one of the three to establish and develop the museum in its early stages, but he was also appointed as the second Director of the Museum in 1906. As we see from his Katalog [4], Dyck was trying also to preserve knowledge about mathematical instruments and models.

How To Draw A Straight Line?

Let us give here an example of a real world problem that was solved by kinematic means but then grew into a more general mathematical problem. It was a very practical problem for James Watt (1736-1819) when he wanted to improve the steam engine: How to change circular motion into straight-line motion and vice versa. Some mathematicians later formulated this problem as *How to draw a straight line?* (see [6, 8, 14]). When using a compass to draw a circle, we are not starting with a model of a circle; instead we are using a fundamental property of circles that the points on a circle are at a fixed distance from a center. Or, we can say that we use Euclid's definition of a circle. Is there a tool (serving the role of a compass) that will draw a straight line? If, in this case, we want to use Euclid's definition "A straight line is a line which lies evenly with the points on itself" it will not be of much help. One can say: But we can use a straightedge for constructing a straight line! Well, how do you know that your straightedge is straight? How can you check that something is straight? What does "straight" mean?

There were several attempts by mathematicians to solve this problem. Several linkages in the Reuleaux model collection are connected with some of the names of 19th century mathematicians, who tried to solve this problem of how to draw a precise straight line. Reuleaux thought that these mechanisms were so important that he designed 39 straight-line mechanisms for his collection, including those of Watt (Figure 8), Roberts, Evans, Chebyshev, Peaucellier- Lipkin (Figure 9), Cartwright (Figure 10), and some of his own design.

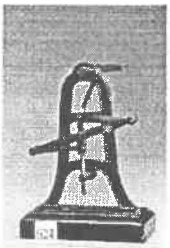

Figure 8: Watt's linkage

Figure 9: Peaucelier-Lipkin linkage

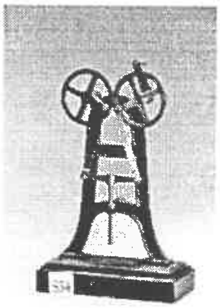

Figure 10: Cartwright mechanism.

If we look at these models we can see that they are showing applications of different mathematical ideas for the same purpose to create straight-line motion. For example, Peaucellier-Lipkin linkage uses the property of a straight line as a circle of infinite radius, Cartwright's straight line mechanism uses bilateral symmetry to produce straight line motion. Therefore, the Reuleaux kinematic mechanism collection is an example of application of mechanisms to understand deeper meanings in mathematics.

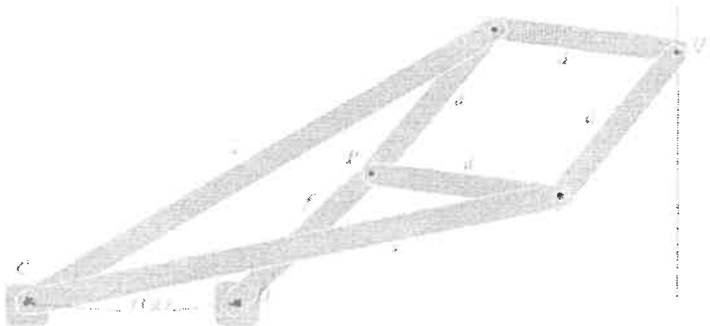

Figure 11: Peaucelier-Lipkin "inversor"

In the Peaucelier-Lipkin In Figure 11 the "inversor" (the links labeled d and s) the points P and Q are inverse pairs with respect to a circle with center C and radius r= {PRIVATE}{PRIVATE} $\sqrt{s^2 - d^2}$ -- analytically, this means that

$$\text{distance } (C, P) \times \text{distance } (C, Q) = r^2.$$

Here, the crucial property of circle inversion is that it takes circles to circles (for details of circle inversions, see [7].) In [9] you can see an interactive movie that shows the motion of the linkage. We can see that P is constrained (by its link to the stationary B) to travel in a circle around B and thus Q must be traveling along the arc of a circle {PRIVATE}{PRIVATE}the radius and center of this circle is varied by changing the position of the fixed point B the length of the link BP, it is shown that the radius of the circle is $r^2 f / (g^2 - f^2)$.) Thus, the Peaucellier linkage draws (at Q) the arc of a circle without using the center of that circle. If the lengths $g(CB)$ and $f(BP)$ are equal then the circle, on which P moves, goes through the center C; and, since points near C are inverted to points near infinity, the circle that Q lies on must go through infinity! How can a circle go through infinity? Answer: Only if the circle has infinite radius. *A circle with infinite radius (and thus zero curvature) is a straight line.* We now have another meaning for straight line, and the Peaucellier-Lipkin linkage is a tool for drawing a straight line using this meaning.

We have developed a learning module on some of these ideas for an upper level undergraduate geometry course that is available on the web [13] and is being used in a course at Cornell University (see the textbook [16]).

CONCLUSION

This KMODDL project started only two years ago, there is still much to do to unwind the mathematics behind the mechanisms and to make them accessible to teachers. There is also a collection of beautiful mathematical models [18] in the Mathematical Laboratory of the University Museum of Natural History and Scientific Instruments, Modena, Italy (a project coordinated by Prof. Mariolina Bartolini Bussi). A difference between the Modena collection and our collection is that their mathematical models were built based on historical drawings and knowledge, while we have the physical models and we are searching for their historical and modern applications. We are doing research on what mathematical ideas lay behind them. In some cases we can find references to the ideas, but for some models we have not been able to find any information about what the ideas were that precipitated the model.

We are continuing work on the KMODDL project now, developing more tutorials for use by teachers and their students. Also, in conjunction with the Museum of Science in Boston we are designing (for secondary school students) didactical exhibits that will involve the possibility of 3-D printing of the models. See Figure 12a and b.

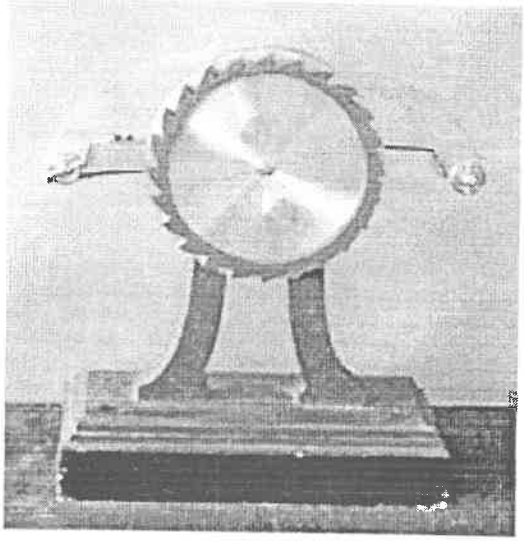

Figure 12a: Ratchet mechanism from Reuleaux models

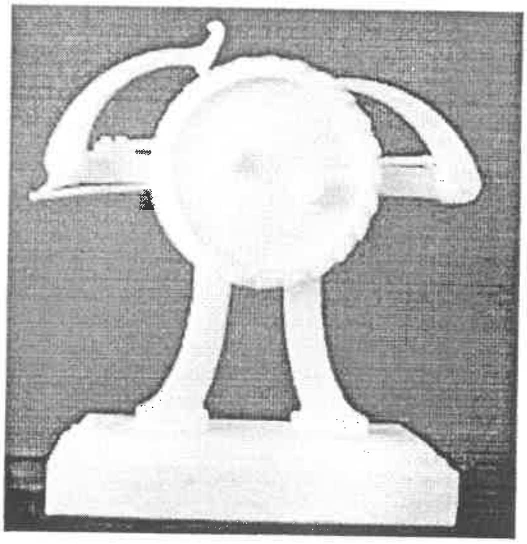

Figure 12b: 3-d printed model of ratchet mechanism

We will continue to work with secondary school teachers to design educational materials that deploy the KMODDL collection in school mathematics, science, and technology instruction. For further discussions of my own experiences working with this collection, see [16]. If others know of examples of the interactions between mathematics and mechanisms, I would appreciate hearing about them.

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Using the Educational Times in the Classroom

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Abstract: *Using The Educational Times as a medium, we encourage a humanistic view of mathematics by introducing relevant problems from the Victorian age. We present problems that are similar to those that contemporary students may encounter in their own courses as well as problems that were popular during the Victorian era. We present problems that were posed and solved by well-known mathematicians as well as problems posed and solved by students of the time. We illustrate some of the academic barriers that students faced a century ago. In particular, we highlight some of the obstacles that were overcome by women of the era and offer the Victorian women featured in this article as role models for our current generation of students. Although they are more than a century old, the problems from the Educational Times can effectively be used to enliven the classroom and stimulate undergraduate research in the twenty-first century.*

Keywords: Cambridge University, Charlotte Scott, Christine Ladd, College of Preceptors, *Educational Times,* Exercises, Girton College, Hertha Ayrton, Mathematics, Victorian Age, Women.

INTRODUCTION

Perhaps more than ever before, mathematics instructors are discovering ways to use the history of mathematics to motivate their students and introduce a humanistic perspective on mathematics. Appropriate integration of mathematics history into the curriculum can lead students to make connections between various mathematical ideas; it can help students appreciate the integral role that mathematics has always played in society; and it can guide students to reach for a deeper and more meaningful understanding of major mathematical concepts. Certainly mathematics journals such as the *American Mathematical Monthly* and *Mathematics Magazine* offer innumerable innovative problems. However, we have also found that historical journals such as *The Educational Times* (*ET*), a popular pedagogical journal of the Victorian era, contain a plethora of relevant problems that can provide rewarding and satisfying problem solving opportunities for students and teachers alike. Indeed, students are often intrigued by the prospect of solving the same mathematical problems that challenged men and women more than a century ago. An added benefit of using material from *ET* is the fact that many of its contributors are mathematicians that our students might encounter in their undergraduate classes. For example, contributors from analysis include Gaston Darboux, Jacques Hadamard, and G.H. Hardy; contributors from abstract algebra include Arthur Cayley, William Burnside, Peter Tait, and J.J. Sylvester; and in combinatorics we find Thomas Kirkman, W.W. Rouse Ball, Eugene Catalan, Edouard Lucas, and and Charles Dodgson (Lewis Carroll). Another distinctive aspect of *ET* is that its problems can be used to introduce women who were mathematically active at a time when the field of mathematics was dominated by men. *ET* served as one of the vehicles through which women were able to successfully establish themselves as competent analytical thinkers during the Victorian era. For those courses in which writing is a component, research on women and mathematics before the twentieth century can provide a much more challenging task than research on male mathematicians.

In the paper that follows we provide a historical overview of the mathematical section of *ET* followed by a brief introduction to the lives and works of some of its female contributors. We conclude with a selection of problems from *ET* that we hope modern students will find as interesting and challenging as did their historical counterparts.

A Historical Overview Of The Mathematical Section Of *The Educational Times*

The Educational Times was first published in the fall of 1847. In 1861 it was adopted as the official publication of the College of Preceptors, which had been established in London by Royal Charter in 1849. The primary goals of the College were to promote sound learning, advance interest in education among the middle class, and provide means to raise the status and qualifications of teachers. Training was offered to those entering the teaching profession and periodic examinations for certification were administered to both teachers and students. A union was formed to make provisions for the families of deceased, aged, or underprivileged members. In addition, the organization endeavored to facilitate better communication between teachers and the public. Monthly meetings of the College were held at Bloomsbury Square in London and open to the public. Here announcements and summaries of important educational movements were promulgated and papers concerning the theory and practice of education were read by the College's members.

Among its pages the reader of *ET* could find notices of available scholarships, lists of successful candidates on examinations given by the College, notices of vacancies for teachers and governesses, book reviews, textbook advertisements, and a section devoted to mathematical problems and their solutions. Space in *ET* was at such a premium that less than a page and a half was normally devoted to the mathematics section. Various departments constantly vied for space with advertisements. On several occasions in the 1850s the mathematics section was completely omitted. However, the mathematics section became so popular, and space for it in *ET* so restricted, that from 1864 to 1918 the semiannual publication *Mathematical Questions with Their Solutions from the 'Educational Times' (MQ)* reprinted problems and solutions that had appeared in *ET* along with many solutions that had not appeared.


W.J.C. Miller

W.J.C. Miller, the first editor of *ET*, served from 1847 until illness forced him to retire in 1897. Miller was mathematical master and vice-principle of Huddersfield College in Yorkshire until 1876 at which time he became Registrar, Secretary and Statistician to the General Medical Council. He was a Fellow of the Royal Statistical Society and a member of the London Mathematical Society. Miller's directions to *ET* contributors were concise: "Make your answers as short as possible, write each question and answer on a separate sheet of paper with your name at the top of each, and remember to pay the postage in full!" Miller was succeeded by Daniel Biddle, a member of the Royal College of Surgeons and Fellow of the Royal Statistical Society. Constance Marks of West Kensington, the third and final editor, served from 1902 to 1918. In 1906 she added subject and author indices to each *MQ* volume.

According to algebraist William Kingdon Clifford, the mathematical problems and solutions section of *ET* did more to encourage original mathematical research than any other European periodical (Biddle 1897). Indeed, the first published works of both G.H. Hardy and Bertrand Russell were solutions to problems from *ET*. Perhaps Clifford's claim stems from the fact that a century ago mathematical textbooks did not typically contain pages of diverse exercises. It was customary for motivated teachers and students to seek out applications of the theory they learned. Hence *ET* proved to be an invaluable source of practice problems to anyone interested in mathematics.

WOMEN'S CONTRIBUTION TO *THE EDUCATIONAL TIMES*

Approximately 1.3 percent of the 18,702 problems posed in *ET* from 1847 to 1915 were contributed by women. More than 3 percent of the published solutions were submitted by women. Table 1 provides a list of the fifteen women responsible for the majority of contributions from women.

	PROBLEMS POSED	SOLUTIONS SUBMITTED	TOTAL CONTRIBUTIONS	ACTIVE *ET* PERIOD
CHRISTINE LADD	53	82	135	1872-1899
SARAH MARKS	22	95	117	1881-1899
BELLE EASTON	26	81	107	1874-1893
ELIZABETH BLACKWOOD	76	23	99	1872-1897
ALICE GORDON	36	41	77	1885-1905
CONSTANCE MARKS	9	50	59	1899-1918
CHARLOTTE SCOTT	9	25	36	1880-1888
EMILY PERIN	4	24	28	1885-1892
KATE GALE	0	21	21	1881-1891
MARGARET MEYER	0	20	20	1882-1885
FRANCES CAVE	0	15	15	1903=1908
ISABEL MADDISON	1	13	14	1886-1899
LIZZIE KITTRIDGE	4	10	14	1873-1892
GERTRUDE POOLE	0	10	10	1887-1888
FANNIE JACKSON	1	7	8	1889-1897

Table 1: Top women contributors to *THE EDUCATIONAL TIMES*

During the nineteenth century, the notion that intellectual capability is independent of gender was a rather revolutionary idea in both British and North American society. However, by the last quarter of that century, women were making important strides toward both social and intellectual independence. This progress can be attributed in part to the inception of women's colleges that offered women access to the higher educational opportunities that had previously been reserved for men. As more women took advantage of their academic alternatives, the number of mathematical contributions made by women to pedagogical journals such as *ET* increased dramatically. Their increasing participation in mathematical pursuits played a role in validating women as intelligent and competent analytical thinkers who were capable of competing on the same mathematical playing field as men. In giving women credit for their mathematical contributions, journals such as *ET* helped promote an emancipated view of women.

For the most part, a good understanding of algebra, trigonometry, calculus, triangle and circle geometry, and basic concepts in mechanics were required to successfully solve the problems posed in *ET*. Women contributors demonstrated that they were acquiring the mathematical backgrounds necessary to solve such problems. Clever solutions and ingeniously posed problems indicate that the women responsible had developed a solid foundation of algebraic, geometric, and analytical reasoning skills. Moreover, the dedication of these pioneering women undoubtedly both inspired and gave a greater appreciation for mathematical thought to their own students and the women who followed in their footsteps.

Girton Class of 1877

Approximately thirty-seven percent of the solutions published by women in *ET* were the work of seven women from Girton College in Cambridge: Sarah Marks, Charlotte Scott, Ada Isabel Maddison, Kate Gale, Margaret Meyer, Emily Perrin, and Frances E. Cave-Brown-Cave. The woman with the greatest number of total contributions, Christine Ladd, was an American. And the mysterious Elizabeth

Blackwood has the greatest number of posed problems to her name. Many of these and other women that appear in Table 1 have rather interesting stories. In the paragraphs that follow we introduce the reader to several of these remarkable women.

Christine Ladd studied mathematics under the supervision of J.J. Sylvester at Johns Hopkins University. Her analytical skills first came to Sylvester's attention via her contributions to *ET*. Before attending Johns Hopkins, Ladd studied at Vassar College and taught in secondary schools in upstate New York and Pennsylvania. At Vassar, under the influence of astronomer Maria Mitchell, Ladd concentrated her studies on physics. Under Sylvester's tutelage at Johns Hopkins, she then redirected her focus to mathematics. Ladd developed a particular interest in the works of Charles Peirce and Bertrand Russell in the area of symbolic logic. Ladd completed her course work and dissertation in 1882. Unfortunately, Johns Hopkins did not recognize degrees for women at the time. She was eventually awarded her Ph.D. from Hopkins in 1926. During her lifetime Ladd authored more than ninety articles in mathematics, symbolic logic, and the psychological aspects of color and vision. In addition, she received the only honorary degree ever awarded by Vassar College.

