MEDITERRANEAN JOURNAL FOR RESEARCH IN MATHEMATICS EDUCATION

AN INTERNATIONAL JOURNAL

A TRIBUTE TO THE WORK OF EDDIE GRAY AND DAVID TALL

Guest Editors
A. Simpson & D. Pitta-Pantazi

Volume 7 No.2 2008

CYPRUS MATHEMATICAL SOCIETY
ΚΥΠΡΙΑΚΗ ΜΑΘΗΜΑΤΙΚΗ ΕΤΑΙΡΕΙΑ
Nicosia - Cyprus

Ed. in Chief
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Mediterranean Journal for
Research in Mathematics Education

Introduction

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DEMETRA PITTA-PANTAZI: Department of Education, University of Cyprus, Cyprus

David Tall and Eddie Gray both retired at the end of 2006. As a team and as individuals they made enormous contributions to the field of mathematics education; introducing and developing theories about mathematical cognition which have influenced our understanding of how people learn (or fail to learn). The scope of those theories stretches from learners' earliest attempts to count through to degree level students making sense of real analysis.

No retrospective can give justice to that scope and the collection of papers in this issue of the Mediterranean Journal for Research in Mathematics Education is designed to give no more than a flavour of their influence. It includes empirical research papers which draw on their ideas, reviews of their impact on the field and critiques which attempt to delineate the reach of their theories.

The two theories which are most often cited are *Concept Image/Definition* and *Procept*. The former, developed by David and Shlomo Vinner was, at the time, one of those profound insights into mathematical thinking which, while simple enough to be expressed in a sentence or two, completely changed the way we thought about students' attempts to make sense of mathematical situations. The core idea is that one may access a mathematical concept through related 'images' (not necessarily pictorial) which have developed from repeated encounters with the concept (or, primarily, examples of the concept) or through a (more or less formal) definition which determines an object by (and only by) given properties. This appears so 'obvious' when we consider the ways mathematics is most commonly presented, that it took significant intellectual prowess to isolate and develop it. Its simplicity belies its power to help us make sense of some of the ways students struggle, particularly with the transition to formal proof and it has particular appeal to practicing mathematicians, normally too busy to show much interest in their students' learning, and has thus become one of the very few ideas from mathematics education to have much currency in the practice of undergraduate mathematics education.

Procept, which Eddie and David developed together from an insight grounded in Eddie's own doctoral study (supervisor: one D. O. Tall), has had an even deeper influence within mathematics education. Similarly seemingly simple, the idea is that mathematical symbols can represent both a process and a concept (hence their portmanteau word 'procept'). Crucially, it is the flexibility provided for a learner who can use it in either form which, they suggest, gives the greatest mathematical power and, as new symbol systems are introduced which are interpreted with this powerful ambiguity by only some people in a class, the proceptual divide opens up.

Developed at the same time as other ideas with which it is often grouped (APOS theory and structural/operational theory), we suggest procept has more scope. Because it proposes no mechanism for the development of processes to concepts (nor, indeed, requires that concepts are necessarily grounded in processes) there seem to be more situations in which it can plausibly account for the apparent dual interpretations students appear to have. Equally, because it places the symbol as the pivot between the interpretations, it highlights the role of symbols and symbol systems in the astonishing compression of meaning that appears to be characteristic of mathematics.

While these theories may be those most commonly associated with David and Eddie, separately and together they have given us much more. We have learned from them (and their co-authors) of the role of visual imagery in early number development; the role well-designed computer packages can have as ‘generic organisers’ of knowledge and the power of a graphical calculator in changing a child’s imagery – not through its graphing functions, but its ability to retain combinations of arithmetical symbols on screen alongside solutions.

However, Eddie and David have been extraordinary in mathematics education not just for their published work and its impact on the field and on practice. They have generated an enormous international community of researchers through their supervision and friendship. They have been involved in the supervision of at least 30 doctoral students (and masters and undergraduates too numerous to mention) and sensitively mentored colleagues. These academic children are beginning to spawn academic grandchildren at a rate which competes even with the prodigious production of their genetic grandchildren!

While both had important careers before this point, David as an academic mathematician and Eddie as a successful teacher and then headmaster, the flowering of their work was within the Mathematics Education Research Centre (MERC) at Warwick University. While MERC was started by Richard Skemp and then developed by Rolph Schwarzenberger, David and Eddie played the lead role in developing it as a world-class institution for the study of mathematical thinking. Through the 1990s it was clearly one of the world’s leading institutions in this area, primarily because of their leadership. Indeed, their retirement roughly coincided with the loss of almost all the other research staff from MERC – a consequence of poor understanding which successive managers in Warwick Institute of Education had of the high quality work being produced in the group in general and by Eddie and David in particular. While we might rue that narrowness of vision, it was difficult to see how MERC would have been able to maintain its position following their retirement.

Their work gave them the respect they deserved in the international community, however. This led them to long-term friendships around the world, which they will maintain beyond their retirement. At international conferences, they were famous not just for intellectually stimulating presentations, but for their bonhomie and harmony singing!

Before the 30th Conference of the International Group for the Psychology of Mathematics Education, in Prague, we were able to hold a celebration bringing together many of their academic friends and family. This reflected the range of influences they have had and, from that wonderful coming together, the idea for this special issue grew.

We must thank the authors of these papers for the efforts they have expended in producing them and the love for David and Eddie which has been put into them. Most of all, of course, we must thank Eddie and David themselves for their years of intellectual leadership, exemplary scholarship and personal friendship. We wish you the happiest of retirements.

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Lessons from the Past and Concerns for the Future

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ABSTRACT: *This paper places the theory that I have shared in developing at Warwick into the broader spectrum of the search for an approach to the teaching and learning of mathematics that would support high levels of achievement. There is no apology that much of the discussion is contextualized within the UK, seen as a typical country that continually questions the general levels of mathematical achievement, implements a variety of initiatives to improve and raise standards but fails to recognize within any of these initiatives that learners do things differently. Without implementing some understanding of the way in which mathematics is learned even the most recent initiatives will need reconsidering.*

Key words: *achievement, historical response, recognising difference, understanding*

INTRODUCTION

As I began to think about this paper I became aware of two UK reports that made me wonder about our ability as mathematics educators to communicate and influence the way that mathematics is taught and thought about.

The first, a publication from the Schools Inspectorate for England and Wales reporting upon the mathematical achievement of 14 to 19 year-olds (OfSTED, 2006), illustrated the distinction between the emphases and outcomes from two forms of teaching. The "best" teaching focused on developing understanding in mathematical concepts to develop critical thinking and reasoning whilst that which focused on teaching a set of arbitrary rules did not motivate students and limited their achievement — "teaching to the test" might ensure that students pass examinations but it would not ensure mathematical flexibility.

The second report was a discussion paper published by the Advisory Committee for Mathematics Education (ACME, 2006). It suggested that top-down pressure on pupil test results imposed by Government, Local Authorities, governing bodies and head teachers has caused primary school teachers to passively and uncritically implement curriculum and pedagogic styles recommended within documents published through the Primary National Strategy, (see for example DfEE, 1999) without due consideration to the needs of their pupils.

The first of these items received considerable attention within the daily press and the BBC. It made me wonder why something, which, because of our research interest is so obvious, should become a news item expressed in such a way that it appeared to be suddenly discovered. It is almost thirty years since Skemp (1976), in slightly different terms, pre-empted the conclusions of the OfSTED evaluation, whilst almost 70 years earlier the Mathematical Association within the UK had been expressing concern that the educational value of arithmetic and algebra could be seriously impaired through a tendency to sacrifice clear understanding to “mere mechanical skills” (Mathematical Association, 1905, p.7). One hundred years separates expressions of concerns on the same theme — teaching the rules of mathematics does not lead to true understanding of mathematics.

GOING FULL CIRCLE: SOME CONSIDERATIONS ON THE CURRICULUM

Hardly a decade seems to pass without concern being expressed about the generally poor level of children's attainment, the nature of the mathematics curriculum or the quality of children's learning. These concerns have resulted in a variety of interest groups, either mathematicians, general educators, politicians or mathematics educators recommending change to either the teaching methods, to the curriculum or to the ways in which we assess children’s achievement. Frequently there has been strong contradiction, even discord, in the way each group has perceived the mathematics curriculum and the emphasis that should be placed upon its content. Should the emphasis be on the acquisition of skill or the development of understanding that will provide conceptual flexibility? Should teaching focus on classes or individuals. The answers to such questions can only be meaningful if we have reflected upon the ways that children learn mathematics and the influences that may support this learning.

One hundred years ago, the mathematics to be taught within the UK was codified within ‘Codes of Regulations’ (a sort of National Curriculum) that specified the criteria children had to meet to fulfil the knowledge requirements of annual inspections. A successful outcome implied that the school had met one of the necessary conditions under which it could be funded. The mathematics specified within these codes was seen as a mathematics for the majority and it was essentially utilitarian — a thoroughly good arithmetic that was taught to satisfy the demands of parents “whatever else had to give way to it” (Ministry of Education, 1958, p. 4).

It was this utilitarian aspect of mathematics that came to dominate elementary schools (those schools that taught children aged 4 to 14) and its justification was confirmed by the Mathematical Association (1932) because it was:

> … only through examples in arithmetic that all children will learn anything of correct mathematical reasoning… and it is perhaps the only subject where mastery is attained. (Mathematical Association, 1932, p.5).

However, even this report did not appear to have taken notice of the caution expressed 10 years earlier

> Arithmetic differs from most subjects in that the hardest step has to be taken right at the beginning. At the very start the child has to leave the world of concrete fascinating realities and concentrate on abstraction, on the creation of human intellect. (Drummond, 1922, p.10)

When I started teaching 45 years ago in a Secondary Modern School —the type of school to which the greater proportion of children aged 11-15, soon to rise to 16, attended if they had not succeeded in the selective examinations at 11 — the words of the 1932 report formed the basis for the mathematics teaching philosophy. It was assumed the children were being taught to think by doing arithmetic, whilst evidence of their mastery or otherwise could be easily identified by the frequency of ticks or crosses. The children left school with no formal recognition of their achievement and were not expected to take part in any mechanism that would give them any. There was no algebra and no geometry — these areas of mathematics were to be within the realm of those who had been successful in the selection process.

However, unknown to me at the time, and apparently to the school I taught in, things were about to change. Recognition of changing world conditions and an increasing appreciation that mathematics was a language, suggested that adaptability and understanding were the qualities that required emphasis more that ever before. It was also recognised that too few children were attaining reasonable and sufficient skills in the computations that they were being taught. Centralisation of the mathematics curriculum and a focus upon the teacher as central to its delivery which had variously dominated the previous 50 years was receding. The curriculum was undergoing rationalization in content, delivery and in its perception of the child as learner.

After making a move to primary schools the mathematics that I taught in the late 1960's and the 1970's was characterised by the belief that mathematics should be broadened beyond arithmetic. It should become embedded in the relationships underpinning the processes that the children used in recognition that mathematics was the universal language of communication. Perhaps the most important aspect of these beliefs was nothing more than an effort to spread an acute awareness of the role of symbolism in mathematics, not only in its conceptual form but also in its operational consequences. It was an exciting time but whilst the resulting decentralization evoked widespread professional interest in the teaching and learning of mathematics it also brought about a high degree of inconsistency in the degree of mathematical provision. But I was lucky, the children I taught were bright, motivated and a delight but the full effect of the inconsistency and my growing insight into an awareness of the role of symbolism had to wait until I had been at Warwick for a few years.

My almost 25 years at Warwick has been characterised by two developments. Firstly my apprenticeship and eventual growth as a mathematics educator and secondly an increasing national concern with children's achievement. The latter appeared to be derived from the aforementioned inconsistency and it eventually lead to recentralisation of mathematics curricula and a growing emphasis on examinations and testing. Within the UK this centralization was invoked through the introduction the National Curriculum in 1989, and the introduction of Standard Attainment Tasks in 1991. However, even by 1998, only 59% of 11-year-olds achieved level 4 (the expected standard for their age) in mathematics. The consequence was a The National Numeracy Strategy (NNS) (DfEE, 1999). This, developed from a Numeracy Project that appeared to be successful mainly because it addressed and seemed to reduce underachievement in mathematics amongst all groups of pupils, was introduced within English schools from September 1999 and incorporated into the National Curriculum in 2000.

It is immediately apparent when the totality of the core document *Framework for Teaching Mathematics from Reception to Year 6* (1999) is considered that throughout the NNS there is a strong bias towards skills, particularly those that are associated with arithmetic. The perception presented in the document is that mathematics consists of a set of facts and procedures. This seems to be derived from an emphasis that seemed to reflect the view of the then Secretary of State for Education, who in a press release in October 1999 said that:

> Numeracy is a vital skill which every youngster must learn properly. Yet for thirty years we have not focused on what we know works. The new daily maths lesson will ensure that children know their tables, can do basic sums in their heads and are taught effectively in whole class settings.

It was almost as if the experiences of the previous hundred years had not existed. We returned to the centralization that previously had promoted the failure of so many with the resulting consequences identified within the reports from OfSTED and ACME. Despite almost a century of initiatives from which the focus of attention had changed from the acquisition of a body of mathematical knowledge and the tempering and broadening of that knowledge to take into account the development of individual children, the political reaction was to centralise, examine and evaluate. Teachers were once again being told what to teach, when to teach and how to teach and both they and their pupils were to be evaluated on the outcome.

I am becoming increasingly convinced that until such time as we can begin to consolidate our sense of the way in which children learn mathematics and the factors that may contribute to this learning and then effectively communicate this to curriculum designers and to teachers a recurring cycle of initiatives designed to address perceived underachievement will continue. Already we have a National Numeract Strategy Phase 2 which attempts to resolve some aspects of underachievement by moving particular topis a year earlier than those of NNS Phase 1.