Sarah Marks is accredited with the greatest number of published solutions among the women contributors to *ET*. As was customary for a typical Victorian woman, Marks managed family responsibilities, nursed ill friends and family members, and was concerned with social issues. However, her passion for science set her apart from many of the women of her era. While a student at Girton, she organized a woman's fire brigade, invented a device for measuring one's pulse, and designed and patented a draftsman's device that could be used for dividing a line into equal parts and resizing geometric figures. Marks took a third class on the 1881 Cambridge tripos examination, but received her B.Sc. degree from the University of London. Before 1948 women were precluded from obtaining a Cambridge University degree — even if they had completed their education at Girton or Newnham and passed the formidable Cambridge mathematical tripos, the passage of which was required for a Cambridge honors degree. To obtain her college degree, a woman such as Marks would have to pass an external examination from a school that granted degrees to women. After receiving her Bachelor's degree, Marks referred to Cambridge as her alma "step"-mater.

Marks pursued her interest in science after leaving Girton. In 1884 she began attending a physics course offered by Professor William Ayrton, F.R.S., at Finsbury Technical College. Ayrton and Marks were married in 1885. In addition to taking her husband's name, Marks changed her first name to Hertha. The name was suggested by some of her friends who compared her to the Teutonic goddess Erda and to the heroine in Swinburne's poem 'Hertha'.

Intrigued by some of her husband's work, Ayrton began experimenting with electric arcs. At the time electric arcs were widely used in lighting. Her work generated significant industrial and commercial interest. Ayrton's research led to the production of more reliable searchlights and to improvements in the performance of movie projectors. She became an acclaimed European expert on the electric arc and was commissioned to write a series of papers for *The Electrician* that formed the basis for her book, *The Electric Arc* (Ayrton 1902). Because of her outstanding contributions to science, she became the first woman elected to a British electrical engineering society and she authored the first paper written by a woman to be read before the Royal Society of London (Ayrton 1901-02).

In 1901 Ayrton redirected her focus to the investigation of wavelike motions and the development of ripple marks on the sea floor. Her discoveries showed how the formation of sand ripples applied to coastal erosion and sandbank structure. In 1902 she became the first woman to be nominated as Fellow of the Royal Society. Although she had her husband's support, the Society rejected her nomination citing, on the advice of council, that "it had no legal power to elect a married woman to this distinction." (Mason 1972). However, in 1906 she was awarded the Royal Society's Hughes Medal for her original research on electric arcs and sand ripples. During her later years she devoted much of her time to women's and social causes, and was an active member of the National Union of Women's Suffrage Societies (Tattersall & McMurran 1995a).

Our next Girton alumna, *Charlotte Angus Scott,* is renowned for her role as the first woman to achieve first class honors on the Cambridge mathematical tripos. In 1880 she scored at the same level as Cambridge's eighth wrangler on this fifty-five-hour examination that was spread over nine days. At the time, women were admitted to Cambridge examinations only by courtesy of the examiners. As a result of Scott's achievement, women were thereafter allowed formal admission to the tripos, their results were publicly announced, and, if successful, they were given certificates of achievement. The certificates, however, were in no way equivalent to a degree from Cambridge University.

Scott remained at Girton until 1885, serving as Lecturer in mathematics. She continued to contribute to *ET* and her contributions demonstrated her strengths in geometry and applied mathematics. In addition to lecturing at Girton, Scott attended Arthur Cayley's lectures in modern algebra, abelian functions, number theory, semi-invariants, and the theory of substitutions. Under his supervision, she took an external D.Sc. degree with honors from the University of London (Scott 1895). In so doing she became the first British woman to receive a doctorate and the second European woman, after Sofia Kovalevskaia, to receive a doctorate in mathematics.

Scott emigrated to the United States and became an active and prominent member of the American mathematical community. She served as chair of the mathematics department at Bryn Mawr for nearly forty years. During this time she supervised seven doctoral candidates (Kenschaft 1986) and influenced and inspired many young students (Maddison & Lehr 1932). She was one of the most active American mathematicians at the turn of the century (Fenster & Parshall 1994). Scott worked in the field of algebraic geometry with a focus on analyzing singularities of algebraic curves and investigating properties of planar curves of degree higher than two. Her work was widely recognized in Europe as well as in America. She was given the curious distinction of being the only woman included in the first edition of Cattell's *American Men of Science* (Cattell 1906). She published an advanced undergraduate geometry textbook (Scott 1894) and worked several years with Frank Morley as co-editor of *the American Journal of Mathematics*, a journal founded in 1878 by Sylvester at Johns Hopkins. Scott served two terms on the Council of the American Mathematical Society and was elected as its vice-president in 1905. Seventy years passed before another woman was elected to that position. Scott was also a founder of the College Entrance Examination Board and served for a time as the Board's chief mathematical examiner.

Elizabeth Blackwood, the woman submitting the most problems, remains a mystery. Her references in *ET* indicate that she held a Bachelor's degree. She sent her submissions from London, New York, and then for the next twenty-one years from Bolougne-sur-Mer. Thirty-five of the problems she submitted went unsolved. Her forte was geometric probability and she devised instruments to check her probabilistic solutions. E.B. Seitz, professor of mathematics at the Missouri Normal School, now Northeastern Missouri State University, solved nine of the problems posed by Blackwood. Seitz, hailed as one of the most distinguished American problem solvers of his day and known for his expertise in geometric probability suffered an early demise at age thirty-seven from typhoid (Finkel 1894).

After completing her studies at Girton, *Kate Gale* received her external Bachelor's degree from Trinity College, Dublin. She went on to serve for two years as assistant mistress at a private school in Brighton, for three years as second mistress at St. John's School in Worcester Park, and for nine years as headmistress at the Blackheath Centre School. Gale then emigrated to South Africa where her work as a mathematical mistress in Wynberg was followed by a long tenure as joint headmistress and co-owner of the Milburn House School in Claremont near Capetown (Tattersall & McMurran 1995b).

Margaret Meyer, another Girton alumna, exhibited a thorough knowledge of geometry, calculus, mechanics, and physics in her *ET* solutions. After completing her undergraduate studies, she acted as assistant mistress at the Notting Hill School for three years before returning to Girton where she served as a resident lecturer and Director of Studies. In 1907 she was awarded a Master's degree from Trinity College, Dublin. Meyer conducted aeronautical research for the British Government during World War I and became one of the first women elected as a Fellow of the Royal Astronomical Society. She was also an avid mountaineer and served three terms as president of the Alpine Club.

Alice Gordon submitted her solutions from the Barnwood House, a private hospital for the insane in Gloucestershire, England. It is not clear whether Gordon was a patient, member of the staff, or a relative of a staff member. Gloucester County records show that in 1891 there was a single female patient, aged 40, with the initials AG residing at the Barnwood House (Turton 1993). Gordon's *ET* references indicate that she held both a Bachelor's and a Master's degree.

Lizzie Kittridge and *Belle Easton* were both graduates of Union School in Lockport, New York. Founded in 1848, Union School was the first regional public high school in the United States. Kittridge graduated with a class of eight in 1873. In 1892, she married Eugene F. Goodman of New York and ceased contributing to *ET.* Belle Easton graduated with a class of seven in 1875. She was influenced by Asher B. Evans, the school's principle and a prolific contributor to the mathematical section of *ET.* Easton demonstrated an aptitude for problems involving geometry, elliptic integrals and Bessel functions. She contributed to ET for nearly twenty years, during which time she obtained a Master's degree.

We briefly mention one additional problem solver. Although *Annie Chartres* is not among top contributors listed in Table 1, we give her an honorable mention since the problem she solved is not only included among our sample problems, but was submitted to *ET* by G.H. Hardy. Unfortunately, that is the extent of our knowledge of Miss Chartres.

Listed below are twelve problems from *The Educational Times* that are representative of the problems we present to our own students. This selection was chosen in part because of the connections between the problems and the women introduced above.

1. The radii of the fore and hind wheels of a coach are r and R, and a is the distance between their centers. A particle driven from the highest point of the hind wheel falls on the highest point of the fore wheel. Find the velocity of the coach. [Christine Ladd, 1874]

2. If *ABCD* is a quadrilateral inscribed in a circle, prove that the incenters of the triangles *ABC, BCD, CDA,* and *DAB* are the vertices of a rectangle. [Sarah Marks, 1886]

3. Into a full conical wine-glass whose depth is a and generating angle is θ, there is dropped a spherical ball of radius r. Determine the radius of the ball that causes the greatest overflow. [Sarah Marks, 1882]

4. If a point is taken at random on the diameter of a semicircle of radius r, a second point is taken at random on the semicircle, and these points are joined with one end of the diameter to form a triangle, determine the mean area of the triangles thus formed. [Kate Gale, 1882]

5. Sum 1 - 2/5 + 3/9 - 4/13 + ··· [Annie Chartres, 1898]

6. Prove that no cube except 8 when increased by 1 can be square. [Alice Gordon, 1885]

7. If a number has the sum of its digits equal to 10, find under what circumstances twice the number will have the sum of its digits equal to 11. [Margaret Meyer, 1884]

8. A candidate is examined in three papers to each of which m marks are assigned as a maximum. His total score on the three papers is $2m$. Determine the number of ways in which this may occur. [Belle Easton, 1881]

9. If n is a seven-digit positive integer, whose sum of digits is 59, then what is the probability that 11 divides n? [Sarah Marks, 1884]

10. If $3n$ zeros are placed between the digits 3 and 7 prove that the number formed is divisible by 37, and if $3n+1$ zeros are placed between the digits 7 and 3 prove that the number formed is divisible by 37. [Kate Gale, 1882]

11. Two points, P and Q, are successively taken at random within a given sphere; show that the chance that the sphere of which P is the center and PQ the radius lies wholly within the given sphere is 1/20. [Elizabeth Blackwood, 1877].

12. If a uniform string of length $2b$ is suspended from two points in the same horizontal line distance $2a$ from each other and the distance between the points is slightly increased by the quantity 2δ, find the height the vertex will ascend. [Lizzie Kittridge, 1881]

Concluding Remarks

The use of problems from *The Educational Times* enhanced with histories of its contributors can be effectively used to motivate students. Our students often become intrigued by the opportunity to match wits with their historical counterparts and will enthusiastically engage in active problem solving. By discussing the educational challenges faced, and met, by women and men a century ago, our own students gain a better appreciation for higher education. An awareness of the important roles played by students and mentors, as well as famous mathematicians, in the evolution of mathematics and mathematics education, will hopefully make modern students aware of their own potential to help meet the academic challenges of today.

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Integrating the History of Mathematics in the Teaching of Mathematics: A Possible Link between Pythagoras and King Tut

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Abstract: *History defines man's need to solve problems he faced, and how man found mathematical solutions for these problems of application is the best way to teach students how to use mathematics. I have found that it is beneficial to capitalize on the limitless curiosity that students have about the mystique surrounding the Pyramids and Stonehenge. Research does indicate that the more varied a lesson structure is, the more interesting the lesson becomes. Maintaining student interest in this MTV age is the problem. Countless stimuli compete for our student's attention. It is our job as mathematics instructors to find methods to reach the students of the information age how to use mathematics. The following article illustrates the methods I have used to get and maintain student interest to foster mathematics learning. Inspired by the historical context in which an event or a possible mathematical application may have taken place, I get the student to see that mathematics is the solution to man's problems, instead of the cause of their problem.*

THE REASON FOR CHANGE

Most current research indicates that the process of educating students has evolved from past practices and will continue to evolve. These changes force mathematics educators to modify their patterns of instruction in both content and delivery. Mathematical instruction requires both context and relevancy in order to become part of the learner's knowledge base. Mathematics education becomes increasingly more complicated owing to the ever-changing needs of society, and technology, the different student learning styles, and the diversity of student backgrounds and culture. According to Sutton, (2002, p.56) "Conclusions from cognitive science indicate that knowledge taught in multiple contexts better supports permanent, functional learning of concepts ". This must become our focus if we are to succeed in educating the next generation of mathematicians and problem solvers.

Meaningful mathematical learning can be accomplished with tactile applications of mathematical concepts. Students need a connection between the theorem, its discovery, and past and current applications. Students need to realize that most mathematics used today in textbooks and classroom environments were actually solutions to application problems of the needs of man throughout history. For years The National Council of Teachers of Mathematics (NCTM) has focused on making mathematics more tactile and application based to allow students a means to connect mathematical concepts to their world. One of the NCTM publications *Connecting Mathematics* (NCTM, 1991) stresses the use of physical modelling and current applications of mathematics in various industries and jobs. Kagan, (1994) also points out that students have three basic ways of sorting new data cognitively: by detail, relationships, and by categorization, and that "all three of these (learning) styles are valuable and students need to have cognitive flexibility enough to see the information in a variety of ways". Kagan is not alone in his beliefs. Saphier and Gower (1997) state, "Skillful teachers are made not born". Saphier and Gower reinforce Kagan's concept that different students respond to the learning environment based on their experiential backgrounds. These statements support modification of traditional lecture modality in educating our young.

Heinemann Kunkel & MacDonald, (2002) while discussing brain research, point to stressing the connections of "meaning and retention" and "making meaning using associations". They state, "The brain sifts through all incoming sensory stimuli and selects those that are then most relevant or meaningful. If the information has no meaning to the students, they will not store the information. One way to make information meaningful is to associate or compare the new concept with a known concept, to hook the unfamiliar with something familiar." (*ibid* p.49) The end result is that professional educators of the next generation of mathematicians and citizens of our world have to realize that we no longer have a captive audience. Teachers have to compete for student interest in learning with an ever changing, highly defined, and marketed multi-media, multi-modal world. Through computer technology and other forms of media, our students are bombarded exponentially with increasing levels of information. The educator who continually modifies his/her classroom procedures and delivers to suit the ever-changing raw material, the modern student, will lead the student to successful mathematical learning.

A Sample Of A New Direction To Consider Changing To

Applying the afore-mentioned strategies, coupled with the simple task of using ropes to measure and define 90° corners, I have found a quite successful way for students to learn and to apply the Pythagorean Theorem. Allowing students to define a 3-4-5 triangle and form 90°, as it has been conjectured[1] that the ancients did in land surveying, has led to a very positive learning environment. The use of a rope with 12 equally-spaced knots enables students to focus on the process of knowledge acquisition by the brain through physical manipulation of the knotted rope in the form of a 3-4-5 right triangle. The display of several transparencies of ancient murals and stone carvings illustrates the context of the use of the 3-4-5 right triangle that might have been made in land surveying, and gives students a focus point on how mathematics is the solution, not the problem, of man's existence. However, it should be noticed at this point that it is controversial whether the Egyptians actually used the 3-4-5 right triangles in surveying the farmland on the Nile and it is a point of contention whether the ancient Egyptians really knew that a 3-4-5 triangle is a right triangle (quotes from several authors on doubt about this belief are given in Gillings 1982; readers can also consult Struik 1967, p.24; Boyer 1968, pp.17-18, Kline 1972, p.20). However, the over 3400 year old mural at Abd-el-Qurna (part of which is shown below) is an illustration of why I believe that such a knowledge was within the possibilities of Egyptian mathematics. Analysis of this mural shows the knotted rope with a surveying team making the measurements. Since the Egyptians usually copied actual events in their art, it might have happened that the Egyptians used the 3-4-5 right triangles to survey the farmland along the Nile. This line of reasoning is close to van der Waerden's analysis of Cantor's conjecture[2].

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[1] According to M. Cantor: "Lets assume for the time being, however without proof, that the Egyptians were acquainted with the fact that three sides of length 3, 4 and 5 satisfy the basic property $4^2+3^2=5^2$ (p. 96) moreover, that they did not fail to notice that a right angle is formed between the two smaller sides of a triangle. Let us consider two poles along the meridian, which are at a distance of 4 units apart. Let us assume that a rope has a length of 12 units and is divided by knots at a distance of 3, 4, 5 and so it is clear (fig. 9) that the rope, stretched at one knot and at the same time the other two are tied to the poles, necessarily forms an exact right angle on one of the poles along the meridian. If this was the main task of the Harpedonaptai [rope stretchers] and might have been their secret job, that is, to place the knots and the sticks in the proper position, we could have at least a reasonable explanation of the fact that there is no information whatsoever in the inscriptions about the process. And so we may perhaps attribute the glory of the "constructions of lines" to them, since they were in the possession of the mysteries of Geometry, which were not passed on to just anybody. And so we may comprehend how their activities presented in the wall paintings were attributed to the King himself in connection with a goddess." (Cantor 1907/1965, pp.105-106; editors' translation).

[2] Referring to this conjecture (see footnote 1), van der Waerden argues: "What is it based on? On two facts and an argument of Cantor. The facts are the following: 'rope-stretchers' took part in laying out an Egyptian temple, and the angles at the base of temples and pyramids are nearly always, very accurately, right angles. Now Cantor reasons as follows: these right angles must

Despite these controversies, however, it is true in general that, putting emphasis on how ancient man actually used, or might have used mathematics to solve his problems reinforces the concept that applying mathematics is important to the continued advancement of society as a whole.

Maintaining student interest in this MTV age is the problem. Since countless stimuli compete for our students' attention, mathematics educators must find a lesson design that will capitalize on student interest and curiosity. Research by NCTM, Kagan, Kunkel and MacDonald, Saphier and Gower support a more varied lesson structure as a way of making the lesson more interesting. The higher the interest level the more likely learning is to become longer lasting. Following this premise in the early 1980's I set about to verify in a limited study if these concepts would work in my high school. With the assistance of two mathematics teachers in my mathematics department, we performed a limited study with two groups of four different sets of students over a four-year period. Each set of 25 students was a ninth grade geometry class. One group of 25 was taught the Pythagorean Theorem in the traditional approach of a passive lecture plus assorted practice problem taken out of the geometry text book by one teacher. The second teacher working with a different group of 25 ninth grade geometry students introduced the students to the replication of a 3-4-5 right triangle. This process took the students out of the classroom into the building. This group used a rope with 12 equally-spaced knots [possibly how the ancients had] to create a 3-4-5 triangle also defines a 90° degree corner.

The students continued to use their new found tool to verify various corners in the school building to determine if the corners were right angles or if they were larger or smaller then 90°. In the next classroom session this group was then introduced to the Pythagorean theorem, and explored various other Pythagorean triples with ropes of different number of equally-spaced knots, (i.e. 30 knots for 5-12-13, etc...) in a very student-based active learning process. This process was repeated 4 times over four consecutive years, using the same teachers with different groups of ninth graders. The evaluation of the students' learning was checked in each group during their final exam at the end of the geometry course and checked again three years later in their senior year by use of a special section of their senior mathematics exam devoted again to Pythagorean Theorem and applications. The groups taught the Pythagorean Theorem using the passive traditional way had an average 58% success rate on the Pythagorean

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have been constructed by the rope stretchers, and I (Cantor) cannot think of any other way of constructing a right angle by means of stretched ropes than by using three ropes of lengths 3, 4 and 5, forming a right triangle. Therefore the Egyptians must have known this triangle." (van der Waerden 1963, p.6) section of their geometry final, and an average 37% success rate on the Pythagorean section in their senior mathematics exam. On the other hand, the 3-4-5-rope stretcher group had an average 78 % success rate on the Pythagorean section of their geometry final, and an average 63% success rate on the Pythagorean section in their senior mathematics exam. The analysis of the students' test results brought about a change in the manner in which most teachers in my department taught mathematics. These results lead us to a more active student-learning environment. This limited study coupled with the various findings in section 1 of this article, led me to the following position on use of manipulative and student active learning during mathematics class.

When students simulate the way that an Egyptian surveying team possibly worked, they not only use the Pythagorean Theorem, but also have the opportunity to see mathematics at work. Using the 3-4-5 right triangle can accomplish sustained interest. As mathematics educators we must foster learning, using whatever means necessary, even if it means crossing over and using elements of another discipline, like history. Following this rationale, it is a virtually possible scenario that the Egyptians used the rope with 12 equally spaced knots divided into the 3-4-5 ratios to define corners and survey land. This scenario may be used in the classroom to excite the students' interest and imagination, keeping in mind to tell them about existing doubts concerning its historical correctness. It should also be noted that many carpenters use the 3-4-5 on their carpenter's square to define and design corners that are 90°. Whether or not the Egyptians did use the 3-4-5 right triangles for surveying, the students enjoy and have success in learning to apply the 3-4-5 right triangles in defining 90° degree angles. The concluding activity for the 3-4-5 classes is to define the current applications of the 3-4-5 right triangles. Students have found uses like: a carpenter square, chalking the batters box in baseball, determining if a wall is perpendicular to the floor, etc… Students who are active learners take ownership of their learning and are more likely to remember concepts in the long term. I have written two books on teaching techniques that capitalize on the limitless curiosity that students have about the mystique surrounding the Pyramids and Stonehenge to learn mathematics (Charette 2003a, b). The Egyptians, the great builders of antiquity, did apply the mathematics developed by the Babylonians. Based on my research I also believe that, "during the time of the ancient Egyptians the right triangle relationship was held in secret trust by their religious leaders. Each time a building was erected, these people, called 'rope stretchers' had to be employed to lay out the corners of the structure so as to form a right angle." (Lewis, 1964, p.132). Perhaps, a second more common application of these rope stretchers dates back to Mendez who unified the upper and lower Nile deltas. The Royal Society of Rope Stretchers was using surveying techniques to define the borderline of adjacent farms in the Nile river basin. These first recorded land surveyors were called annually to measure the Nile river valley after the rich silt was deposited by the annual flooding.

The Egyptians solved their unique problem, by first placing square pillars marking the farm's inland corner. Some of these original pillars are still visible today south of the Hydroelectric dam near Giza Plateau. At the top of these pillars the Egyptians placed a GOLD pin which might have acted as the key vertex for the 3-4-5 right triangle used by the land surveying team. (During Napoleon's attack on Egypt, his soldiers stole the gold pins that were replaced by metal pins soon after).

I have found that using the ancient mural at Abd-el-Qurna, the first recorded image that might be related to a possible application of the 3-4-5 right triangle by the Royal Society of Rope Stretchers to survey the land for wheat crops in the Nile river basin, is an excellent tool in getting students attention and interest.

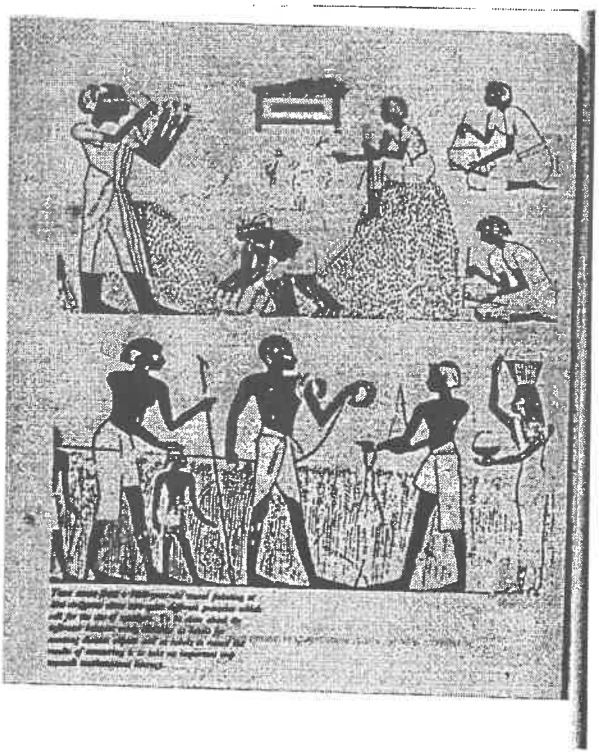

Photo 1: The ancient mural at Abd-el-Qurna

It is possible to use in the classroom a hypothetical rational reconstruction of the method that might have been used by the Egyptian surveying team, which is simple, yet highly accurate in settling many property line issues: The surveying team had three members and they used only a rope. The rope, twelve cubits long[3], was equally divided by twelve knots, each one cubit in length.

To create the right angle needed to define the corner of each farm, the surveying team used the same process. One member of the team would count out three knots, holding the third knot on the pin in the stone pillar. The second member of the team would then walk four knots toward the rising sun (the Nile). The third member of the team would then connect the end of the five-knot section with the end of the three segments. At this point, the students may be informed about the high accuracy attained by the Egyptians in constructing right angles, as it can be seen in the great Pyramid of Cheops with an error of the order of 0.1%

I have found that allowing students to experience using the 3-4-5 right triangle surveying technique, elevates the learning of mathematics to higher levels of application, and being related to a historical context adds meaning to the concept. This process is based on research; mathematics learned in context (who- when- where and why) has a higher degree of student learning and retention success. The successful student has to continue to be our major goal, if we are to improve the process of mathematics education. Numerous educational studies have also indicated that hands-on and tactile student-based active lessons have more meaning and make the necessary connections to long-term retention of learning. By taping onto the limitless curiosity about the mystique surrounding ancient Egypt's Royal Society of Rope Stretchers and their role in land surveying and the pyramids, mathematics teachers can make progress on competing with the many forces vying for the student's attention and focus. The 3-4-5 right triangle possible land surveying technique has been able to grab students and hold the interest and attention.

An additional resource in maintaining student interest is to use a photo of ancient carvings and murals to give students a multi-modal and visual connection to past uses of mathematics. Students see these as evidence and a mirror into the past of mathematics.

Traveling further south down (actually up the Nile) to LUXOR affixed in stone for all time the image of the Pharaoh his throne supported by the two larger figures binding a bundle of wheat. The two larger figures are members of the Royal Society of Rope Stretchers.

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[3]Twelve times the distance from the bent elbow to the longest finger.

Photo 2: The LUXOR carving

Notice the relatively small size of the Pharaoh in the center of this massive carving, compared with the three times larger Rope Stretchers symbolically tying the wheat with the rope. This is yet another illustration defining the importance of the land surveyors in the daily life of ancient Egyptian society.

Both of these images define the great value and prominence that the applied mathematics of the Royal society of Rope Stretchers had in ancient time.

CONCLUSIONS

I concur with the research listed in the first section of this article, which indicates that meaningful long term student learning can be enhanced and strengthened by use of a student-based activity lessons. The coupling of the tactile [manipulating the 3-4-5 rope triangle] with the visual experience in historical context gives students a connection to the past. This connection to the past allows students to develop an understanding that mathematics has been used for millennia to solve man's problems and enhance man's life. The use of the illustration of the 3-4-5 right triangle land surveying technique has been extremely successful in the learning and retention of the Pythagorean Theorem.

The use of images such as the previously illustrated images of the Rope Stretchers, is an effective way of getting students actively involved and interested with their learning by making connections between man's past uses of mathematics theorems and formulae and current mathematics applications. The difference the Pythagorean Theorem problems on the exam grades at the end of one year (20%) and then again at the end of four years with a 26% difference is an indication of success learning. During my 17-year tenure as mathematics department chairman in the Wallingford, Connecticut school system, I have conducted similar studies on activity-based learning. One direction of my focus was Stonehenge and the use of the circle in construction of arches. This study had similar findings and led me to write "Circles Rolling Through Time" (Charette 2003b) an active based lesson design text.

In my 37-year career, I have encountered numerous former students who have indicated their enjoyment for the way they were taught. They often comment on how they remember the Pythagorean Theorem, or Stonehenge and circles activity-based learning exercises. Based on my opinion and research the end result is more learning and understanding, as well as, retention of knowledge and application of mathematics. Man needs only look to the past to see how mathematics has evolved to be used by mankind to solve and define solutions to the issues that had to be faced. There are countless other mathematical historical events that can be used to stimulate students to learn; I hope you enjoy the task in finding and applying them.

Photos by R.J.Charette

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How History Fuels Teaching for Mathematising: Some personal reflections

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Abstract: *According to Freudenthal (1973), to learn mathematics is to re-invent mathematics, the process of which he called 'mathematising'. Coupled with Wittmann's perspective of mathematics education as 'design science' (1995), the study of mathematics education becomes the study of instructional designs, which facilitate the process of mathematising. This paper outlines two examples, one on the exploration of properties of unit fractions, and the other on the teaching of common factors and multiples, where historical materials do shed light on the instructional designs that aim to engage pupils in the process of mathematising. Some interesting observations in the classroom are reported and discussed.*

INTRODUCTION

Freudenthal (1973, 1991) coined the term 'mathematising' in his great works and gave full elaboration of its meaning in connection with his viewpoint of mathematics as a human activity. In brief, he regarded mathematics as an activity where people organize reality, in ways where the interplay of form and content of mathematics is respected. To learn mathematics is to re-invent mathematics. "Children should repeat the learning process of mankind, not as factually took place but rather as it would have done if people in the past had known a bit more of what we know now. (1991, p.48)" To teach mathematics is to guide the learner through the process of re-inventing mathematics. This activity of guiding reinvention means "striking a subtle balance between the freedom of inventing and the force of guiding, between allowing the learner to please himself and asking him to please the teacher. (1991, p.48)"

During the process of reinventing mathematics, the learner is engaged in an activity where experience is described, organized, and interpreted by mathematical means. This activity, according to Freudenthal, is 'mathematising'.

Wittmann (1984, 1995/1998, 2001) proposed a systemic-evolutionary approach to mathematics education, which rested upon the study and development of substantial learning environment (abbreviated SLE). A substantial learning environment is a teaching/learning unit with the following properties:

1. It represents central objectives, contents and principles of teaching mathematics at a certain level.

2. It is related to significant mathematical contents, processes and procedures *beyond* this level, and is a rich source of mathematical activities. (italics original)

3. It is flexible and can be adapted to the special conditions of a classroom.

4. It integrates mathematical, psychological and pedagogical aspects of teaching mathematics, and so it forms a rich field for empirical research. (2001, p.2)

He maintained that:

"The design of substantial learning environments around long-term curricular strands should be placed at the very centre of mathematics education. Research, development and teacher education should be consciously related to them in a systematic way." (bold-face original, 2001, p.4)

If we put Wittmann's view of mathematics education as a design science, and Freudenthal's notion of 'mathematising' together, we readily see that the study of mathematics education is the design and investigation of substantial learning environments where students are engaged in the process of mathematising. In what follows, I outline two examples of SLEs, the design of which is enlightened by historical materials. The first one has been tried out in elementary classrooms since 1999 while the second one is still awaiting classroom implementation.

EGYPTIAN UNIT FRACTIONS

Ancient Egyptian's representation of rational values as a sum of unit fractions has attracted considerable attention of both mathematicians and mathematics educators. Questions of investigation range from the solution of certain specific type of Diophantine equation where unit fractions are involved (Sylvester 1880; Kellogg 1921; Curtiss 1922; Beck et al. 1969; Gardner 1978; Brenton & Bruner 1994; Anne 1998), to how classroom investigations can be organized around the application of unit fractions (Hurd 1991; O'Reilly 1992, 1995a, 1995b). More information on Egyptian fractions is available at

http://mathworld.wolfram.com/EgyptianFraction.html

Among the wide range of questions that may be worth exploring by elementary school pupils, the following tasks are considered more manageable:

- Express a proper fraction of positive integers as a sum of distinct unit fractions (the reciprocals of positive integers) — referred to as 'Egyptian representation' for simplicity.

- For a given proper fraction of positive integers, find as many Egyptian representations as possible.

- Find those proper fractions that have an Egyptian representation of just TWO terms.

- Find those proper fractions that do NOT have any Egyptian representations of just TWO terms.

- Investigate the existence of Egyptian representations for EVERY proper fraction of positive integers.

Before going into the technical details of how lessons can be organized, the teacher has to perform a thought experiment where elementary pupils are led through the process of investigating some or all of the tasks above, based on knowledge acquired by the pupils, giving full attention to possible obstacles and ways to overcome them. To begin with, the exploration cannot start before pupils have a good grasp of how to compute addition and subtraction of fractions with different denominators. In addition, a good knowledge of converting between equivalent fractions may be an advantage. In Hong Kong, this is the case for Grade 6 or Grade 5, and occasionally Grade 4 pupils. Described below are results that were obtained from such a category of pupils.

Inevitably, this investigation activity has to be conducted in a purely mathematical context. This is not only unavoidable, but also essential. Given the short attention span of, and the high pressure that is put on, pupils, the development of the lessons must be very elegant and focused. Less related issues have to be put aside, or else we run the risk of having pupils drop out of the discussion before any fruitful attempt is made. Besides, the question of "why bother to do such stupid thing" may arise during the course, and has to be tackled somehow. Another thing is the mismatch between the previous exposure of pupils where they were exclusively asked to work from the left side of $\frac{1}{2}+\frac{1}{5}+\frac{1}{10}=\frac{4}{5}$ to the right, and the current demand where they are required to work along the opposite direction! This is a big jump. It is easy to imagine the desperate feeling of pupils when seemingly none of what they learned before could be applied. The design of the lessons must take into account the possible effect of the intention to abandon working when the number of obstacles grow faster than their being removed.

Having considered the issues above, the entire exploration takes the following major steps:

1. The teacher shows some photos taken from the Rhind Mathematical Papyrus (Robins & Shute, 1987) and begins by highlighting the peculiar way of recording fractional values adopted by Ancient Egyptians. A typical opening remark is: "More than 3500 years ago, the Egyptians did not learn fractions as we do nowadays. However, they managed to represent EVERY fraction you learned by expressing them as a sum of distinct fractions with unit numerator (and occasionally together with the special fraction $\frac{2}{3}$). Today you will enjoy a time-travel back to this era where you will experience the Egyptian style of handling fractions!"

2. Two worksheets (Appendices 1 and 2), each containing four questions for pupils to explore, form the framework of exploration. Worksheet 1 (Unit Fraction 1) invites pupils to begin their exploration by working on simple cases. In particular, Question 1, a warm-up exercise, is followed by Question 2 that pinpoints the main theme --- to express fractional values as a sum of distinct unit fractions. After completing Question 2, the exploration may steer along pupils' interests and suggestions, and hence the other questions may be tackled in a less prescriptive order.

This design was tried out by 4 teachers coming from 3 different schools in Hong Kong from 2000 to 2004, where pupils were considered having below average to above average mathematical achievement. Some other unreported experiments existed. Teachers chose to participate voluntarily either because they were interested in the topic itself, or because they wanted to know what their classroom would look like when authentic mathematical activity, which tapped on the creative capability of pupils, was carried out. As expected, some who were informed of the design were not confident enough of their own mathematical training to conduct lessons that contained very unfamiliar contents. Still some others worried about their pupils' ability to cope with the heavy demand that would likely prevent them to persevere. Constrained by availability of class time, only a total of 2 to 5 teaching periods, about 35 minutes each, spreading across not more than 1 teaching week, were spent on the topic.

As seen from classroom interaction, many pupils, though confronted with a series of difficult tasks, did not give up early as some teacher predicted. Some studied hard even after class. Still some involved their family members in the pursuit. In a classroom where the teacher had successfully developed an open atmosphere for mathematical discussion, many pupils gave their own opinion clearly and confidently, albeit making mistakes frequently. In order that pupils could arrive at something clear and substantive, free discussions among pupils were not only allowed, but essential. Besides, the teacher had to give guidance (in terms of questioning and rephrasing pupils' suggestions etc.) and hint from time to time. Thus, any authentic discovery within the classroom should be better taken as a result of the collective effort of both the pupils (with possibly some help from their family members) and the teacher, instead of the effort of individual pupil alone. Contrasted with the different performance of pupils, one phenomenon was common: the first 20 minutes of the exploration was exclusively a period of frustration for many. It was observed that many pupils need this amount of time to get to a stage when they could write down something. In other words, had the teacher allowed pupils to choose to terminate the activity, all the attainment given below would not ever exist.