INSIGHT INTO THE QUALITY OF LEARNING

The observation that some individuals are more successful than others in mathematics has been evident for generations. It could perhaps be explained through Piaget's novel method of interpreting empirical evidence to hypothesise that all individuals pass through the same cognitive stages but at different paces but I have become more and more doubtful that this is the case. The philosophy behind the National Numeracy Strategy suggests that as long as children are taught the subject matter in the correct order and as long as teachers teach properly then all will not only be well but that learning targets for each year of a child's time within school can be identified. Not only the order but the pace is given and, as in the earlier Codes, children are then examined against specified outcomes. We had been here before.

I like to think that our work at Warwick refutes the philosophy behind the current curriculum initiatives and I believe it has made some contribution to our insight into the ways children learn mathematics.

After arriving at Warwick I was given time. Fresh from being head teacher of a large school my supposed expertise as administrator and teacher were the things that were used and appeared important to the department. I had no solid research agenda and neither did I possess much more than superficial research training but I used my time to visit schools, observe teaching and talk to children about they ways they thought about mathematics. There was no explicit research philosophy — it was simply an open-ended experience that involved talking to children about their mathematics and how they solved simple arithmetical problems. But how the children talked was eye opening. At one extreme the children were very precise offering clear explanations of their attempts to solve questions and address issues raised in conversation. At the other they demonstrated some considerable mathematical confusion; I was taught how to count on my fingers, how to count if I didn't use fingers and how to count in my brain. On one occasion I listened with a sense of sadness as Jonathan (aged 9), attempted with extreme difficulty to use mental images of fingers to add 4 + 3 explaining:

> I like trying to do things in my mind. I like them to be harder because when I grow up I will be able to do harder things.

Jonathan thought that even the best children in his class counted fingers in their heads. They were just quicker than him.

I watched with amusement and incredulity as Tara (also 9) suddenly took her shoes off to use her toes to support counting back to 2 from 11 and I listened with awe as Maria (4) gave me the answer to a wide range of number combinations to 20 and all of the table facts to 10 x 10. I also watched the delight when Simon, also four, who when asked to use a calculator to give me two numbers that make eight responded:

> Look at that. See. One million take away nine hundred and ninety nine thousand nine hundred and ninety two — that makes eight.

I listened incredulously as children talked about moving the calculators in their head, combining a red 4 with a yellow 5 to get a blue 9, effectively solved three digit subtraction problems by taking largest from smallest (and mentally recorded the difference so that an adjustment could be made later). And I watched as teachers cast aside such novel approaches to teach standard algorithms which, temporarily at least, caused considerable confusion.

The outcome of these almost random observations began to lead to some tentative assumptions:

- Children can develop mathematically at different rates but it was not to be assumed that given enough time all children will reach the same level of achievement.
- Children make qualitatively different interpretations of mathematical activity and thus give different meanings to subsequent activity.
- There are several key points in the construction of mathematical knowledge where the consequences of a divergence in thinking may make the next stage of development easy for some but very difficult, if not impossible, for others.

The key issue that guided my later ways of working was to consider these assumptions in the context of the question "What is it that children are doing differently and why". In establishing some insight into making a response to these questions the seminal work of Richard Skemp and developments with, and by, David Tall together with the studies of an increasingly large group of PhD students, contributed towards an increasingly wide range of evidence that provided insights into the mathematical development of learners across the spectrum of learning from elementary school to University.

Many of the key outcomes from this work have been reported in a variety of individual papers whilst summaries are presented within Tall et al (2001) and Gray & Tall (in print) but here I would like to highlight several features relevant to issues associated with the development of children's thinking within the primary school, that is schools for children with median ages from 4.5 to 10.5.

The divergence reported by Gray (1991) was later conceptualized within the notion of procept (Gray & Tall, 1994). Conceptualising this construct not only permitted the articulation of the essence of mathematical symbolism in the way that was attempted by the curriculum reforms of the 1960's and the 1970's, but it also provided insight into a fundamental paradox: mathematical symbolism gives the power to think mathematically but can be the source of considerable complexity for those who are trying to learn it.

Differing interpretations of the conceptualised notion of procept enabled us to see the operational consequences of different forms of mathematical thinking that can lead to a proceptual divide. An interpretation of the evidence drawn from Demetra Pitta's PhD (Pitta, 1998) would seem to suggest that particular frames of reference may be associated with learners moving along each arm of this divide (Gray & Pitta, 2006). The disposition of low achievers to articulate the descriptive properties of items, without any reference to their more relational characteristics, would seem to be related to their tendency to focus upon procedures that are fundamentally episodic. More recently, the contribution that styles of teaching may contribute to procedural conceptions of mathematics have been indicated by Md Ali (2006) and to children's descriptive and episodic interpretations of representations by Doritou (2006).

Gray & Tall (in press) suggest the evidence coming from the full range of studies carried out at Warwick fit a theoretical framework that sees abstraction as a natural process of mental compression. As Bills (2002) indicated, not all pupils abstract the same things from their experiences and neither do all children abstract the same experience from common activities — if existing mental constructions are different then new activities will be construed in different ways.

The indications are that moves towards increasingly sophisticated concepts often fail if pre-requisite concepts, ('thinkable concepts') are not available to make sense of the new materials. Howat's (2006) study of the outcomes of a remediation programme, developed to bridge the gap between low achievers current knowledge and the knowledge required to participate more fully within designated whole class teaching phases, amply illustrates this. When working individually with the children it became clear that some of the children's cognitive difficulties were so elementary that these children would require much more than a simple remediation programme.

CONCLUSION

One of the features that seems to emerge from an examination of the variety of mathematics curricula proposed for schools over the past 100 years is that they appear to be conceived of by people who are successful. Those who are mathematicians think in sophisticated ways because of their ability to compress knowledge. Their mathematics is seen from a mature viewpoint in which the structures have great richness and interiority and they therefore have a perception of simplicity in which this structural richness plays an implicit fundamental role.

Learners do not yet have this conceptual richness, but the belief is that as long as teachers explain to them how to manipulate sophisticated thoughts they are giving them power and strength.

In seeking to design a contemporary curriculum for the UK the over-riding emphasis has been placed upon the achievement of results that seem to illustrate improvement in mathematical skill. This impetus would appear to be derived from the politicians need to see success in a way that they believed was apparent in past generations. Perhaps they also have misguided notions of equality since they invoke attempts to teach a whole population increasingly sophisticated mathematics. However, the response from curriculum designers not only meets this desire by identifying ‘What’ should be taught but also indicates ‘When’ and ‘How’. But, as we have seen, short term need in the form of procedural growth tends to take precedence over long term development.

Within the current mathematical climate there is a tendency to provide practice to develop and confirm “understanding” but we need to provide practice which disconfirms and requires the search for alternative approaches. The more we work at remembering procedures the more we are likely to use them. If all of our effort goes into this solution to our problems it is perhaps the case that the more we will remember procedures but, paradoxically, we may possess less understanding. Failure becomes a more distinct possibility than long term success.