What pupils did during class time was captured by the worksheets, but only partially. Some pupils refrained from putting down doubtful or immature things and chose to write on a separate sheet of paper (and did not hand in at the end), while some copied what was written on blackboard (resulting from collective effort of the class). As seen from what was collected, most pupils could follow the exploration. Disregarding Question 1 (the warm-up question) of Unit Fraction 1, and putting aside Question 4 of Unit Fraction 1 and Question 3 of Unit Fraction 2, which were mainly dealt with orally, pupils' work for other questions are consolidated below.

Typical answer for Question 2 of Unit Fraction 1 is shown in Figure 1. Figure 2 is a summary of results for Question 3 of Unit Fraction 1 obtained during a lesson. One devoted pupil even wrote down many answers (Figure 3) with increasingly large denominators! He, being a Grade 4 pupil who had not learnt multiplication and division of fractions, further explained to the whole class that after he changed $\frac{1}{4}$ to $\frac{1}{5}+\frac{1}{20}$, he noticed that $\frac{1}{8}$ is half of $\frac{1}{4}$, hence should be equal to halving each of the terms of $\frac{1}{5}+\frac{1}{20}$. By multiplying the denominators by 2, he got $\frac{1}{10}+\frac{1}{40}$. Putting everything together, he found

$$\frac{3}{8}=\frac{1}{4}+\frac{1}{8}=\frac{1}{5}+\frac{1}{20}+\frac{1}{8}=\frac{1}{5}+\frac{1}{20}+\frac{1}{10}+\frac{1}{40}.$$

$$\frac{2}{3}=\frac{1}{2}+\frac{1}{6} \qquad \frac{9}{20}=\frac{1}{4}+\frac{1}{5}$$

$$\frac{4}{5}=\frac{1}{2}+\frac{1}{5}+\frac{1}{10}$$

$$\frac{7}{10}=\frac{1}{5}+\frac{1}{2}$$

$$\frac{5}{12}=\frac{1}{4}+\frac{1}{6}$$

Figure 1

$$\frac{3}{8}=\frac{1}{3}+\frac{1}{24}$$

$$\frac{3}{8}=\frac{1}{5}+\frac{1}{8}+\frac{1}{20}$$

$$\frac{3}{8}=\frac{1}{5}+\frac{1}{10}+\frac{1}{20}+\frac{1}{40}$$

$$\frac{3}{8}=\frac{1}{5}+\frac{1}{10}+\frac{1}{25}+\frac{1}{50}+\frac{1}{100}+\frac{1}{500}$$

$$\frac{3}{8}=\frac{1}{4}+\frac{1}{10}+\frac{1}{40}$$

Figure 2

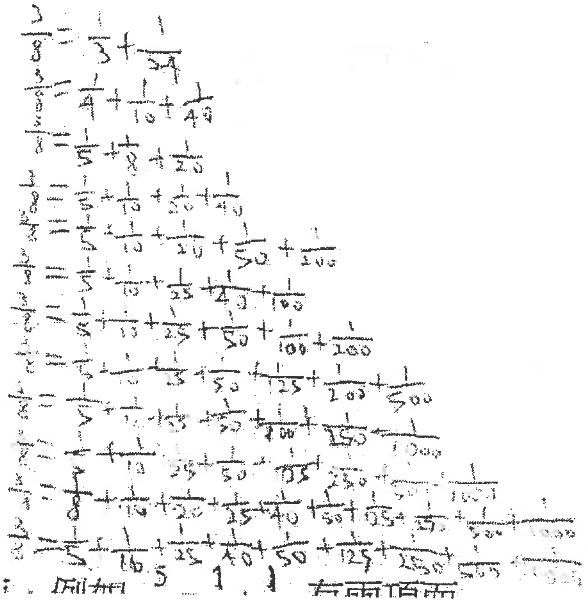

Figure 3

When dealing with Question 1 of Unit Fraction 2, some pupils (from each of the classroom episodes) did manage to identify the 'trick' that the difference between two consecutive unit fractions is again a unit fraction, and applied it to generate answers (Figure 4). Yet still some were happy about the guess-and-check strategy (Figure 5). There were a few who managed to describe the 'trick' by oral or written expression (Figure 6).

$$\frac{1}{3}=\frac{1}{4}+\frac{1}{12}$$
$$\frac{1}{4}=\frac{1}{5}\ \frac{1}{20}$$
$$\frac{1}{5}=\frac{1}{6}+\frac{1}{30}$$
$$\frac{1}{6}=\frac{1}{7}+\frac{1}{42}$$
$$\frac{1}{7}=\frac{1}{8}+\frac{1}{56}$$

Figure 4

$$\frac{1}{3}=\frac{1}{12}+\frac{1}{4}$$
$$\frac{1}{4}=\frac{1}{12}+\frac{1}{6}$$
$$\frac{1}{5}=\frac{1}{30}+\frac{1}{6}$$
$$\frac{1}{6}=\frac{1}{18}+\frac{1}{9}$$
$$\frac{1}{7}=\frac{1}{56}+\frac{1}{8}$$

Figure 5

$$\frac{1}{6}=\frac{1\times(6+1)}{6\times(6+1)}=\frac{6+1}{42}$$
$$=\frac{6}{42}+\frac{1}{42}=\frac{1}{7}+\frac{1}{42}$$

Figure 6

Question 2 of Unit Fraction 2 was well attempted by pupils. Many answers were obtained (Figure 7a and Figure 7b).

$$\frac{5}{6}=\frac{1}{2}+\frac{1}{3}$$
$$\frac{5}{6}=\frac{1}{2}+\frac{1}{4}+\frac{1}{12}$$
$$\frac{5}{6}=\frac{1}{2}+\frac{1}{6}+\frac{1}{9}+\frac{1}{18}$$
$$\frac{5}{6}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{18}+\frac{1}{36}$$
$$\frac{5}{6}=\frac{1}{2}+\frac{1}{7}+\frac{1}{6}+\frac{1}{42}$$
$$\frac{5}{6}=\frac{1}{2}+\frac{1}{4}+\frac{1}{16}+\frac{1}{48}$$
$$\frac{5}{6}=\frac{1}{2}+\frac{1}{6}+\frac{1}{9}+\frac{1}{18}+\frac{1}{54}$$

Figure 7a

$$\frac{5}{6}$$
$$=\frac{1}{2}+\frac{1}{3}$$
$$=\frac{1}{2}+\frac{1}{4}+\frac{1}{12}$$
$$=\frac{1}{3}+\frac{1}{6}+\frac{1}{4}+\frac{1}{12}$$
$$=\frac{1}{3}+\frac{1}{7}+\frac{1}{42}+\frac{1}{5}+\frac{1}{20}+\frac{1}{13}+\frac{1}{156}$$
$$=\frac{1}{4}+\frac{1}{12}+\frac{1}{8}+\frac{1}{56}+\frac{1}{42}+\frac{1}{6}+\frac{1}{30}+\frac{1}{20}+\frac{1}{13}+\frac{1}{156}$$

Figure 7b

$$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$$
$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{12}$$
$$= \frac{1}{2} + \frac{1}{5} + \frac{1}{20} + \frac{1}{13} + \frac{1}{156}$$
$$= \frac{1}{2} + \frac{1}{6} + \frac{1}{30} + \frac{1}{21} + \frac{1}{420} + \frac{1}{14} + \frac{1}{182} + \frac{1}{157} + \frac{1}{24492}$$

Figure 8

Coupled with the 'trick' they learned from Question 1 of Unit Fraction 2, some even produced expressions with surprisingly many terms and shockingly large denominators (Figure 8). Some were able to go further to assert that there are infinitely many such expressions (Figure 9).

$$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$$
$$\frac{5}{6} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{20}$$
$$\frac{5}{6} = \frac{1}{4} + \frac{1}{5} + \frac{1}{8} + \frac{1}{15} + \frac{1}{20} + \frac{1}{30} + \frac{1}{40} + \ldots$$ [illegible]

有無限個

Figure 9

Limited by time, rarely did Question 4 of Unit Fraction 2 receive significant attention. In most cases, it was not discussed at all, except possibly some devoted pupils might have brought it into a private discussion with the teacher inside or outside the classroom. Only a few pupils could explicitly say that the two-term representation of $\frac{5}{6}$ is unique while providing another representation for $\frac{3}{20}$ (Figure 10).

$\frac{5}{6}$不能，$\frac{3}{20}$能

$$\begin{aligned}\frac{3}{20} &= \frac{1}{40} + \frac{1}{8} \\ &= \frac{1}{40} + \frac{5}{40} \\ &= \frac{6}{40} \\ &= \frac{3}{20}\end{aligned}$$

Figure 10

It can be easily imagined that throughout the lessons, a major portion of the time was spent on either doing calculations or correcting computational mistakes. In fact, one of the aims of such exploration is to engage pupils in doing calculations with fractions. Beyond that, pupils were also involved in an inquiry of 'deep' mathematics, which could hardly be developed without addressing the communication aspect of mathematics learning. Regarding Question 4 of Unit Fraction 1 and Question 3 of Unit Fraction 2, pupils' ideas and ability to express were better than many would have thought. In two of the classes, there were pupils who managed to supply orally an argument saying that $\frac{1}{2}$ and $\frac{1}{3}$ are the largest two unit fractions (less than 1), and they add up to $\frac{5}{6}$, a value still less than $\frac{6}{7}$, implying it is not possible to express $\frac{6}{7}$ as a sum of TWO distinct unit fractions. In one of the cases, it was followed by a pupil, making use of this result, to claim:

Theorem 1

Any proper fraction greater than $\frac{5}{6}$ cannot be a sum of two distinct unit fractions.

To add, one pupil successfully identified that $\frac{4}{5}$ is an example of proper fraction, which is not a sum of two distinct unit fractions. It was a pity that neither its relation to the converse of Theorem 1, nor how it could be deduced was examined.

The search for a general method to find Egyptian representation took the largest amount of classroom time spent. Apart from trial-and-error, pupils from all the classes, irrespective of teacher and grade level, in one way or another, arrived at the 'primitive' method having the following steps:

1. Multiply both the denominator and numerator by a positive integer.

2. Try to express the new numerator as a sum of distinct factors of the new denominator.

3. If successful, split into several terms and reduce each fraction to a unit fraction. The final expression is required. If failed, go back to Step 1 and try to multiply by another positive integer.

Although this does not turn into an algorithm automatically, with slight modification, it will become the greedy algorithm of Sylvester (1880), which is essentially successive subtraction of the largest unit fraction not yet used from the remaining value until a unit fraction is left (which is guaranteed by a strictly decreasing sequence of numerators consisting of positive integers). In the classrooms, most pupils exercised their own preference both on choosing the multiplier and when splitting the 'blown up' numerators, thus arrived at different expressions. Even more surprisingly, one pupil from Grade 4 mentioned Sylvester's algorithm, although he admitted that he did not know what to do if he could not get an Egyptian representation within a few steps. This is unusual. As observed, most pupils began with the given proper fraction, hoping that by juggling with it, they could arrive at an answer. He was the only one who considered the given proper fraction and the sequence of unit fractions together.

On the affective side, pupils were generally positive towards the experience. Based on questionnaires collected, over 70% agreed that the exploration was interesting and challenging. Around 20% felt that the tasks were too difficult. Yet some others explicitly mentioned that the tasks were difficult but interesting. Most of the free responses requested that more experience of this type should be included in their mathematics lessons.

EUCLID'S TREATMENT OF COMMON FACTORS AND MULTIPLES

In his great work *Elements*, Euclid used lengths to represent numbers (Heath, 1956). Unlike what we commonly do in a number theory course where deduction is almost entirely done algebraically, Euclid derived results of factors and multiples by measurement activities. In Book VII Proposition 2, he derived his famous Euclidean algorithm for finding the highest common factor of two positive integers. Based on the notion of co-primeness, he developed, in Proposition 34, a method to find the least common multiple of two positive integers, which is essentially dividing the product of two numbers by their highest common factor. Pages in between Proposition 2 and Proposition 34 were used to pave the way for the proof of Proposition 34. The geometric flavour of his proofs diminished significantly throughout those pages in the book when his measurement activities were replaced more and more by deductive arguments, which were based on previous propositions. In his proof of Proposition 34, he invoked seven propositions established previously in the book.

Nowadays, with the concept of prime factor, the fact that the product of two numbers is equal to the product of their highest common factor and least common multiple is often deduced via the unique factorization theorem. What Euclid might not know is that his measurement activities may well be applied to derive, in a simple and direct way, not only a recursive algorithm to find the highest common factor of two positive integers, but also a recursive algorithm to find the least common multiple of two positive integers! The following basic result is a starting point.

Theorem 2

Let a, b be positive integers. If r is the remainder ($0 \leq r < b$) when a is divided by b, the common factors of a and b are identical to those of b and r.

This Theorem 2 can be easily derived by measurement activities where anything measuring both a and b is demonstrated to measure indeed both b and r and vice versa. It establishes a recursive method of finding the highest common factor of two positive integers by reducing the task to one involving two smaller numbers. The Euclidean algorithm for finding the highest common factor of two positive integers is but a corollary. Other basic results concerning factors may well be derived analogously.

In order that Euclid's Proposition 34 can be derived directly from measurement activities without side-tracking to concepts like 'prime' or 'co-prime', we need to turn the following algebraic relations into some corresponding phenomena of measurement activities: (i) m is a common multiple of a and b, and (ii) m is the least common multiple of a and b.

Unlike the case of Theorem 2 where we actually reduce the set of all common factors (rather than the highest common factor itself), here we reduce the calculation of the least common multiple instead. At the classroom operational level, we need the following definitions, which correspond to (i) and (ii) listed above:

Definition I

Let M, A, and B be lengths that represent respectively positive integers m, a, and b. If the following phenomenon occurs (Figure 11), we say that M is an end-match of A and B, or A end-matches B at M, or B end-matches A at M.

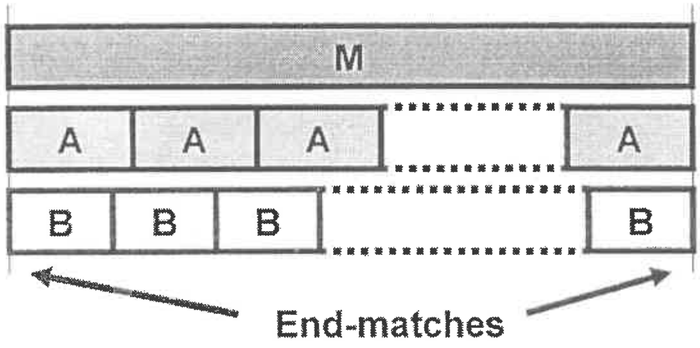

Figure 11

Definition II

Let M, A, and B be lengths that represent respectively positive integers m, a, and b. If M is the first (hence shortest) end-match of A and B, we say that A first end-matches B at M, or B first end-matches A at M.

If B (first) end-matches A at A, b is a factor of a. Conversely, if B (first) end-matches A at B, a is a factor of b. In general, the first end-match occurs with both A and B appear at least twice, equivalent to saying that neither a nor b is a multiple of the other.

To derive Proposition 34, let us consider the case when A first end-matches B at M. Suppose A is longer than B, we will see, in general, a gap C existing between A and the maximum number of B falling short of A (Figure 12). It corresponds to the remainder c when a is divided by b. The crucial step to note is that the number of times C appears here is the same as both the number of times A appears in its first end-match with B, and the number of times C appears in its first end-match with B (or else a smaller number of B is required to first end-match A).

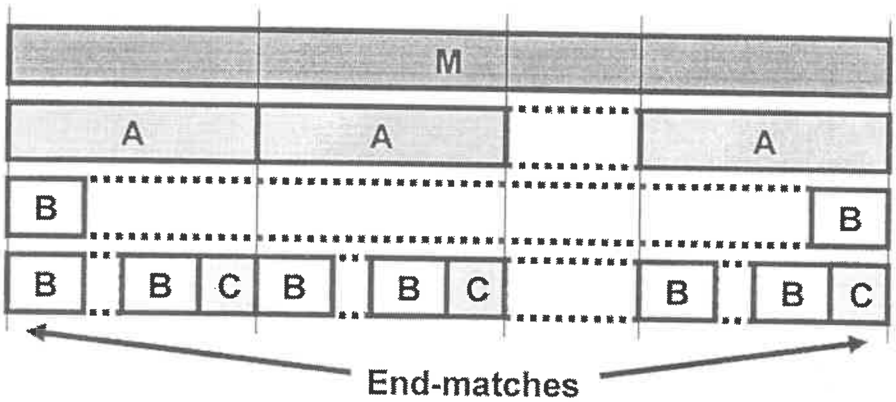

Figure 12

Thus, if C first end-matches B at N, we have the following:

$$\begin{aligned}\text{length of } M &= \text{length of } A \times \text{number of } A \text{ that first end-matches } B \\ &= \text{length of } A \times \text{number of } C \text{ that first end-matches } B \\ &= \text{length of } A \times \frac{\text{length of } N}{\text{length of } C}\end{aligned}$$

It translates to the following recursive relation:

Theorem 3

If a, b, c are positive integers such that c is the remainder when a is divided by b, we have:

$$\text{the least common multiple of } a \text{ and } b = a \times \frac{\text{the least common multiple of } b \text{ and } c}{c}$$
$$= \frac{a}{c} \times \text{the least common multiple of } b \text{ and } c$$

By carrying out the Euclidean algorithm for finding the highest common factor of two numbers, we get a sequence of strictly decreasing sequence of positive integers $a_1, a_2, a_3, \ldots, a_{n-2}, a_{n-1}, a_n$ where a_{k+2} is the remainder when a_k is divided by a_{k+1} (k=1, 2,...n-2), and a_n divides a_{n-1} (which means that a_n is the highest common factor of a_1 and a_2, and a_{n-1} is the least common multiple of a_{n-1} and a_n). Invoking Theorem 3 repeatedly, we get

$$[a_1, a_2] = [a_2, a_3] \times \frac{a_1}{a_3} = [a_3, a_4] \times \frac{a_2}{a_4} \times \frac{a_1}{a_3} = [a_4, a_5] \times \frac{a_3}{a_5} \times \frac{a_2}{a_4} \times \frac{a_1}{a_3} = \ldots. =$$
$$[a_{n-1}, a_n] \times \frac{a_{n-2}}{a_n} \times \frac{a_{n-3}}{a_{n-1}} \times \frac{a_{n-4}}{a_{n-2}} \times \ldots.. \times \frac{a_3}{a_5} \times \frac{a_2}{a_4} \times \frac{a_1}{a_3} = \frac{a_1 \times a_2}{(a_1, a_2)}$$

where $[a,b]$, (a,b) denote respectively the least common multiple and the highest common factor of a and b.

It is not clear, at this moment, to what student group is such an approach effective. With suitable manipulatives, the reasoning does not go very much beyond the capability of elementary school pupils. Apart from the results, which may well be memorized by any keen learner without ever digging into the details, the measurement activities upon which the results are constructed via the finger-tips serve to illustrate a powerful, yet elementary way of doing mathematics. The treatment of Euclid, despite its old-fashioned packing, does open up an alternative way to develop a whole batch of results which otherwise may not be accessible to many.

EPILOGUE

My personal experience with the design and implementation of the two SLEs above provides much for teachers to ponder. Firstly, when pupils' performance was reported in Hong Kong, most teachers were surprised by what pupils did. Some explicitly said that they would never have imagined anything close. Many began to re-examine their belief regarding the interest and ability of their pupils. It is particularly thought-provoking to see that the division between attentive and inattentive pupils changed dramatically when they were confronted with the unit fraction investigation. Some of those who were believed to be conscientious pupils lost the attention soon, while some regularly inattentive ones concentrated on the tasks with all their might. Secondly, do the lessons we conducted for our pupils provide them with the environment and tools to re-invent mathematics? Are they given the opportunity to get their hands dirty before being put to learn 'clever' routines, which probably take years to evolve in the human culture? Why is that some teachers find it painful to teach even the mechanical computation of fraction addition while some manage to involve pupils in deep mathematical investigation like Egyptian unit fractions without much difficulty? Can we see a clear path through which pupils mathematise?

Perhaps we should insist on developing mathematics from what is genuinely in the minds of our pupils, not from what we assume to be in their minds. And for the sake of mathematising, let us temporarily forget about whether our pupils have arrived at the most elegant (hence most polished) version of the mathematics they are working with, and focus on whether they experience the making of mathematics in its most primitive and naïve way, as long as they value and make sense of what they are doing, together with the pain and gain that embrace it.

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Appendix 1: Unit Fraction 1

Unit fractions are fractions whose denominators are positive integers, and whose numerators are '1', viz. $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6} \ldots$

1. Use the following unit fractions (each not more than once) to make sums, what proper fractions can you get?

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$$

2. Ancient Egyptians used to represent proper fractions as a sum of a certain number of distinct unit fractions, for examples

$$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}, \quad \frac{6}{7} = \frac{1}{2} + \frac{1}{3} + \frac{1}{42}.$$

Try to express the following proper fractions as a sum of distinct unit fractions.

$$\frac{2}{3}, \frac{4}{5}, \frac{7}{10}, \frac{5}{12}, \frac{9}{20}$$

3. There may be more than one representation into sum of distinct unit fractions, for example $\frac{5}{6} = \frac{1}{2} + \frac{1}{3} = \frac{1}{2} + \frac{1}{4} + \frac{1}{12}$.

Try to find another representation of $\frac{3}{8} = \frac{1}{4} + \frac{1}{8}$.

4. Representation as a sum of distinct unit fractions may have different number of terms, for examples there are two terms in $\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$ while there are three in $\frac{6}{7} = \frac{1}{2} + \frac{1}{3} + \frac{1}{42}$.

   Try to investigate if $\frac{6}{7}$ has a representation of two terms.

   Can you find two proper fractions for which two-term representation does not exist?

Space in between questions for pupils to write out their work is omitted here.

Appendix 2: Unit Fraction 2

1. Unit fractions themselves can be represented as a sum of two distinct unit fractions, for example $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$.

   Try to express $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$ as a sum of two distinct unit fractions.

2. Write down as many representations of $\frac{5}{6}$ as a sum of distinct unit fractions as possible and examine how many representations of this kind can be found.

3. Apart trial-and-error, can you tell a method where you can express a proper fraction as a sum of distinct unit fractions?

4. Is it possible that a proper fraction has more than one representation as a sum of two distinct unit fractions? That is, can you find representations other than $\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$, $\frac{3}{20} = \frac{1}{10} + \frac{1}{20}$ that express the respective values as a sum of

two distinct unit fractions?

Space in between questions for pupils to write out their work is omitted here.

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Two Examples from the Natural Sciences and their Relationship to the History and Pedagogy of Mathematics

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Abstract: *Mathematics and physics have been closely intertwined since ancient times. The development of optics, especially the study of the phenomena of reflection and refraction of light, provides an interesting perspective on how mathematics plays a vital role in science. Chemistry has come to depend heavily on mathematics too; enzyme kinetics is a particularly striking example. From a pedagogical perspective, an integrated approach to teaching, in which mathematics and science interact with each other, is a viable and desirable option. Also, a genetic approach provides an adequate historical framework that can enhance the learning process.*

Keywords: Calculus, differential equations, enzyme kinetics, historical perspective, integrated approach to teaching, principle of least time, reflection and refraction of light, steady-state hypothesis

INTRODUCTION

The close relationship between mathematics and physics goes as far back as the time of Euclid and Archimedes in the third century before the Common Era. Euclid wrote a book about optics and Archimedes did important work on statics and hydrostatics. This relationship has been especially strong since Galileo established the modern conception of the scientific method, wherein mathematics plays a crucial role, in the first decades of the 17^{th} century. After Galileo many notable mathematicians have made contributions to physics: Descartes, Fermat, Leibniz, the Bernoulli brothers in the 17^{th} century, Euler, D'Alembert and Lagrange in the 18^{th} century, Laplace, Cauchy, Gauss and Riemann in the 19^{th} century, Poincaré and von Neumann in the 20^{th}

century. Newton himself made outstanding contributions to both mathematics and physics.

A deep connection between mathematics and chemistry, on the other hand, started in the 19th century; some two hundred years after a similar relationship had been consolidated between mathematics and physics. Nonetheless, despite this late interaction, important branches of chemistry, such as thermodynamics and chemical kinetics, have come to depend heavily on mathematics.

More recently, the biological sciences have been subject to a process of mathematization. Indeed, the degree to which mathematics is used in a certain discipline often is a measure of how far it has become a mature quantitative science. What happens is that the scientific method, with its three stages of formulating hypotheses, obtaining consequences, and contrasting them with reality, uses mathematics at its very core. As we will see, a distinctive feature of the scientific method is its demand that a model should make predictions to be tested against the real world. The latter is the "supreme judge" in the process of validation.

We have chosen examples from optics and chemical kinetics to provide a framework for discussion of the interplay between mathematics and the natural sciences across time. Some pedagogical issues will be analyzed as well. An underlying theme is the close relationship between the history of mathematics and the history of science.

OPTICS

For more than two millennia scientists have been using mathematics in the field of optics. During the Hellenistic era, a roughly 600-year period that starts in 331 B.C.E. with the founding of Alexandria, optics was one of the first scientific theories to be developed. Besides Euclid's "Optics" (300 B.C.E.?) and Heron's "Catoptrics" (circa 100 C.E.), a notable work was "Optics" by Ptolemy (circa 150 C.E.). In a later era, at the height of Muslim mathematics and science we have to mention "Treasury of Optics" by al-Hassan ibn al-Haytham, also known as Alhazen (986-1050 C.E.). We should add that during the 17th century Descartes, Fermat, Huygens, and Newton occupied themselves with the phenomenon of light.

We start by analyzing a purely mathematical problem, namely the characterization of the path of least distance, and then use it to study the law of reflection on a plane mirror from a geometrical perspective. Afterwards we compare this approach with the calculus alternative.

Given points P and Q above a certain line L we wish to find a point O, lying on the line, such that the distance $PO + OQ$ is minimal. Draw the perpendicular segment $\overline{QS}$ and extend it up to T so that $ST = QS$ (figure 1). Let O be the point where $\overline{PT}$ intersects the given line L. We claim that POQ is the path of least distance. Let us prove this claim: First of all we note that $\Delta QSO \cong \Delta TSO$ (SAS property of congruence). Thus $OQ = OT$. Let $PO'Q$ be any other path and draw $\overline{O'T}$. The above-mentioned property of congruence allows us to conclude that $\Delta QSO' \cong \Delta TSO'$, consequently $O'Q = O'T$. Then we apply the triangle inequality to $\Delta PTO'$ and obtain $PT < PO' + O'T$. Thus $PO + OT < PO' + O'T$, which in turn leads to $PO + OQ < PO' + O'Q$.

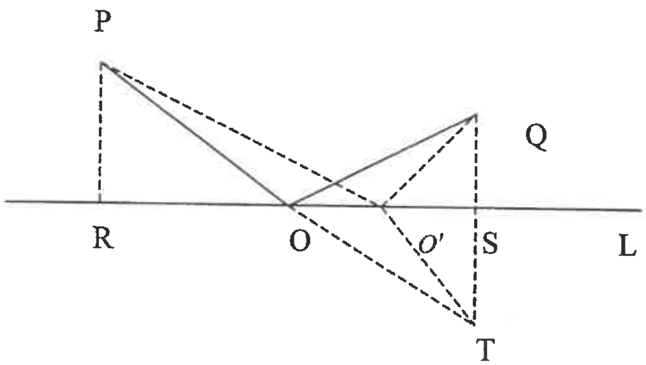

Figure 1

The law of reflection

Let us suppose that light chooses the path of least distance when traveling from P to Q, after touching the mirror. This is a minimization principle that appeared in Heron's "Catoptrics", the first principle of its kind ever published (Russo 2004). It is our task to show that the angles of incidence and reflection are congruent. According to the characterization of the path of least distance we can ascertain that light will choose to go through O (figure 2), where $QS = ST$ ($\overline{PR}$ and $\overline{QT}$ are perpendicular to $\overline{RS}$). Since

$\Delta QSO \cong \Delta TSO$ it follows that $\angle QOS \cong \angle TOS$. But vertical angles have the same measure, consequently $\angle POR \cong \angle SOT$. Thus $\Delta POR \approx \Delta TOS$. However, $\Delta TOS \cong \Delta QOS$. Therefore $\Delta POR \approx \Delta QOS$, which in turn leads to $\angle RPO \cong \angle SQO$.

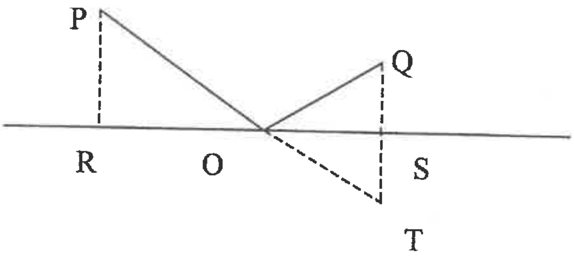

Figure 2

Next assume that $\alpha = \beta$ (the measure of the angles of incidence and reflection). Can we prove that under these circumstances light has chosen the path of least distance? The ray of light chooses a certain path *POQ* (figure 3). Prolong $\overline{PO}$ so that it meets the perpendicular ray $\overrightarrow{QS}$. According to the characterization property all we have to do is show that *QS* = *ST*. First of all we note that

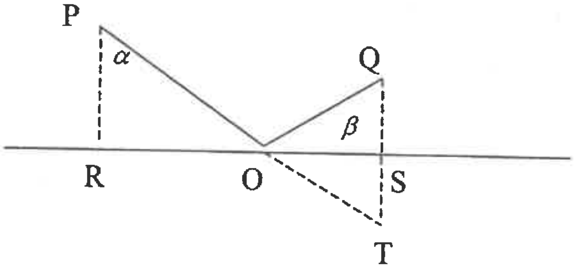

Figure 3

$\Delta POR \approx \Delta QOS$. Therefore $\angle POR \cong \angle QOS$. But $\angle POR \cong \angle TOS$ (they are vertical angles!). By the ASA property of congruence it follows that $\Delta QOS \cong \Delta TOS$. Consequently $QS = TS$.

A calculus perspective of the phenomenon of reflection of light

First let us pose a purely mathematical question: among all possible paths between *P* and *Q*, touching the line $\overleftrightarrow{RS}$, is there a shortest one? Draw coordinate axes with center at R (figure 4) and define $f(x)=\sqrt{a^2+x^2}+\sqrt{(d-x)^2+b^2}$, for all real numbers x. This is the distance function we have to work with. The chain rule and a little bit of algebra leads to $f'(x)=\frac{x}{\sqrt{a^2+x^2}}-\frac{d-x}{\sqrt{(d-x)^2+b^2}}$,

$$f''(x)=\frac{a^2}{(a^2+x^2)\sqrt{a^2+x^2}}+\frac{b^2}{((x-d)^2+b^2)\sqrt{(x-d)^2+b^2}}$$

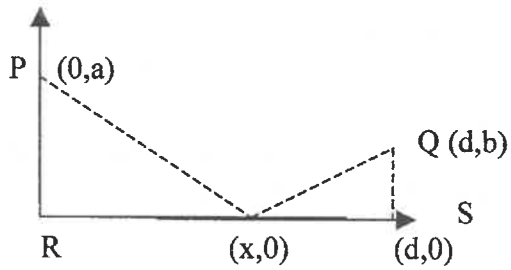

Figure 4

Since $f''(x)>0$ it follows that $f'(x)$ is strictly increasing, in particular it is one-to-one. We have $f'(0)<0$ and $f'(d)>0$, thus by the intermediate value theorem there exists c between 0 and d such that $f'(c)=0$ (such a point is unique because f' is one-to-one). But $f''(c)>0$, so we can conclude that f adopts its absolute minimum at c. We have given an affirmative answer to the question stated at the beginning of this section.

Accepting that light chooses the path of least distance, a ray that starts at P and then goes through *Q*, after touching the mirror, will choose path *POQ*, where *O* = (c,0) (figure 5). Since $f'(c)=0$ we will have $\frac{c}{\sqrt{a^2+c^2}}=\frac{d-c}{\sqrt{(d-c)^2+b^2}}$, so $\sin\alpha=\sin\beta$.

Then $\alpha=\beta$.

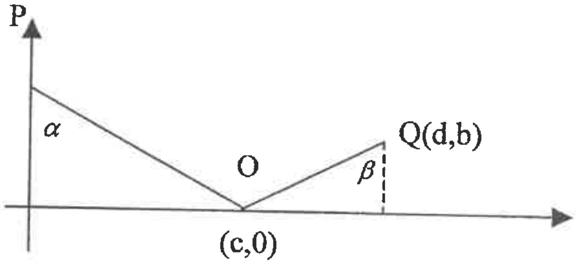

Figure 5

Now let us deal with a different proposition. Namely, under the hypothesis $\alpha = \beta$ we wish to prove that light has chosen the shortest path (figure 6). Recall that the function

$f(x) = \sqrt{a^2 + x^2} + \sqrt{(d-x)^2 + b^2}$, for any real number x, has a unique absolute minimum at a certain point c. Let r be the point where the ray of light hits the mirror. Since $\alpha = \beta$ we get

$\sin\alpha = \sin\beta$. Therefore $\dfrac{r}{\sqrt{a^2+r^2}} - \dfrac{d-r}{\sqrt{(d-r)^2+b^2}} = 0$, that is to say $f'(r) = 0$. But f' is one-to-one, so $r = c$.

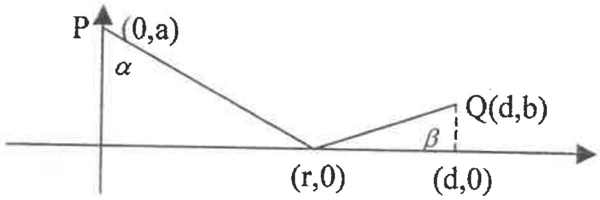

Figure 6

Refraction of Light

At the time of Euclid the phenomenon of reflection of light on a plane mirror was already well understood, in the sense that the basic law of reflection had been determined. But the law of refraction could not be found despite the toils of first-rate scientists such as Ptolemy and Kepler. Finally, after a long quest, Snell found around 1620 that $\sin\alpha/\sin\beta = K$ where α is the angle of incidence and β is the angle of refraction. This law appeared in print for the first time in 1637 in Descartes' opus magnum "Discourse on the Method" (appendix on optics). The constant K depends on the nature of the media being considered (say air and water), and for Descartes and later for Newton this constant was equal to v_2/v_1 where v_1 is the velocity of light in air and v_2 is the velocity of light in water. On the basis of the principle of least time, which happens to be equivalent to the principle of least distance when only one medium is being considered, Fermat (1601-1665) reached the formula $\sin\alpha/\sin\beta = v_1/v_2$ in 1662. Evidently only one of them can be true. It is experimentally determined that $\alpha > \beta$, so $\sin\alpha > \sin\beta$. Thus Fermat's formula implies that $v_1 > v_2$ while Descartes-Newton's formula implies $v_2 > v_1$. In the 17th and 18th century it was not possible to determine, through experiments, whether the velocity of light in air is higher or lower than the velocity of light in water. In 1850 Foucault was able to conduct an experiment using a new and very precise technical device, reaching the conclusion that $v_1 > v_2$. The result of this crucial experiment was one more reason for the demise of the corpuscular theory of light and the universal acceptance of Fermat's principle of least time. The latter was deemed to be valid mainly because of its ability to make predictions that were confirmed by experiments.