We as members of a mathematics education community are all interested in the development of mathematical knowledge. We recognise some of the strengths and weaknesses of curriculum implementation within our own countries. We also have insight into the difficulties and misconceptions that may be associated with particular areas of mathematics. We support our own subject knowledge with an ever-evolving pedagogic content knowledge but how good are we at actually communicating, not simply to ourselves but to those who “set standards” and those who attempt to help children reach them. Without such communication and some acceptance of our message I would suggest that what it will not be long before we undergo another cycle of change and then yet another until it is recognized that what is being attempted is undoable until such time as the nature of the sophistication required to compress complicated ideas is grasped. I believe that our studies at Warwick have made a contribution towards explaining why this may be the case but how do we make it part of mainstream thinking.

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The Retirement Era - The End of Mathematics Education

(A paper in honor of David Tall and Edie Gray on the occasion of their retirement)

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ABSTRACT: *The paper describes the primary directions of research in mathematics education at the time when the International Group for the Psychology of Mathematics Education was formed. It also describes new directions which have become popular in the community of mathematics education researchers (philosophy, history, psychology, anthropology, etc.). As an additional direction it suggests the educational direction. It recommends using some elements which are typical to mathematical thinking (like analysis and control) to everyday life situations. These are also elements of rational thinking. The paper designates rational thinking as an ultimate goal of mathematics education. It claims that rational thinking with an appropriate emphasis may lead to a value oriented behavior, which is the goal of education.*

Key words: *control, education, educational values, mathematical thinking, rational thinking.*

If life is a journey, the metaphor can be used in order to distinguish between two types of people. Those who look back from time to time and reflect on the events and landscapes they have gone trough during this journey, and those who always look forward. I myself belong to the first type. I like to reflect on my own development and also on the development of other people. We, the retired generation of this group (which can roughly be described as PME veterans), started our career in mathematics education as a research discipline, when this notion (mathematics education) was quite clear to us. We believed that the goal of this discipline was to investigate the processes of learning and teaching mathematics, to describe them, to analyze them, to understand them and , as a result of all these, to make some suggestions how to improve the practice of mathematics education. As a consequence of this belief, several mathematical concepts and topics were chosen and investigated. Here is a partial list: Fractions, equations, geometrical figures, limits, functions, derivatives, problem solving, definitions and proofs. After a while, for some reasons, this direction of research became unsatisfactory for some people, veterans as well as new comers. That is a fact. The explanation can be that these people found the above direction too narrow. They wanted to theorize on wider domains and to relate to more general questions than those which were investigated in the original domain. Thus, to the primary direction of mathematics education as a research discipline, some new aspects were added: sociological aspects, anthropological aspects, historical aspects, and philosophical aspects. Epistemological and ontological questions were raised. All of a sudden we were involved with questions like: what are the origins of mathematics? What is it? What are the mathematical objects? How were they formed in our mind? And so on. Recalling a line from Julius Caesar, some of these questions were always Greek for me. However, many people were fascinated by these questions, and, even more, by the answers. Were Tall and Gray part of this movement? I think it is better to ask them to elaborate on this. An answer to such a question depends also on interpretations to some given texts. Interpretations, as we all know, depend very often on the desires of the interpreter. My desire is not to consider Tall and Gray among those who chose the Greek direction. I prefer to believe that they have stayed in the famous English path of common sense. Therefore I do not interpret their PROCEPT theory as an attempt to explain how mathematical objects came into being. In my view, it is an attempt to point at the fact that some central mathematical symbols are ambiguous. They denote both an operation (a process) and the product of that operation (a concept). In fact, it is already claimed by the title of their paper: Duality, Ambiguity, and Flexibility (Tall & Gray, 1994). Thus, comprehension of mathematical situations requires flexible mind. Human beings with such flexibility form a minority in our society and therefore only few people are capable of studying mathematics. Fortunately, we are, of course, among the few who are capable, so we believe. Unfortunately, because we are so few, like all minorities, we do not have simple life.

Since this paper is not about Tall and Gray, but in honor of Tall and Gray, I would not discuss any further their contribution to mathematics education (that is dealing with thousands of printed pages, not to mention teaching, supervision and other services to the community of mathematics education). I would rather use this occasion to confess that I myself, few years ago, like other people whom I mentioned earlier, started to move in a direction different from the original direction of mathematics education. I have done this not because the original direction became too narrow for me. I did it because, as a result of accumulated research in the original direction, I started to have serious doubts about the necessity and importance of teaching mathematics the way it is taught, to the extent it is taught, to the entire population in almost the entire world. The result of the common teaching of mathematics is that the decisive majority of the world population does not know mathematics and very often also hates mathematics. On the other hand, this lack of knowledge has no negative impact on the life of the mathematically ignorant, or even on the society in which they live. The mathematical needs of this society are supplied by few people who have studied mathematics properly and can use it professionally whenever it is needed. A reasonable conclusion of this state of affairs is to give up mathematics as a compulsory subject in schools. However, even expressing this view is dangerous to any speaker who dares to express it in any educational forum. Why is it so? First of all, it contradicts the common rhetoric about the importance of knowing mathematics. People hate to give up rhetoric. The rhetoric helps us to make our life meaningful. It is quite frustrating to realize that we are involved in meaningless activities. In addition to that, mathematics has a very important role in the educational system. It is an excellent selection tool. By means of mathematics scores, the society can select those who will be accepted to a higher stage of education, whether this is a junior high school, high school, college or special prestigious disciplines as law, medicine, economics or business administration. The claim that in order to study some disciplines one should know mathematics is only partly true. Medicine does not require mathematics. If you have any doubts about that, examine your doctors' mathematical knowledge. I have done it and, frankly, I could not care less about the poor results. I was concerned about their medical knowledge and this was, fortunately, quite satisfactory. As I said, one can argue about the wisdom of the current practice, but since any attempt to change it will raise brutal objection, there is no point even to mention it.

So, if this is the case, what are we supposed to do? Well, retirement is a reasonable option if you are old enough. But before elaborating on retirement, I would like to share with you more of my reflections. They relate to my attempt to reach a global view on the overall disciplines which are taught in school and the special role mathematics plays there. In fact, the achievement aspect of other school disciplines is not more impressive than the achievement aspect of mathematics. It is only less traumatic. But if this is the case, why do we have schools at all? Of course, we need them as a giant baby-sitting system. We need them, people remind me, for the sake of socialization. We need them, I eventually recall, in order to educate people. Quite funny - to spend almost my entire life on mathematics education without thinking about its relation to general education! But better late than never, and since in the near future mathematics will be a compulsory subject in schools, and since we have a huge population of mathematics teachers and mathematics educators, why won't we use this constellation and try to improve education?