A calculus perspective of the phenomenon of refraction of light

The year 1684 marks the appearance of the first paper ever published on the calculus. Its author, Gottfried Leibniz, discusses – among other topics—the deduction of Snell's law on the basis of Fermat's principle of least time. Fermat's proof is long and rather complicated while Leibniz was able to present a proof that is short and direct thanks to the new mathematical techniques that he had invented independently of Newton. Leibniz was so proud of his proof that at the end he wrote: "Many learned men have sought in many devious ways what people versed in this calculus can accomplish, in a few lines, as by magic" (Fauvel & Gray 1987). Besides Fermat's original proof (Sabra 1967) there are proofs that are elementary -- in the sense that calculus is not used -- but none of them seems to be simple (Schiffer & Bowden 1984, Helfgott & Helfgott 2002). A modern proof, inspired on Leibniz' work, goes as follows: Consider two media separated by a plane surface, say air at the top and water at the bottom. Let v_1 be the velocity of light in air and v_2 the velocity of light in water. A ray of light starts at A (in air) and reaches B (in water), hitting the surface of separation at C (figure 7). According to Fermat's principle, light goes from A to C and then from C to B employing the least amount of time. Our goal is to show that

$\sin\alpha/\sin\beta = v_1/v_2$ (α and β are the measures of the angles of incidence and refraction, i.e. $\alpha = m\angle OAC$, $\beta = m\angle CBD$, where $\overline{AO}$ and $\overline{BD}$ are perpendicular to $\overline{OD}$).

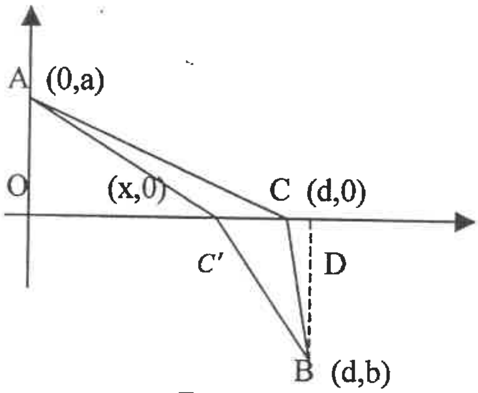

Figure 7

Let us draw a pair of Cartesian axes, as shown in figure 7, and define the time function

$$t(x) = \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(d-x)^2}}{v_2}$$

For any x this is the time it takes a ray of light if it were to follow the path $AC'B$. Next we will make an entirely similar analysis to the one done when we discussed the phenomenon of reflection of light for plane mirrors. We have

$$t'(x) = \frac{x}{v_1\sqrt{a^2+x^2}} - \frac{d-x}{v_2\sqrt{b^2+(d-x)^2}}$$

$$t''(x) = \frac{a^2}{v_1\sqrt{a^2+x^2}\,(a^2+x^2)} + \frac{b^2}{v_2\sqrt{b^2+(d-x)^2}\,(b^2+(d-x)^2)}$$

We will see that this function does have an absolute minimum between 0 and d. Indeed, since $t''(x)>0$ we can conclude that $t'(x)$ is strictly increasing, thus it is one-to-one. Since $t'(0)<0$ and $t'(d)>0$, by the intermediate value theorem there exists a unique c between 0 and d such that $t'(c)=0$. Since $t''(c)>0$ we can conclude that $t(x)$ attains its absolute minimum at c. According to Fermat's principle of least time, the ray of light will choose to go precisely through c. Since $t'(c)=0$ it follows that $(d-c)/v_2\sqrt{b^2+(d-c)^2}=c/v_1\sqrt{a^2+c^2}$. Thus $\sin\beta/v_1=\sin\alpha/v_2$, that is to say $\sin\alpha/\sin\beta=v_1/v_2$. We have achieved the goal set at the beginning.

Next we will try to prove something different, namely: If $\dfrac{\sin\alpha}{\sin\beta}=\dfrac{v_1}{v_2}$, with $R=(r,0)$ the point where the ray of light hits the surface of separation of the two media (figure 8), then the ray has chosen to travel through the path of least time. Let us recall that the function $t(x)=\dfrac{\sqrt{a^2+x^2}}{v_1}+\dfrac{\sqrt{b^2+(d-x)^2}}{v_2}$ attains its absolute minimum at a point c, which is unique. Thus, all we have to do is prove that $r=c$. By hypothesis $\dfrac{r}{v_1\sqrt{a^2+r^2}}-\dfrac{d-r}{v_2\sqrt{b^2+(d-r)^2}}=0$, therefore $t'(r)=0$. Since $t'(x)$ is one-to-one and $t'(c)=0$ (recall that $t'(x)$ adopts its minimum at c) we can conclude that $r=c$.

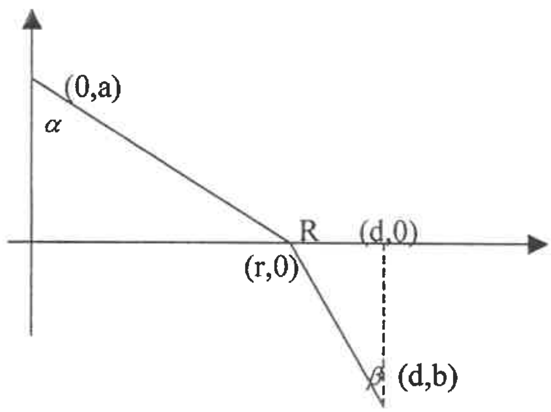

Figure 8

Huygens' proof

Christiaan Huygens, the greatest Dutch scientist of the 17th century, dealt with the phenomenon of refraction in his book "Treatise on Light" (Huygens, 1690/1982). Huygens proved that if Fermat's version of Snell's law is true then a ray of light will follow the path of least time when traversing between two points located in different media. The phenomenon of refraction can best be studied using calculus through a modern version of Leibniz's original analysis of the problem in 1684, but Huygens purely geometrical approach is accessible to students that have taken trigonometry only; his ingenious proof provides a historical perspective about the initial doubts raised by the new calculus techniques.

Assume that a ray of light goes from A in air to C in water (figure 9) in such a way that $\sin\theta_1/\sin\theta_2 = v_1/v_2$ (v_1 is the velocity of light in air, v_2 is the velocity of light in water). Huygens proved, as we will see, that under these circumstances $|AB|+|BC|<|AF|+|FC|$, where B is the point where the ray crosses the surface of separation, F is any other point on this surface, and the symbol $|XY|$ denotes the time it takes light to go from X to Y (this symbol was introduced by Michael Golomb (Golomb, 1964)). In other words, if Fermat's version of Snell's law is true then light chooses the path of least time when going from a point in air to a point in water. Assume F is to the

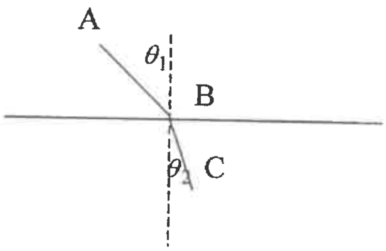

Figure 9

right of B. First of all let us draw a line that starts at F and is parallel to $\overline{AB}$; then we draw the perpendicular segments $\overline{AO}$, $\overline{BH}$, and $\overline{FG}$. Finally we build $\overline{FC}$ (figure 10). We have $\sin\alpha = \dfrac{HF}{BF}$ and $\sin\beta = \dfrac{BG}{BF}$. But $\alpha = \theta_1$ and $\beta = \theta_2$. Why?

Just note that the two quantities $\theta_1+\mu$ and $\alpha+\mu$ are equal to $90°$; similarly $\theta_2+\gamma$ and $\gamma+\beta$ are equal to $90°$. Therefore $BF=\frac{HF}{\sin\theta_1}=\frac{BG}{\sin\theta_2}$,

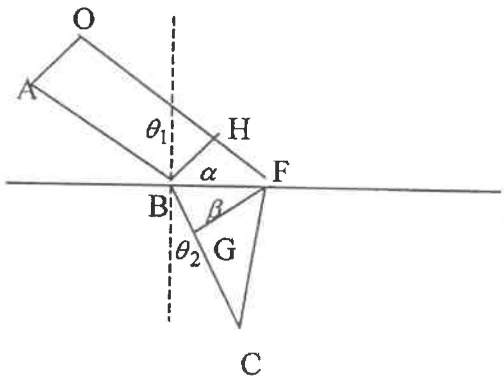

Figure 10

which in turn leads to $\frac{HF}{BG}=\frac{\sin\theta_1}{\sin\theta_2}=\frac{v_1}{v_2}$. Thus $\frac{HF}{v_1}=\frac{BG}{v_2}$ and consequently $|HF|=|BG|$. That is to say, $|HF|-|BG|=0$. Let us recall that our goal is to show that $|AB|+|BC|<|AF|+|FC|$. Indeed, $|AB|+|BC|=|OH|+|HF|-|BG|+|BC|=|OF|+|GC|$. But in any right triangle a leg is smaller than the hypotenuse, so $|OF|<|AF|$ and $|GC|<|FC|$. Therefore $|AB|+|BC|<|AF|+|FC|$. The proof is essentially the same if F is to the left of B. It is interesting to note that Huygens does not mention Leibniz's 1684 paper, with which he was probably acquainted. We may surmise that this was due to the fact that the newly developed techniques of the calculus seemed "highly suspicious" to such masters of Euclidean geometry as Huygens; they lacked "Euclidean rigor".

Enzyme Kinetics

The basic model of enzyme kinetics, namely $S+E\leftrightarrow C\rightarrow E+P$ with parameters k_1, k_{-1}, for the reversible part of the reaction and k_2 for the irreversible part, was developed in the first decade of the 20th century. The substrate S combines with the enzyme E giving birth to an intermediate compound C through a reversible reaction. The intermediate compound in turn decomposes into the product P and regenerates the enzyme. The values of the parameters are unknown, and thus have to be determined by analyzing the data once the model has been validated. The simpler model $S + E \rightarrow P + E$ had to be discarded because it implies, accepting first-order kinetics, that $P'(t) = kS(t)E(t)$ and in particular $v(0) = P'(0) = kS(0)E(0)$, thus the initial velocity of the reaction could be increased, without bounds, by increasing $S(0)$. But such a prediction is not in agreement with experiments: it is known that $v(0)$ cannot be increased beyond a certain amount of $S(0)$.

According to the law of mass action, a basic law of chemistry, the acceptance of first order kinetics leads to the following differential equations:

$$C'(t) = k_1 E(t)S(t) - k_{-1}C(t) - k_2 C(t) \tag{1}$$

$$S'(t) = -k_1 E(t)S(t) + k_{-1}C(t) \tag{2}$$

$$E'(t) = -k_1 E(t)S(t) + k_{-1}C(t) + k_2 C(t) \tag{3}$$

$$P'(t) = k_2 C(t) \tag{4}$$

Furthermore,

$$E_T = E(t) + C(t) \tag{5}$$

where E_T denotes the total amount of enzyme in the process. A fraction of enzyme is free, namely $E(t)$, while the rest is bounded to the intermediate compound.

The aforementioned system of differential equations looks quite complicated, and it is complicated indeed! In 1925 two British scientists, Briggs and Haldane, put forward a daring hypothesis in a remarkable two-page article (Briggs & Haldane, 1925): At the beginning of the experiment, substrate and enzyme combine quite rapidly giving birth to C and thereafter a "steady-state" ensues, during which time the amount of the intermediate compound remains practically constant. This hypothetical situation takes place because we may surmise that whenever a molecule of P is formed by a rearrangement of C, a molecule of the enzyme is regenerated and combines rapidly with a molecule of substrate—there is a high affinity between both of them. This mechanism lasts while there is substrate left. Thus, during considerable time one should expect that $C'(t) = 0$.

Let us see what happens to the system of differential equations if we were to accept the steady-state hypothesis:

From (1) and (5) we have $C'(t) = k_1(E_T - C(t))S(t) - k_{-1}C(t) - k_2C(t)$, that is to say

$C'(t) = k_1E_TS(t) - (k_1S(t) + k_{-1} + k_2)C(t)$. So $0 = k_1E_TS(t) - (k_1S(t) + k_{-1} + k_2)C(t)$. Therefore

$$C(t) = \frac{E_TS(t)}{K_m + S(t)} \tag{6}$$

where we define $K_m = \frac{k_{-1} + k_2}{k_1}$. Furthermore, using (4) and (6) we arrive to

$$P'(t) = \frac{k_2E_TS(t)}{K_m + S(t)} \tag{7}$$

But $P'(t)$ is precisely the reaction velocity $v(t)$. The graph of v(0) as a function of S(0) corresponds to a well-known mathematical function (figure 11). By adding more and more substrate, at the beginning of the experiment, we can increase the speed of the reaction. However, when $S(0)$ is big enough compared to K_m we will have $v(0) = k_2E_T$, so it makes sense to write $v_{\max} = k_2E_T$. A

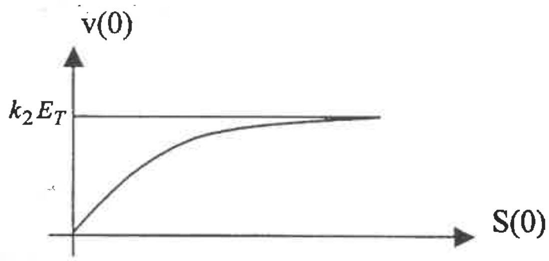

Figure 11

little bit of algebra leads to the equality $-S'(t) = P'(t)$ provided $C'(t) = 0$. That is to say, under a stationary-state scenario we arrive at the differential equation

$$S'(t) = \frac{-v_{\max} S(t)}{K_m + S(t)} \tag{8}$$

known in the literature as the Micaelis-Menten equation, honoring Leonor Micaelis and Maude Menten for their pioneering work in biochemistry at the beginning of the 20th century. Let us pay a closer look at this equation. We notice that it is a separable variables differential equation. We can separate variables and then integrate, reaching

$$K_m \int_0^t \frac{S'(x)dx}{S(x)} + \int_0^t S'(x)dx = \int_0^t -v_{\max}\, dx$$

Hence

$$K_m [\ln S(x)]_0^t + S(t) - S(0) = -v_{\max} t$$

Therefore

$$\frac{1}{t} \ln \frac{S(0)}{S(t)} = -\frac{1}{K_m} \frac{S(0) - S(t)}{t} + \frac{v_{\max}}{K_m} \tag{9}$$

Our goal is not to solve for S(t), a difficult task due to the fact that S(t) appears "inside" a logarithm and all by itself, but rather to obtain consequences of (9) that can be compared with experimental

values. The last equality predicts that if $\frac{S(0) - S(t)}{t}$ is placed on the x-axis and $\frac{1}{t} \ln \frac{S(0)}{S(t)}$ on the y-axis, the points $(\frac{S(0) - S(t)}{t}, \frac{1}{t} \ln \frac{S(0)}{S(t)})$ are to be distributed around a straight line with negative slope. This is a crucial fact, which helps in the process of validation of the original model and the steady-state hypothesis. Very many experiments have been carried out, all of them confirming the above-mentioned distribution.

Once the model has been validated, our job is to calculate the parameters K_m and $v_{\max}$, which happen to vary according to the type of enzymatic reaction being analyzed. We draw the line of best fit that goes through the points $(\frac{S(0)-S(t)}{t}, \frac{1}{t}\ln\frac{S(0)}{S(t)})$, calculated from a table of experimental values $(t, S(t))$. From (9) we can observe that $v_{\max}/K_m$ is the y-intercept, while the x-intercept is found from the equality $0 = -\frac{1}{K_m}\frac{S(0)-S(t)}{t} + \frac{v_{\max}}{K_m}$. Thus $\frac{S(0)-S(t)}{t} = v_{\max}$. Using the line of best fit we can find $v_{\max}$ and $v_{\max}/K_m$. So, $v_{\max}$ and K_m can be calculated. On the other hand, since $v_{\max} = k_2 E_T$ we are able to find k_2. We cannot calculate k_1 and k_{-1} yet, despite the fact that K_m is known already. We will do so next by employing the elementary theory of second order differential equations.

Evidently, it is enough to calculate k_1 because $k_{-1} = k_1 K_m - k_2$. With this purpose in mind we will analyze the basic model of enzyme kinetics before the steady-state, a very short period of time, at the beginning of the experiment, during which it is a good approximation to assume that S(t) is practically S_T; the latter is the total amount of substrate with which we started (Bartholomay, 1972).

We differentiate equation (4). Taking into account (1) we can ascertain that $P''(t) + a_1 P'(t) = a_2$, where $a_1 = k_1 S_T + k_{-1} + k_2$, $a_2 = k_1 k_2 E_T S_T$. It should be mentioned that we have used the equality $E(t) = E_T - C(t)$ and the approximation $S(t) = S_T$. In front of us there is a second order linear homogeneous differential equation. Its characteristic polynomial is $r^2 + a_1 r$ with roots $r_1 = 0$ and $r_2 = -a_1$. By inspection of the differential equation it is evident that the function $\frac{a_2}{a_1}t$ is a particular solution. So its general solution is given by $P(t) = c_1 + c_2 e^{-a_1 t} + \frac{a_2}{a_1}t$. The initial conditions are $P(0) = P_0$ and $P'(0) = k_2 C(0) = 0$. These lead to the algebraic system $c_1 + c_2 = P_0$, $-a_1 c_2 + \frac{a_2}{a_1} = 0$ in the unknowns c_1 and c_2. Then

$P(t) = P_0 - \frac{a_2}{a_1^2} + \frac{a_2}{a_1^2}e^{-a_1 t} + \frac{a_2}{a_1}t$. However, when t is very small we have the approximation $e^{wt} = \sum_{n=0}^{\infty}\frac{(wt)^n}{n!} \approx 1 + wt + \frac{w^2t^2}{2}$. Therefore

$$P(t) = P_0 - \frac{a_2}{a_1^2} + \frac{a_2}{a_1^2}(1 - a_1 t + \frac{a_1^2 t^2}{2}) + \frac{a_2}{a_1}t$$

Simplifying this expression we arrive to $P(t) = P_0 + \frac{a_2}{2}t^2$. Consequently

$$P(t) - P_0 = \frac{k_1 k_2 E_T S_T t^2}{2}$$

Thus

$$k_1 = \frac{2(P(t) - P_0)}{t^2 v_{\max} S_T} \tag{10}$$

In summary, the basic model of enzyme kinetics predicts that if it is possible to measure P(t) before the steady-state, then the expression to the right of (10) will be constant. When around 1950 scientists developed experimental techniques to deal with very short spans of time they could corroborate that the above-mentioned expression remains practically constant. The parameter k_1 was then obtained from the corresponding line of best fit. It is to be emphasized that important parameters such as $v_{\max}$, k_2, K_m, k_1, and k_{-1} are evaluated after the model is validated.

The prediction that stems from (10) was a driving force in developing advanced techniques in what is now called ultra-rapid enzyme kinetics. After all, scientists wanted to know whether the prediction could withstand experimental verification. It might not be an exaggeration to assert that enzyme kinetics provides the opportunity to see the scientific method in its full splendor and power.

What do optics and enzyme kinetics have in common? The principle of least time and the steady- state hypothesis are accepted on the basis of the wealth of predictions that follow from them and are later confirmed by experiments; there is no a priori reason why either has to be true. We may add that although the phenomena of reflection or refraction require different mathematical tools than those encountered in enzyme kinetics (the latter relies heavily on ordinary differential equations, while the former uses mainly geometry and calculus), both provide a vision on how mathematics plays a crucial role in the development of models in the realm of physics and chemistry. Moreover, a historical approach to both subjects cannot but advance the teaching of mathematics and the natural sciences.

Pedagogical Issues

On several occasions in this paper we employed more than one way to present a certain topic, be it reflection on plane surfaces through geometry or calculus, or refraction through calculus or trigonometry. We did so deliberately, in the belief that students benefit from a multiple perspective of a topic. That is to say, adopting different points of view we can improve understanding. We may add that not everything needs to be proven; for instance, the intermediate value theorem for continuous functions, which appears when one wishes to use Huygens' proof to show that the principle of least time implies Snell's law, should be accepted due to its geometrical interpretation (a proof is more appropriate in a real analysis course).

Although we have not discussed the use of technology in the mathematics classroom, graphics calculators play an important role in optics (Helfgott 1998, Helfgott & Simonsen 1998), as well as in enzyme kinetics, especially when we have to calculate the line of best fit (also called the regression line). A proper use of technology can greatly facilitate the task of learning and teaching mathematics.

What pedagogical ideas stem from the close partnership between mathematics and the natural sciences? Until the 1960's, at most universities, students majoring in mathematics, or prospective High School mathematics teachers, were required to take several calculus-based courses in physics, because physics, and to a lesser extent chemistry, is a source of many interesting and illuminating examples that help in the understanding of mathematics. Unfortunately, a trend toward the elimination of the natural sciences from the curricula of the above-mentioned students has been gaining momentum; nowadays it is fairly common in the U.S. that prospective High School teachers of mathematics get their degree without ever having taken a single course in physics.

To remedy this situation several mathematics educators advocate an integration of the teaching of mathematics and science starting from the school level. The problem of integrating mathematics and science in the K-12 curriculum in the U.S. has a long history that goes back to E.H. Moore at the turn of the 20th century. Moore, Professor of Mathematics at the University of Chicago, recommended teaching mathematics in close relationship to problems in physics, chemistry, and engineering. After almost a century a great deal of evidence has been accumulated (Czerniak et al. 1999). It is to be noted that several college courses have been developed in the recent past that blend mathematics and the natural sciences (for instance Helfgott 1990, Jean and Iglesias 1990); some also provide a historical perspective (Helfgott 1995).

L. A. Steen (Steen 1994) considers various possibilities of change, among them:

[1] *Employing scientific examples and methods thoroughly in mathematics instruction, taking necessary steps to coordinate the curricula of the two subjects. This too would have great benefit both for mathematics and for science, but in uncommon ways. For mathematics, it would reinforce the perspective of investigation, exploration, and experimentation that is so important to all contemporary curricular recommendations. For science, it would help underscore the importance of careful data analysis, logical thinking, and modeling as part of the scientific method.*

[2] *Employing mathematical methods thoroughly in science, and scientific methods thoroughly in mathematics, coordinating both subjects sufficiently to make this feasible. This is, I submit, an ideal situation. Each discipline, science and mathematics, would accrue benefits from an infusion of methods of the other, but neither would lose its identity or distinguishing features in an artificial effort at union. There are, after all, important differences between science and mathematics, both philosophical, methodological, and historical. These should not be lost in a misguided effort at homogenization.*

This process of integration, along the lines established by Professor Steen, could benefit not only prospective schoolteachers but future mathematicians too. After all, only a small segment will specialize in branches of mathematics that are entirely removed from any foreseeable applications to the real world. We advocate the use of the natural sciences as a pedagogical tool in the mathematics classroom whenever possible, while keeping a historical perspective. The genetic approach (Tzanakis 2000, Mosvold 2003), wherein one looks for the origins of a concept or idea without necessarily giving step-by-step historical details, is ideally suited to achieve such a goal. The examples on optics and chemical kinetics discussed in previous pages illustrate our thoughts on this matter.

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New Developments and Trends in Secondary Mathematics Education

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INTRODUCTION

10th International Congress on Mathematical Education (ICME-10)

Under the auspices of ICMI (International Commission on Mathematical Instruction), the 10th International Congress on Mathematical Education (ICME-10) was held in Copenhagen, Denmark from 4-11 of July 2004. The aim of the ICME congresses is to show what is happening in mathematics education worldwide, in terms of research as well as teaching practices, exchange information on the problems of mathematics education around the world and learn and benefit from recent advances in mathematics as a discipline.

In particular, the scientific programme for ICME-10 included the following elements:

I. eight Plenary sessions (P),

II. 80 Regular lectures running parallel in five time slots (R),

III. 29 Topic Study Groups (TSG) with four time slots each,

IV. 24 Discussion Groups (DG) working in three time slots,

V. a Thematic Afternoon (TA) with five parallel mini-conferences, five National Presentations (NP), the Small Group Activities (SGA) consisting of 46 Workshops, and 12 Sharing Experiences Groups,

VI. a Poster Exhibition with more than 220 posters, and meetings in ICMI-Affiliated Study Groups (ASG) and presentations of ICMI-Studies

Topic Study Groups

As the name suggests, a Topic Study Group is designed to gather a group of congress participants who are interested in a particular topic in mathematics education. The word "study" suggests that the activities of the groups include presentations and discussions of important new trends and developments in research or practice related to the topic under consideration. Each TSG is organised by a team of five prominent experts in the field. Two chairs are appointed for each team. The purpose of the TSGs is to provide both an overview of the current state-of-the-art in the topic, and expositions of outstanding recent contributions to it, as seen from an international perspective. By their very nature, some of the topics are focused more on research than on practice. For others the opposite will be the case, whereas several topics will have a fairly equal balance of the two.

To some extent, a number of the topics in the list for the Topic Study Groups have mutual overlaps with other topics. Even so, each topic has a well-defined and specific "centre of gravity" that makes it clearly discernible from the others in the list.

The list of topics is organised into four clusters. The first cluster, TSG 1-7, consists of Topic Study Groups that focus on the target groups of mathematics teaching as reflected in educational levels and special categories of students. The next cluster, TSG 8-21, focuses on matters and issues pertaining to content-related aspects of mathematics curricula, across educational levels, and to teaching and learning in relation to those aspects. The cluster formed by TSG 22-27 deals with the overarching perspectives and facets of mathematics education that are present across different educational levels and different curricula. Finally, TSG 28-29 are the Groups in which meta-issues concerning mathematics education itself, as a field of practice and a discipline of research, are the focus of attention.

The ICME-10 Topic Study Group 2 (TSG2) - New Development and Trends in Secondary Mathematics Education

The ICME-10 Topic Study Group 2 (TSG2), *New Development and Trends in Secondary Mathematics Education,* aimed at discussing and sharing opinions, experiences, and research results within the ICME-Community related to this broad theme.

There are several trends and projects in the world that represent the reform of mathematics education at the secondary level. These include policy, curriculum or textbook developmental research; developing teaching practices based on classroom research such as lesson studies and the development of teaching-learning environments for mathematics using new technologies, as well as the results and the impact of international comparative studies.

TSG2 focused on future movements in mathematics education at secondary level and exemplarily illustrated these movements by presentations on:

1. Research projects for curriculum development having the potential to influence mathematics education in the next decades;
2. Policies of secondary schools' reforms having the potential for new trends in secondary mathematics education;
3. Developmental studies of teaching new contents in mathematics;
4. Developmental studies of new ways of teaching mathematics;
5. Influential research results in mathematics education for secondary school level.

In the first session, internationally known specialists presented their ideas on three main issues related to the central theme of TSG2, namely the impact of new technologies, curricular developments, and the role and results of international comparative studies. In the second and third session, papers around two themes were grouped. The first, *Curricular Developments and New Contents* provided our community with a description of current events in different parts of the world and the second, *Learning from Research and Classroom Practice* provided us with some of the means to critically reflect on the new trends. In the fourth and final session, the chairs and members of the Organizational Team presented their personal opinions and reflections on *New Development and Trends in Secondary Mathematics Education* and on the different papers that were presented in the fist three sessions.

The chairs and members of the Organizational Team were Dirk de Bock and Masami Isoda, Juan Antonio Garcia Cruz, Athanasios Gagatsis, and Elaine Simmt. The Organizational Team took the initiative to publish Proceedings of TSG2 making the full text of all papers available for interested parties (De Bock, Isoda, Cruz, Gagatsis, & Simmt, 2004).

PAPERS OF THE ICME-10 TOPIC STUDY GROUP 2 (TSG2) - NEW DEVELOPMENT AND TRENDS IN SECONDARY MATHEMATICS EDUCATION

Keynote Presentations

The first session consisted of three keynote presentations by Paul Drijvers, Florence Glanfield and Ross Turner who presented their ideas on the impact of new technologies, curricular developments, and the role and results of international comparative studies.

Paul Drijvers in his paper *The Integration of Technology in Secondary Mathematics Education: Future Trend or Utopia?* addressed the integration of technology in mathematics education at secondary level. By means of a brief retrospective, he noticed that the impact of technology on teaching and learning was not as big as was expected. Infrastructural arrangements, adequate research, curriculum development and teacher training were identified as factors that affect a productive future integration of ICT in mathematics education.

In her paper *Secondary Mathematics Education Curriculum Developments: A Canadian Perspective*, Florence Glanfield described the current trends, the background of these trends, and posed some possibilities for the future of secondary mathematics curriculum development in Canada. Glanfield argued that, in Canada, the secondary mathematics curriculum development has been in the revolution of addressing the availability of "new and available technologies" and "humanizing mathematics" and gave some examples concerning applied mathematics, consumer mathematics and the role of computer technologies. She also argued that the role of technology will continue to be a focus in secondary mathematics curriculum development in Canada because their focus will be to continue to "humanize" mathematics.

In the third keynote presentation (*PISA and Secondary Mathematics Education)*, Ross Turner described the OECD's Program for International Student Assessment, an international comparative study that has the potential to influence secondary mathematics education policy and practice. Some of the objectives and features of PISA were outlined, comparisons with TIMSS were briefly discussed, some results and outcomes were presented regarding the impact of PISA on mathematics education, and some limitations of PISA were briefly explored.

Paper Presentations

In the second and third session, nine papers which are grouped around two themes: *Curricular Developments and New Contents* consisting of five papers about current events in curricular development in different parts of the world and *Learning from Research and Classroom Practice* consisting of four papers that critically reflect on the new trends.

Curricular Developments and New Contents

In the first theme group, on curricular developments, two of the papers provide a description of a mathematics education reform in China. The first paper concerning

China's mathematics curriculum reform by Kwok-cheung Cheung *New Development in Mathematics Education at Obligatory Education Level in People's Republic of China* sought to introduce new developments in mathematics education at the obligatory education level (grade 1-9) based on the Mathematics Curriculum Standards (Experimental Version) released by the Ministry of Education of People's Republic of China. Basic concepts, design considerations, curriculum objectives, curriculum contents, curriculum implementation and mathematics background knowledge recommendations were explicated in detail.

Guo Rong Xu and Stephen Lerman focused on the problems in Chinese education reform in their paper *The Small Tip of a Large Iceberg? The Problems in Chinese Education Reform.* The Chinese reform was influenced by Western educational ideologies and focused at changing the classroom practice. However, like previous reforms in Chinese mathematics education, this reform, according to Xu and Lerman, seems to be ineffective in implementing substantial changes in mathematics classroom practices. This study looked at the actual impact of the reform on classroom practice and attempted to identify and analyse some factors that hindered it.

Maitree Inprasitha described in his paper the *Movement of Lesson Study in Thailand.* Lesson study, a Japanese form of professional development, is a well-known approach to improve teacher practice. In his paper, he introduced how to use lesson study for another purpose, namely to improve the new launched 5-year program for educating mathematics teachers at all faculties of education in Thailand. In his concluding remarks, he stated that the lesson-study approach has begun to have great influence on the reform program for professional development in Thailand.

In her paper, Sofia Anastasiadou described the *Perceptions – Attitudes – Conducts of Greek Mathematicians for Statistics in Secondary Education.* The objective of this research was to define and to determine the perceptions, attitudes and conducts of the mathematicians in Secondary Education towards statistics. In conclusion, the author argued that the mathematicians show both positive and negative attitudes and representations towards statistics. The presenter suggested that the negative attitudes were a product of the durable absence of teaching this science, which possibly creates repugnance, anxiety and disdain towards this science.

Allan Tarp's paper *Adding Per-Numbers* used post-modern "sceptical Cinderella" research to look for new ways to teach mathematics at the secondary school to solve the relevance paradox in mathematics education. The paper introduced addition of per-numbers as a more user-friendly approach to the traditional subjects of proportionality, linear and exponential functions and calculus.

Learning from Research and Classroom Practice

As Wim Van Dooren, Dirk De Bock, An Hessels, Dirk Janssens and Lieven Verschaffel argued, although recent curricular documents in many countries underline the role of modelling in secondary mathematics education, educational practice and research in the last decades uncovered many difficulties and systematic errors that may cross students' learning of a mathematical modelling disposition. In their paper *Studying and Remedying Secondary School Students' Modelling Skills: A Case Study,* they reported on a research-based teaching experiment with 8th graders aimed at remedying one of these errors, namely students' tendency to see and apply the linear model everywhere. The different studies that are briefly reported in this paper have empirically demonstrated the strong and irresistible tendency among secondary school students to over-generalise the linear (or proportional) model when working on applied geometrical problems about the lengths and the area or volume of similarly enlarged geometrical shapes. Moreover, it was evidenced that even with considerable support – both in a testing condition and in a short-term teaching experiment – only (very) few students made the shift to the correct non-proportional reasoning. In the authors' opinion, it might be appropriate to intervene much earlier in students' school career in order to prevent – rather than remedy – the illusion of linearity and to continue this preventive effort throughout the mathematics curriculum.

In their paper *Students' Improper Proportional Reasoning: A Multidimensional Statistical Analysis*, Modestina Modestou and Athanasios Gagatsis investigated the predominance of the linear model in 12-13 year old Cypriot students, while solving non-proportional word problems involving area and volume of rectangular figures. Using three different kinds of tests, related to the context of the word problems presented, they attempted to identify a differentiation in students' responses. Two different statistical analyses were used on the data: Factor analysis and the Implicative statistical analysis. Both statistical analyses suggested the same grouping of students' responses and confirm the existence of improper proportional reasoning.

In the third paper *Developing Interdisciplinary Activities in Secondary School Classrooms,* Yuriko Yamamoto Baldin, José Antonio Salvador and Pedro Luiz Aparecido Malagutti reported on Project Pró-Ciências. This project was carried out by Universidade Federal de São Carlos, Brazil, in 2001 and 2002, with collaboration of elementary school authorities and governmental educational agencies, aiming the professional development of secondary school teachers and updated with modern requirements of school curriculum. The project grounded on the National Curriculum Standards and focused on the understanding, planning and execution of interdisciplinary activities, connecting Mathematics to other Sciences and the real world.

In a last paper presentation *A Comparative Study on Composite Difficulty Between New and Old Chinese Math Textbooks*, Jiansheng Bao compared the old middle school maths syllabus to the newly published National Mathematics Standards. Numerous changes, both to curriculum and to mathematics contents were noticed, leading to the following questions: What precisely are the differences between the new and old maths textbooks? How do these differences affect the styles of mathematics teaching and learning? In order to answer these questions, he used a self-developed model to evaluate the composite difficulties of new and old eighth grade maths textbooks using five factors of difficulty so as to highlight some initial findings.

Discussion Papers

In the fourth session, the chairs and members of the Organizational Team presented their personal reflections on *New Development and Trends in Secondary Education* and on the different papers that were presented in the fist three sessions.

In his paper *Mathematics Education in the 21st Century: New Trends and Developments* Dirk De Bock ran over some major recommendations expressed in reform documents of the eighties', which made a strong plea for reforming mathematics education in all areas and how they were implemented in school curricula. In parallel, he asked some questions about future trends that seem to appear in the field of mathematics education. More specifically, he focused on developments related to (1) the role of modelling, (2) probability and data analysis and (3) technology.

The second paper by Masami Isoda (*Mathematics Activity as a Human Endeavor Project: Exploring Secondary School Mathematics via Historical and Innovative Tools*) described the "Mathematics Activity as a Human Endeavor Project" and illustrated this project with an example of studying an ellipse compass. This project has been developed with historical and technological materials based on four ideas; mathematization, mediational means, theory of embodiment and hermeneutics.