The title of my talk is a modification of a famous book title by Neil Postman (1996): *The End of Education*. It is an ambiguous title since "end" can be understood as a goal, purpose, etc., and it can also be understood as termination. The bottom line of Postman's book is that if we do not set goals for education it will die. Some people claim it is already dead. For me, Postman's title is a warning on a wall (I am referring here to a biblical event described in the Book of Daniel). However, as in the biblical story, people do not take seriously such warnings. They prefer to go on with the party. It is more convenient to follow inertia, old habits and current routines than to initiate change. And what are the goals of education anyway? An academic attempt to answer this question will imply infinitely long discussions, will raise bitter controversies and will last for ever. In order to avoid this, I would like to suggest non-academic and simple approach. The goal of education, according to this approach, is an educated person. Since this looks circular I will add: an educated person is a thoughtful person. "Thoughtful" in English, like the above mentioned "end," is ambiguous. It means contemplative as well as considerate. I suggest that to be considerate in behavioral terms means to follow the rule: *what you hate - do not do to other people*. This was the answer which was given in ancient Jewish text to somebody who asked one of the sages to summarize the entire Jewish moral theory in one sentence. Few years ago, I saw in London underground a poster saying: To be considerate means not to carry your backpack on your shoulders. This is a small example of the above rule and it is quite typical. Each day, in case we have interaction with other people, we face dozens of situations with similar characteristics. If we have decided to be considerate, we should be aware of the factors involved in the situation, foresee whether an action of ours can disturb another person and avoid such an action, in case we understand that it disturbs somebody. This requires analytical thinking and control. Can this be related to mathematics education? Can it be related to any other school discipline? Very often we speak in mathematics education about analytical thinking and control. We certainly do it in mathematics education conferences. Some of us may speak about it even in their mathematics classes, in case they teach mathematics. It is considered as a meta cognitive activity or a reflective activity. In teacher training we encourage our students to include it in their future teaching. I assume that very few of us and very few mathematics teachers, when they speak about it in their mathematics classes, point at the fact that these are also relevant to our every day life. If the notions "moral behavior" or "moral values" were not so emotionally loaded, I would have said that the issues at stake at this point are moral behavior and moral values. Unfortunately, when people hear these notions they think immediately about issues like abortions or homosexual marriages and so on. In order to avoid these connotations, I will use the terms "educated behavior" and "educational values." Unfortunately, in most cases, these are not part of any discipline taught in schools. So, on what ground do we expect our graduates to behave in a civilized manner?

My claim is that educational values should be integrated in school disciplines by means of the principles and the nature of these disciplines. Usually, it does not happen. Moreover, even reflection on thinking in general does not occur. Just a small illustration: One of the courses I have taught in many years was topics in mathematics education. It was a graduate course, the students of which were middle school and high school teachers. In the beginning of each year I asked them about the goals of mathematics education. Many of them claimed that one of the main goals of teaching mathematics was to develop the students' thinking. However, when I asked them to elaborate on it or to demonstrate it by means of some examples, their decisive majority practices its legal right to remain silent. I realized that *developing the students' thinking* has become a common rhetoric associated with mathematics teaching. But for the majority of mathematics teachers it is not clear at all what it really means and how their actions in class can help their students to develop their thinking. At this point I start a class discussion. I try to draw my students' attention to the fact that there are many kinds of thinking: religious thinking, scientific thinking, intuitive thinking, dogmatic thinking, mystical thinking, analytical thinking and more. - *Which kind of thinking do we want to develop in our students?* I ask. - *Mathematical thinking* (the answer does not hang around too long). With the previous frustration and being afraid of some more frustrating answers, I do not ask the student, who gave the answer, to elaborate on it this time. *Well,* I say, *isn't mathematical thinking a too narrow goal? How will it help our students in the rest of their lives outside the mathematics classes?* Silence again. *Well,* I say after a while, *it is unfair to expect you to read my thoughts. What I have in mind is **rational thinking**.* At this point, twenty pairs of cold eyes are staring at me. - *All right,* I am trying a new direction. - *What is the teacher's main activity?* - *Teaching* (another answer which does not hang around too long). - *This is correct, but can you be more specific?* - *Making silence in the class.* -*Again,* I say, *it is unfair to expect you to read my thoughts. The main activity of the teacher is explaining.* The twenty pairs of cold eyes become hostile at this point. - *Consider a certain event,* I suggest, like *somebody becomes sick. Assume you want to explain it. Can you think about two explanations to it, a religious one and a scientific one and tell the difference between them? - Yes. The religious explanation will use arguments of punishment and award, where as the scientific explanation will use medical arguments.* Something starts to move in the class. - *And which one do you prefer?* I ask and add immediately, *please, don't tell me, because if you prefer the religious explanation you have a problem as science teachers. - But we are not science teachers, we are mathematics teachers. Didn't you tell us that mathematics is not about the real world, it is about an abstract world which exists only in our mind? - I am glad you remember this,* I reply, *but didn't you tell me that one of the main reasons for teaching mathematics is that science desperately needs mathematics in order to develop its theories? Are you trying to tell me that you work so hard serving science without believing in it? - Is this discussion related to mathematical thinking?* (a skeptic reaction which comes directly from the seats of the opposition). - *Well,* I conclude with my own question, *isn't it related to rational thinking? And isn't rational thinking related to mathematical thinking?*

So far with this illustration. I would like to return now to my original line of thought. Mathematical thinking and scientific thinking are parts of rational thinking. However, rational thinking is broader. Rational thinking is the kind of thinking which is needed to maintain our society. By "our society" I mean the liberal democratic society with its values, its various institutions, its science, technology and medicine. Other kinds of societies may need, perhaps, other kinds of thinking.

It seems, at this point, that if I recommend rational thinking as the main goal of teaching mathematics I have to define it. Well, definitions in the mathematical sense of the word exist mainly in mathematics. (This is another lesson that we should teach our mathematics students; namely, when they are outside the domain of mathematics and they want to clarify some notions, then looking for definitions is not necessarily a useful activity.) This does not necessarily have to be a problem. There are many notions for which we do not have strict definitions and yet we use them in academic discussions. But rationality is enormously broad, ambiguous and vague notion which has become a research topic for several academic domains like philosophy, psychology, economics, game theory, etc., and it will be extremely hard to uniquely characterize such a notion without arousing objection from scholars of different disciplines or of different personalities. It is quite typical to the academic community that controversies about central concepts of scholarly research never end. On the other hand, it seems that for ordinary people, using ordinary language, the concept of rationality is quite clear. In everyday situations, people recommend to each other to behave rationally. When they do it, it is quite clear what they mean. It is true that very often we fail to behave rationally, but this happens not because we do not know what it means to be rational. It happens because we are driven by strong impulses that we fail to control. This reality does not imply that we should give up rationality as an educational goal. On the contrary, education is about overcoming and controlling negative tendencies. The claim that human beings are irrational is irrelevant to education exactly as the claim that human beings are evil (or in the biblical formulation: *The desire of man's heart is evil from his youth* (Genesis 8, 21)). The end of education is to teach us how to control our evil desires. Moreover, even the claim that human beings are irrational is quite inaccurate. Some people use the collective work of Tversky and Kahneman to establish this claim. As a matter of fact, what Tversky and Kahneman show is that under certain circumstances, when people are given some intellectual tasks they fail to reach the correct answer. It happens because they make all kinds of mistakes. It does not happen because they do not care about rationality. To be irrational, in my opinion, means that we know what should be a rational decision in a given situation and in spite of that, we decide not to follow it. Such examples are gambling, smoking, eating frequently at McDonald's and so on and so forth. Therefore, I would rather claim that what Tversky and Kahneman show us is that people even try to be rational but they fail because they do not have suitable tools to deal with the problems posed to them. An indication to this claim is the title of Kahneman's Nobel Prize lecture (2002): Maps of bounded rationality. (More about this issue can be found in Leron & Hazzan (2006) and in Stanovich (1999)). Therefore, if this is the case about people's attempt and failure to be rational, why won't we teach them?

When discussing rationality, some researchers prefer to speak about rational behavior rather than rational people. This is quite reasonable, because a person can behave rationally under certain circumstances and irrationally under different circumstances. Robert Auman, in his Nobel Prize lecture (December 2005) suggested the following definition for rational behavior: *A person's behavior is rational if, when given his information, his behavior is in his best interest*. This definition was given within the framework of game theory, and we see that rationality is related here only to means and not to goals. It can be applied to moral as well as amoral goals. A criminal planning a perfect robbery, having all the necessary information about the place he is going to rob, is demonstrating a rational behavior. Even if he is caught later on, because he was not aware of some alarms in the place of the crime, his behavior can still be considered as rational because Auman's definition is relating to the information the person has. What I am saying here only demonstrates what I claimed earlier that the moment a definition is suggested, it arouses infinite discussions which never reach a conclusion. In addition, Auman's approach is not suitable for us because we want to speak of rationality in the context of education. Therefore, I suggest to use a non technical approach to the notion of rationality, namely, to use it as it is used in colloquial language. I am aware of the possibility that even here it might be hard to obtain a consensus. I have no illusions about that, and if somebody does not agree we can just agree to disagree.

One way to clarify the meaning of a notion in a colloquial language is to look it up in a dictionary. The Merriam-Webster dictionary suggests that to be rational is to be *reasonable*. *Rationality is the quality or state of being agreeable to reason.* Rationality is applied to *opinions, beliefs and practices*. About being reasonable, the dictionary adds that reasonable is *not extreme or excessive* and it is *moderate and fair*.

If we wish to bring a broader support for the dictionary suggestion, we can collect various excerpts from newspapers, magazines, everyday discourse, and political statements and examine the meaning of rationality in these contexts. I have done this in my own language but this is not necessarily a support for the use of "rationality" in English. A third resource can be looking it up in an encyclopedia. Generally, encyclopedias do not deal with colloquial terms. And indeed, the Encyclopedia Britannica (the 1974 edition) does not have "rationality" as an entry. It discusses "rationalism" in its scholarly style. However, "rationalism" is not the technical equivalence for "rationality" in the colloquial language. On the other hand, Wikipedia, the free encyclopedia of the ITERNET has "rationality" as an entry. In addition to its explication for "rationality" it also elaborates about the use of the term "rational." It suggests that *in a number of kinds of speech, "rational" may also denote a hodge-podge of generally positive attributes, including: reasonable, not foolish, sane and good* (Wikipedia, 2006). I hope that this reference can be used as an additional support to my claim about the meaning of rationality, at least, in colloquial English. I would like to add that the interpretation of rationally which I am suggesting includes also one characterization of rationalism that *regards reason as the chief source and test of knowledge* (Encyclopedia Britannica, 1974; rationalism). On the other hand, I am not considering "rationality" as a rival to empiricism (which is true about "rationalism"). On the contrary, I claim that **to be rational implies taking into account science, medicine and technology.** A person behaves rationally if he or she takes into account all the scientific information which is relevant to their decision making. Thus, rationality is a relative notion. A rational behavior in Newton's era is not necessarily rational in our era since science has been changed enormously.

At this point, I hope that I have clarified my interpretation to the notion of rationality and I want to assume that it is quite close to its interpretation among people who have at least a college degree in Western society (especially, our mathematics and science teachers). I expect them to distinguish between rational thinking and other kinds of thinking (dogmatic, mystical, divine, etc.). However, I take into consideration that the gap between my expectation and the practice might be huge and therefore I suggest that rationality in the above sense should be discussed in pre-service and in in-service teacher training. What I want to avoid is a university requirement of another dry academic highly sophisticated course in philosophy, which deals with rationality in a technical way. Such a course will, probably, discuss rationalism as a philosophical movement rather than rationality in the above sense.

I feel that after the above discussion about rationality I can return to mathematics teaching and suggest for it an educational goal. As I claimed earlier, education is primarily about values. I suggested that an educated person is a thoughtful person where the emphasis is on being considerate. My claim was that in order to be considerate we should use rational thinking. Please, note that my only concern here is about being considerate. This is not because I am unaware of other potential values. It is because I am aware of potential controversies over additional values that I can suggest. Even about being considerate some people may claim that it is inappropriate as an educational goal. They believe that selfish behavior is more useful for their interests (and note that it is quite coherent with Auman's definition of rational behavior and with the view that the human being is *homo economicus* (economical man) and as such he or she is logically consistent but immoral). However, within the educational domain, I hope, nobody will dare to recommend selfish behavior as an educational goal. Now, values and rationality are not part of any mathematics curriculum. They are not part of any curriculum at all. My suggestion is that teachers will address them in their classes at suitable moments. This requires a different state of mind from the teachers. It requires that while dealing with a mathematical issue, the teacher is expected to be aware of the possibility to add a reflective dimension to the lesson and integrate in it a discussion about rationality and its importance to values. It requires from the teacher some improvisation skills (of course, one can plan it or partly plan it, but improvisation is better). What I am suggesting here is clearly in contradiction with the current conception of mathematics lessons which are supposed to be strictly task oriented, structured and planned to the last detail.

For example: when dealing with fractions one can discuss sharing as a value. Sharing is one of the principle educational values mentioned by Postman quoting Robert Fulghum's book: *All I ever really needed to know I learned in Kindergarten*. Fulghum's list is the following: ***sharing everything***, *playing fair, don't hit people, put things back where you found them, clean up your own mess, wash your hands before you eat, and flush* (Postman, 1996, p.46). Space restrictions, again, prevent me from presenting more examples in which particular mathematical topics can be used in order to discuss educational values.

So, my slogan can be**: From covering the syllabus as a main goal to a value oriented meaningful learning.** I am aware of the difficulties of such a move. The official educational systems will reject it strongly. They developed a system in which learning achievements have the first priority. The common belief is that they also developed tools to measure these learning achievements. Unfortunately, measuring has become the ultimate criterion for any educational project. *I measure therefore I exist* can replace Descartes' *cogito ergo sum.* As long as the authorities of the educational system will not change their *covering the syllabus* policy, teachers won't change their teaching style. The question is whether in one school or several schools, which will decide to have an autonomous approach to education, will the teachers be able to teach in the above recommended style? I believe that this is quite feasible. As I said earlier, some training is necessary, but teachers are sensitive human beings who care about educational values (if they are not, they chose a wrong profession). Their majority prefers meaningful teaching to meaningless teaching. Many of them are quite frustrated with the current style of teaching and learning. However, they feel they can do nothing about it. But even in the current state of affairs, if they take only five minutes from each lesson to deal with values in the way recommended above, this will be an important change. Thus, this is not a huge change in the system. It needs only some amount of approval and encouragement from the educational authorities.