In her paper *The Illusion of Linearity and New Trends in Secondary Education* Elaine Simmt suggested that the illusion of linearity exists for more than just students of mathematics. Indeed this illusion is prevalent in teachers', researchers' and policy makers' interpretations and understandings of curricula. In this discussion paper, she proposed that new trends in secondary education discussed over the course of ICME–10 challenged this illusion. More specifically, she focuses on developments related to mathematics in context, reflection and recursion, cycles and complex collective learning systems.

In the fourth paper *The Role of Representations in Secondary Mathematics Education*, Athanasios Gagatsis argued that concern has been growing about the role of representations in learning and teaching mathematics. His paper was an attempt to exemplify upon the different roles representations can and should play in meaningful mathematics learning and generally mathematics education. The presented paper was linked to the framework of the ongoing discussions about the nature and importance of mathematical representations. New curricular approaches, instructional innovations and reforms as well as recent research studies consider representations as a central issue in mathematics education from different perspectives. These emphases on the particular issue indicate that a better understanding of representations and their role in learning mathematics may enrich our conceptions of learners' sense making, facilitate instruction aiming at helping students develop mathematical understanding in various ways and generally serve the advancement of the field of mathematics education.

Finally, Juan Antonio García Cruz made *A Reflection on Some Aspects of Mathematics and Mathematics Education.* His main concern was the way mathematics and mathematics education is reported in the media and the mathematics classroom practice. According to Cruz, the new mathematics reform implemented from the late eighties in Spain has not succeeded as expected in changing the current mathematics classroom practices. In his opinion, one of the main reasons is the current exam system. Cruz argued that the teacher's beliefs are an important factor, which will affect the future of the relationship between information technologies and mathematics education. Reflecting on the role of PISA and other international comparative studies, such as TIMSS, he argued that while PISA levels keep in a paper-and-pencil base, it is hard to see how it could influence and modify classroom practices. He concluded that if we want to improve classroom practice by modifying the current practice, we have to change teacher's attitudes and beliefs and also the way mathematics practice is perceived in our society.

Reference

De Bock, D., Isoda, M., Garcia Cruz J. A., Gagatsis, A., & Simmt, E. (Eds.) (2004). *Proceedings of ICME-10 – Topic Study Group 2: New Developments and Trends in Secondary Mathematics Education.* Copenhagen: ICME-10.

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SEMT '05

International Symposium
Elementary Mathematics Teaching

to be held in
Prague, Czech Republic
at the Faculty of Education of Charles University
August 22 - August 27, 2005

First Announcement

Invitation

We cordially invite you to participate in the seventh bi-annual conference on Elementary Mathematics Teaching SEMT '05 which is to be held from August 22 - 27, 2005 in Prague. The programme will be focused on the teaching of mathematics to children within the age-range 5-11 years.

The theme of SEMT '05 is

'Understanding the environment of the mathematics classroom'.

The Symposium comprises plenary lectures, presentation of papers, workshops and discussion groups.

SEMT '05 International Programme Committee: Graham Littler (Great Britain) Chairman, Brian Doig (Austrailia), Alena Hospesova (Czech Republic), Jarmila Novotná (Czech Republic), Demetra Pitta (Cuprus), Louise Poirier (Canada), Ewa Swoboda (Poland), Marie Tichá (Czech Republic), Pessia Tsamir (Israel)

SEMT '05 PLENARY speakers are:
Nadine Bednarz (Canada), Marja van den Heuvel-Panhuizen (The Netherlands), Bernard Sarrazy (France) & Jarmila Novotná (Czech Republic), Heinz Steinbring (Germany)

Call for papers

This is an invitation to colleagues who would wish to present their paper at the conference.

Papers may be of TWO types:

1. Research papers which have a rigorous theoretical underpinning;
2. Papers from teachers and others on research/experimental work which has been done in the classroom.

ALL papers submitted must relate to the theme of the conference and will be reviewed by a panel of experts, provided the guidelines and timetable as set out below ares met.

Guidelines for papers

1 *Research papers* should meet the following guidelines:

- An explicit statement of the aim of the research should be given
- A sustained argument based on theory and referenced to relevant publications should be made
- The methodology used in experimental research should be made explicit
- The results should be presented and analysed
- Conclusions should be drawn which include the implications of the research to teachers or teacher training.

2 *Classroom research papers* should contain the following:

- The aim of the research
- References to similar work or theory on which the work is based
- Details of the experiment(s) used in the research
- Details of the methodology used when using the experiments
- The presentation of the results and their analysis
- Conclusions, implications for the classroom and future work.

General matters

All papers should adhere to the following principles:

- **Clarity** including using standard references and giving definitions of crucial terms
- **Organisation:** the content must be well organised according to precise headings such as Aim(s), Theoretical Framework, Methodology etc.
- **Understandable** and submitted in word processor proof English
- **Conform** to the following specification:

  Should not be longer than 8 pages
  Should be printed in Times New Roman or similar font together with the use of Symbol font where necessary;
  The page should be suitable for transcription to A4 size, with text dimensions being 16cm x 24.7 cm (ie a 2.5 cm margin on A4 paper)
  The font size to be used is 14 with 'single' spacing between lines and 6 point space between paragraphs.

Deadlines for authors:

- The full text of the papers wishing to be presented should be submitted by March 5, 2005.
- Papers will be submitted to referees by April 1st, 2005.
- Authors will receive referees comments not later than April 30, 2005.
- Final deadline for submission of papers following any necessary modifications requested by the referees is June 1, 2005. Papers received after June 1, 2005 will not be accepted.
- Proposals for posters of one A4 page must be sent to the organizers by March 31, 2005.

The accepted papers will then be published in the conference proceedings which will be handed out to participants at the start of the conference.

Student Teachers

Student-teachers may participate at the conference. We offer these students a reduced conference fee. If requested, a session for these students can be arranged.

Symposium language: English

Organizing committee address: SEMT '05
Dept. of Mathematics and Mathematical Education
Charles University, Faculty of Education
M.D. Rettigové 4, 116 39 Praha 1, Czech Republic
e-mail: jarmila.novotna@pedf.cuni.cz
nada.stehlikova@pedf.cuni.cz
Website address: http://www.pedf.cuni.cz/kmdm/index.htm

CERME 4

Fourth Congress of the
European Society for Research in Mathematics Education

17 - 21 February 2005 in
Sant Feliu de Guíxols, Spain

YERME day: 16 February

http://cerme4.crm.es/

INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

Annual Meeting

FIRST ANNOUNCEMENT

10 – 15 July, 2005

Melbourne, Australia

http://staff.edfac.unimelb.edu.au/~chick/PME29/

Learners and Learning Environments

The Third International A.S.I. Conference
Implicative Statistic analysis
Palermo, 6-8 Octobre 2005

Association pour la Recherche en Didactique des Mathématiques (ARDM)

PerformanSe

Università di Palermo

Objective

Following the preceding conferences (Caen June 2000, Sao Paulo July 2003), the third ASI conference will allow the presentation of the advancement of the theoretical studies of the method of analysis of data, Implicative Statistic Analysis, as wel as remarkable number of its applications in different fields such as the Sciences of education, the psychology, the extraction of the Knowledges, the economics and the biology.

Let's remember that this method allows to discover and to structure in the form of rules a data set crossing subjects (or objects) and variables beginning from a statistic modeling of the almost-implication: *if the variable or the conjunction of variable **a** is observed in the population, then the variable **b** is in general also observed in the population.* The variable can be of different types: binary, modal, numerical, of interval, fuzzy. The whole sets of rules *that are* obtained can be structured according to different complementary approaches (implicative graph, directed hierarchy). The visual representation of the results as their interpretation, are made easier by the software C.H.I.C. (Classification Hiérarchique Implicative et Cohésitive).

Submission

The proposals of communications must not exceed the 4000 words or 12 pages. The first page it will bear the title of the communication, the names and the affiliations of the authors and a summary (in French and in English) of maximum 10 lines. The texts typed in Word, Times 12, line spacing 1,5, must be addressed before March 1 st 2005 in the two following versions:

- an electronic version must be sent to the following address : rencontreASI05@polytech.univ-nantes.fr
- ***a printed* version *must be mailed* to**
Régis GRAS
14 avenue de la chaise
35170 BRUZ
France

The proceedings will be published.

Committees

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Sponsors

Association pour la Recherche en Didactique des Mathématiques (ARDM) http://www.ardm.asso.fr/
Société Francophone de Classification (SFC) http://sfc.enst-bretagne.fr
Association Extraction et Gestion des Connaissances (EGC) http://www.polytech.univ-nantes.fr/associationEGC/
Société PerformanSe, SA, Nantes-Carquefou www.performanse.fr
International Association for Statistical Education (IASE) http://www.cbs.nl/isi/iase.htm
Groupe "Enseignement de la statistique" de la Société Française de Statistique (SFdS) http://www.sfds.asso.fr
Università di Palermo www.unipa.it
G.R.I.M. (Gruppo di Ricerca sull'Insegnamento delle Matematiche), Dipartimento di Matematica, Università di Palermo http://math.unipa.it/~grim/
Società Italiana di Statistica http://w3.uniroma1.it/sis/index.asp

Projet de Programme

Jeudi 6.10.2005 Journée « Applications de l'A.S.I. à diverses disciplines »
9h00-9h30 Accueil
9h30-10h00 Ouverture du colloque. Parole au président de l'université de Palerme, à un représentant de l'**ARDM** (A Kuzniak), de **EGC** (H.Briand), de la **SFDS** (J.C. Régnier), de la **SIS** (L La Tona), aux **organisateurs**

10h00-11 h Conférence de **Régis GRAS** : "Panorama du développement de l'A.S.I. à travers des situations fondatrices"

Communications à planifier :
Gérard Ramstein : Vers une analyse implicative des données issues de puces à ADN
Letizia La Tona et Angela Alibrandi : Il concetto di implicazione nell'ambito della statistica applicata : differenti approccia a confronto
Raphaël Couturier : Un système de recommandation base sur l'A.S.I
Serge Baquedano, Vincent Philippé, Jacques Philippé, Régis Gras, Olivier Guillaume : L'analyse implicative pour l'élaboration de référentiels comportementaux
Ingrid Verscheure et Catherine-Marie Chiocca : Réseaux implicatifs de représentations d'élèves de 1ère concernant les activités physiques et sportives
Jérôme David, Fabrice Guillet, Vincent Philippé, Henri Briand et régis Gras : Validation d'une expertise textuelle par une méthode de classification basée sur l'intensité d'implication
Patrick Leconte : Représentation sociale, apprentissages et structuration de la gestion des compétences. Apports de l'ASI
Catherine-Marie Chiocca et Susana Murillo : CHIC, on corrige !
Nadja Acioly-Régnier et Jean-Claude Régnier : Repérage d'obstacles didactiques et socio-culturels au travers de l'ASI des données issues d'un questionnaire
Eugen Barbu, Pierre Héroux, Sébastien Adam, Eric Trupin : Mining document images for association rules

11h00-11h30 Communication
11h30-11h45 Pause
11h45-12h45 Travaux pratiques CHIC (traitement des variables-intervalles)
12h45-14h30 Pause déjeuner
14h30-16h30 Communications
16h30-17h00 Pause
17h00-18h00 Communications
18h00-19h00 Communications

Vendredi 7.10.2005 Journée « Applications de l'A.S.I. à la didactique des mathématiques
9h00-10h00 Conférence de **Filippo Sapgnolo : "L'Analyse Statistique Implicative : une des méthodes d'analyse des données en didactique"**

Communications à planifier :
Gérard Frossard : CHIC et les études docimologiques
Aldo Scimone et Filippo Spagnolo : The importance of supplementary Variables in a case of an educational research

Iliada Elia et Athanasios Gagatsis : Le rôle des représentations dans la résolution des problèmes du type additif
Iliada Elia, Athanasios Gagatsis et Régis Gras : Can we "trace" the phenomenon of compartmentalization by using the I.S.A. ? An application for the concept of function
Anastasiadou et Athanasios Gagatsis : Attitudes des étudiants, futurs enseignants de l'école primaire grecque
à l'égard de la statistique
Elsa Malisani et Filippo Spagnolo : Incognita o « cosa che varia » ? L' A.S.I. in una ricerca in didattica delle matematiche
Françoise Jore : Analyse de tests de géométrie plane dans le cadre de la formation des futurs professeurs des écoles
Pilar Orus et Pablo Gregori : Des variables supplémentaires et des élèves "fictifs" dans la fouille de données avec CHIC
Dominique Lahanier-Reuter : Validation d'une expertise textuelle par une méthode de classification basée sur l'intensité d'implication. **Saddo Ag Almouloud** : L´analyse statistique de données multidimensionnelles:outil révélateur des conceptions d´enseignants en formation
Alain Kuzniak : Espace de travail géométrique personnel : une approche didactique et statistique

10h00-11h00 Communications
11h00-11h30 Pause
11h30-12h30 Communications
12h30-12h45 Questions aux intervenants
12h45-14h30 Pause déjeuner
14h30-16h00 Communications
16h00-16h30 Pause
16h30-17h30 Communications
17h30-18h30 Communications
18h30-19h00 Questions aux intervenants

8.10.2005 Journée « Variations sur la modélisation de règles d'association »
9h00-10h00 Conférence de **Pascale KUNTZ** : « Classification hiérarchique orientée en A.S.I."

Communications à planifier
Julien Blanchard, Fabrice Guillet, Henri Briand et Régis Gras : Une version discriminante de l'indice probabiliste d'écart à l'équilibre pour mesurer la qualité des règles
Gilbert Ritschard : De l'usage de la statistique implicative dans les arbres de classification
Stéphane Lallich, Philippe Lenca et Benoît Vaillant : Variations autour de l'intensité d'implication

Xuan-Hiep Huynh, Fabrice Guillet et Henri Briand : ARQUAT : une plateforme d'analyse exploratoire pour la qualité des règles d'association

10h00-11h00 Communications
11h00-11h30 Pause
11h30-12h30 Communications
12h30-14h30 Déjeuner
14h30-16h30 Discussions-Bilan

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INFORMATION FOR AUTHORS

Manuscript Submission

For the purpose of reviewing, articles should be submitted in five hard-copy printouts to the Editor-in-chief:

Prof. Athanasios Gagatsis
Mediterranean Journal for Research in Mathematics Education
Department of Education
University of Cyprus
P.O.Box 20537
1678 Nicosia
CYPRUS
Email: gagatsis@ucy.ac.cy

Manuscript Presentation

The language of the journal is English. The manuscripts should conform to the conventions specified in the Publication Manual of the American Psychological Association (5th Edition, 2001). The publication is available from the American Psychological Association, 750 First Street, NE, Washington, DC 20002. They should be printed on A4 paper, one side only, leaving adequate margins on all sides. Please double-space all material, including references in order to allow editorial commentary. Manuscripts most acceptable length is 15-25 typewritten pages or between 4500 and 7500 words. Pages should be numbered consecutively and the first page should contain the following information:

- running head (shortened title)
- title
- author(s)
- affiliation(s)
- full address for correspondence, including telephone, fax number and e-mail address.

The journal has adopted a blind reviewing policy. Authors should use separate pages for all identifying information (name, affiliation, references or footnotes to one's own work, etc). These pages will be removed before the manuscript is sent to the reviewers. The reviewers will also remain anonymous.

Abstract

Please provide a short abstract of 100 to 250 words in English. This abstract should be printed at the beginning of the paper.

Key Words

Five to ten key words or short phrases should be provided in alphabetical order.

Figures and Tables

All figures and tables must be submitted on different pages. The author should specify where s/he wishes them to appear in the text.

Processing of Manuscripts

After the receipt of the manuscript, the Editor-in-chief examines whether it falls within the journals domain and complies with its style requirements. If it does not fall in the journal's domain the Editor-in-Chief returns the manuscript to the author without further consideration. If the manuscript falls in the journal's domain but does not comply with the journal's style requirement, it is returned to the author before the reviewing process continuous.
If the manuscript is considered appropriate, the Editor-in-chief notifies the author and sends it to at least two reviewers. The reviewers are chosen based on their expertise on the subject. When the reviewers complete their reviews, the Editor-in-Chief informs the author whether the paper:

- Has been accepted
- Needs some substantial revision and will be re-evaluated by the reviewers.
- Has been rejected.

The Editor-in-Chief will also forward the reviewers´ evaluations to the author. If the manuscript is accepted a final version of the paper will be requested in a 3.5 inch disc and a hard copy.

Final submission of manuscripts

Once the manuscript has been accepted, the author should submit the final version of the manuscript on 3.5 inch diskette along with a final hard copy of the paper. Please note the following:

- Send 3 hard copies when first submitting your paper.
- When your paper has been refereed, revised if necessary and accepted, send a 3.5 inch diskette containing the final version and a final hard copy. Please label your diskette properly, giving exact details on the name(s) of the file(s). For the sake of safety, always retain a backup copy of your file(s).
- Manuscripts should be submitted in PC Word.
- Figures and tables should be submitted in separate files.
- The file should follow the general instructions on style.
- The file should use the wrap-around end-of-line feature (i.e. no returns at the end of each line). All textual elements should begin flush left, no paragraph indents. Place two returns after every element such as title, headings, paragraphs, figure and table.
- Proofs are sent to the first-named author. Authors should return their corrected proofs within 7 days of receipt by mail (airmail overseas) in order to avoid delay in the production schedule.

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MEDITERRANEAN JOURNAL FOR RESEARCH IN MATHEMATICS EDUCATION

Volume 3 No. 1-2 2004

THE ROLE OF THE HISTORY OF MATHEMATICS IN MATHEMATICS EDUCATION

CYPRUS MATHEMATICAL SOCIETY
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