As to teacher training, I recommend that it will include discussion of rationality and values in the above mention form. Since I am recommending *spontaneous discussions* on these issues as a tool in school teaching, I also recommend that these issues will be dealt in teaching training by means of spontaneous discussions and not by lectures. It is quite important that a teaching style which we recommend in teacher training will be presented to future teachers in that recommended style. I recall a student in teacher training who, in a private conversation, expressed criticism on one of her teachers who recommended to his students to use many teaching styles in their classes but the only style he used was lecturing.

My recommendations to deal with values in mathematics classes can certainly be applied to other school disciplines. It can clearly be applied to all natural science disciplines where rational thinking is an essential issue. However, it can also be applied to social sciences and humanities. There, rationality is perhaps not a first priority. However, since they all deal with human beings they can easily be tied to human values. A mathematics education framework is perhaps not the place to advocate this claim, but I would like to tell you one short story from my own biography which is relevant to my recommendation. When I was a high school student I had a history teacher who was considered by the class as a lazy person. When I analyzed his behavior years later I understood that the reason for this view was that very often he avoided covering the syllabus. We were not worried by it because history was not one of our matriculation disciplines. Very often this teacher discussed with us football games and westerns (which he called "historical films"). Football games and westerns were not my cup of tea even as a teenager. However, I enjoyed these discussions because they save me writing down academic comments in my history notebook. Now, in historical thinking there is an extremely important principle. It is the distinction between a fact and an opinion. A fact is a fact, I am trying to be short here, but an opinion can be true or false. This abstract principle requires some concrete examples. My teacher's generic example was the following (and I am making a necessary adaptation of it for my current audience): *That Arsenal lost the game against Barcelona is a fact. However, that their game was weak is an opinion.* Many years have passed since then and when I compare now my historical knowledge to the historical knowledge of my friends who studied history with teachers who covered the syllabus, I see no difference. All of us know almost nothing. However, some of them still cannot tell the difference between facts and opinion. I consider it as a thought defect which can be labeled, in the terminology of Tversky and Kahneman, as bounded rationality. Moreover, every week I face situations, in politics, economics, psychology and more, where the people involved do not distinguish between facts and their own opinions (in fact, sometimes, the distinction is not so easy). I am grateful to my history teacher for the distinction he taught me in his special humoristic way. It serves me in my everyday life not less than in my academic life. It helped me also resisting post modernism in which the notion of fact vanished in order to promote the ultimate notion of narrative.

Another example is taken from my own teaching experience. In a research methodology course which I used to teach, I very often reminded my students that *a correlation is not necessarily a causal relation.* This is quite a subtle idea, since it does not exclude the possibility of a causal relation. It only calls us to examine very carefully the situation before deciding about a causal relation. It also tells us that in many cases we cannot be sure and that in many cases we even won't be able to know. This is clearly part of rational thinking. It is not part of mathematical thinking. It is undoubtedly part of scientific thinking (whether it is in natural sciences or in social sciences). I believe that this idea, if internalized, can lead to a moderate behavior and to more balanced analysis of various situations, and therefore to a more desirable behavior which is the end of education. Nevertheless, every week, as in the case of fact and opinion, I face situations in which people do not take into account this idea. They are politicians, economists, police investigators, lawyers and medical doctors. All of them can do excellent jobs in their field of expertise, but sometimes, when ignoring the above idea, they can make fatal mistakes when drawing conclusions.

At this point I would like to call my diverging recommendations and analyses to converge. My suggestion was to set rational thinking as a main goal of mathematics education by means of which it can be also tied to educational values. It is not an entirely new idea. Processes as control, analysis, planning, meaning negotiation, reasoning and more have been recommended long ago by mathematics educators. They are all part of rational thinking. What is new is that all these should be presented to the students as part of rational thinking together with additional educational values which I mentioned earlier. Teachers are recommended to discuss all these in their classes in appropriate contexts which can come out by reflection from mathematical situations. It should be emphasized that rational thinking is the best way of thinking for our liberal democratic society. We should remember that rational thinking has produced science, medicine and technology, and where would we be without them nowadays. Rational thinking is also the best way of thinking for solving conflicts, whether these are conflicts between individuals, groups or nations. Rational thinking is the best way of thinking for our students. It can lead them to worthy destinies and help them choose desirable goals for their lives. Although game theory researchers, as well as some other researchers, are reluctant to relate rationality to goals, we do not have to accept it in the domain of education. Here, I believe, we are allowed to speak also about rational goals. If somebody needs a philosophical support for this approach, we can send them to Kant and Aristotle.

Ingmar Bergman, in his television series *Scenes from a Marriage,* where the hero's behavior is extremely irrational, complains that schools taught him all kinds of things like the Pythagorean Theorem or the capital of Brazil, but they did not teach him how to treat other people. Thus he finds himself at a certain moment in a terrible conflict with his wife, which turns into a physical violence. If Ingmar Bergman were an educator he would recommend schools which teach students how to treat each other. Since

Bergman is an artist he avoids making a formal recommendation. However, many great artists have implicit educational recommendations in their art. A school which teaches its students how to treat each other is a great vision. It is a huge project. Certainly, it is not a project for some old retired mathematics educators. However, it can be recommended as their will.

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Distinctive Ontogeny: Ontological Musings Inspired by two discerning gentlemen of distinction(s), David Tall & Eddie Gray

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ABSTRACT: *The notion of a procept is used to contribute to observations about how distinctions come into existence (their ontogeny), what makes some distinctions useful and others not (the robustness of their ontology). Proceptual thinking, Concept Images and Advanced mathematical Thinking are used to illustrate these questions, exemplified in the notion of infinite unfolding of arithmetic and algebraic expressions.*

Key words: *procept, concept image, advanced mathematical thinking, reification, distinctions, ontology, ontogeny.*

INTRODUCTION

Where is mathematics education headed? Are we making progress, or are we going round in circles, re-discerning and re-labelling each other's distinctions? Is mathematics education developing as a coherent articulation of increasing insight, re-articulated and refined in each generation, or is it turning into a morass of the same or similar insights repeatedly put forward as new and different, often in fresh language but otherwise the same?

In this tribute I wish to demonstrate my high regard and admiration for the work of my two friends and colleagues, David Tall and Eddie Gray, by musing on their place in history as a case study in the ontogeny and ontological status of distinctions in mathematics education: how certain distinctions come into existence and, at least for a time, are taken up and used; how they become embedded in the fabric of 'taken as shared' by the community at large, while other distinctions wither into obscurity. I shall focus primarily on distinctions signalled by *procepts, concept images* and *advanced mathematical thinking* whose origins and development have been due largely to David and Eddie's ongoing programme of enquiry, aided by colleagues and students over the years. Indeed I will do most work on procepts, for reasons which will emerge in the rest of the paper. I will leave to the reader further contemplation of my opening questions.

I want to see how these particular three distinctions manage to bridge the supposed chasm between theory and practice, that unholy distinction which is, in my view, wholly unhelpful in making sense of a complex world. In other words, some distinctions are helpful by informing future action, while others trap us in artificial tensions. Some are distinctive in their longevity; others vanish quickly without a trace. Apart from socio-cultural forces which priviledge some people's distinctions over others, I suspect that the reason that some survive is pseudo-Darwinian, fit-for-purpose: where they enrich people's sense of meaning and understanding of complex situations, and where they inform actions, they survive and are passed from generation to generation.

LASTING LEGACIES

I suggested that many distinctions and their labels are forgotten within a few years, much less a generation. Here however we are celebrating contributions of two discerning scholars whose distinction is that some, at least, of their distinctions have already influenced several generations of researchers, and are in almost common parlance within mathematics education. I refer to the notions of *procepts, concept images* and *advanced mathematical thinking*.

Procepts

The label *procept* signals a distinction between the use of a technical mathematical term as a label for a process and for a concept. Thus someone can engage in a process but be unaware, at least explicitly, that there is a process, or what that process consists of (Piaget, 1973, p.25). This is consistent with the notion of *theorem in action*: an action carried out which assumes or makes use of some theorem or property of which the user is entirely unaware (Vergnaud, 1981). Classic examples include counting-on (Gray & Tall, 1994), adding (or any other binary or unary operation, perhaps most spectacularly division), angle (experienced as rotation), the quadratic formula (result of algebraically completing a square), and rotation (or any other transformation).

The contribution of the term *procept* is that it reminds teachers and researchers of an important distinction between learners carrying out a process, possibly without even being aware of their actions as a process, and learners sufficiently aware of a process to be able to refer to it and begin to encapsulate it as an object of thought, something to be studied or contemplated in its own right.

Some people argue that since all concepts are ultimately based on sense-based metaphors, all mathematical concepts have a proceptual basis (see for example Lakoff & Nunez, 2000). Others, myself included, would suggest that a concept such as *ring* might be a counter-example, because the notion of a ring arises not from an action but as an abstraction through identifying and isolating properties describing relationships discerned in particular instances. The abstraction comes about through what I believe is a peculiarly mathematical move: taking identified properties and making an almost ontological assertion that what is to be studied are objects with those properties (and what can be deduced from them). Of, course, mathematically it is necessary to verify the ontology by proving that at least one such object exists.

Because it is hard to imagine what it is like *not* to be aware of a process as itself an object of study, it may be worthwhile to consider a specific and perhaps not entirely familiar example.

Unfolding

The fact that $0.\dot{9}$ is another decimal name for 1 seems to be quite hard for learners to accept, being tied up with the notion of limit and the notions of 'completed' and 'ongoing' infinity. Consider the following fact:

$$1 = 0.9 + \frac{1}{10}(1)$$

Now substitute this fact about 1 into the 1 in brackets on the right:

$$1 = 0.9 + \frac{1}{10}\left(0.9 + \frac{1}{10}(1)\right)$$

Now repeat the substitution

$$1 = 0.9 + \frac{1}{10}\left(0.9+\frac{1}{10}\left(0.9+\frac{1}{10}(1)\right)\right)$$

Now imagine doing this on and on forever (infinite unfolding):

$$1 = 0.9 + \frac{1}{10}\left(0.9 + \frac{1}{10}\left(0.9 + \frac{1}{10}\left(0.9 + \frac{1}{10}(...)\right)\right)\right)$$

Here the … is a symbol introduced by Isaac Newton and taken up by John Wallis, signalling a process of 'continuing in the same manner'. Usually people start by contemplating and wondering what point nine recurring might mean. Even though learners are familiar with different fractional names for the same rational, they find the idea of infinitely repeating 9s hard to encompass as another decimal name for 1. By starting with the specific and gradually introducing iterated unfolding before converting to an infinite process, a slightly different perception is available: learners start with the familiar so as to reach out to the unfamiliar rather than starting with the unfamiliar and trying to relate it to the familiar. Treating the expression on the right as an unknown number whose value is sought, texts often use algebraic reasoning:

$$x = 0.9999... \quad \textit{so } 10x = 9+x \textit{ so } x = 1.$$

Using the unfolding you can write

$$x = 0.9 + \frac{1}{10}\left(0.9 + \frac{1}{10}\left(0.9 + \frac{1}{10}\left(0.9 + \frac{1}{10}(...)\right)\right)\right) = 0.9 + \frac{1}{10}(x) \quad \textit{so again } x = 1.$$

Here there are two potential processes: unfolding (which can be extended to infinite unfolding) and expressing the whole in terms of itself (recognising *x* inside its own expression). There are, of course, analytic problems of whether the infinite unfolding has meaning and, following the legacy of Weierstrasse and Cauchy, this leads to questions about limits. However one can also follow the legacy of Euler and ask what happens if you permit infinite unfolding to produce formal objects, analogous to formal power series. This can lead to dramatic results!

Consider for example the number 1.618033988749894848204587… which may be familiar as the beginning of the decimal expansion of the golden ratio

$$\frac{1+\sqrt{5}}{2}$$

As a contribution to gaining proceptuality with infinite decimals, that is, with appreciating both completed and ongoing infinity, it may be helpful to observe that $\sqrt{5}$ and, consequently, the golden ratio both 'know all their decimal places'. It is just that they cannot tell us, and we will never know them.

Less familiar presentations also make use of unfolding. Here I present infinite unfolding, expressions recognising the recursive use of the expression, and an unfolding sequence.

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\dots}}}} \qquad\qquad \begin{aligned} x &= 1+\frac{1}{x} \\ x^2 &= x+1 \end{aligned}$$

Of course in expressing the unfolding you can also see the expression inside itself in other ways:

$$x = 1+\cfrac{1}{1+\cfrac{1}{x}} = \frac{2x+1}{x+1}, \qquad x = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}} = \frac{3x+2}{2x+1}, \qquad x = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}}} = \frac{5x+3}{3x+2}, \qquad \dots$$

Attention to the process, watching what you do as you calculate the next fraction by making use of what you already know, reveals the coefficients of the rational polynomial as Fibonacci numbers. Becoming aware of a process, and treating the result of a process already completed as an object, leads to familiar territory and makes the simplification a great deal easier than starting from scratch each time!

Even less familiar may be

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\dots}}}} \quad \textit{where } x = \sqrt{1+x} \text{ so } x^2 = 1+x\,,$$

or

$$x = \sqrt{1+\sqrt{1+x}} \text{ so } \left(x^2-1\right)^2 = 1+x$$

which raises questions about which root one chooses for ‘the value’ if this even makes sense in every case.

Nothing ventured nothing gained, the fundamental equation can be rearranged in other ways:

$$x = -1 + x^2 \text{ so perhaps } x = -1+\left(-1+\left(-1+\left(-1+(\ldots)^2\right)^2\right)^2\right)^2$$

These are simplicity itself compared to a more vigorous version of unfolding which in view of the purpose of this paper might be called *DOTTY EG* due to the promiscuous use of the symbol … (dots) as an example of a whole class of objects accessed through the process of unfolding, yet resonant with David Tall’s initials D. O. T. and Eddie Gray’s initials E. G..

$$1+\frac{1+\dfrac{1+\dfrac{1+\dfrac{1+\frac{1+\ldots}{\ldots}}{\frac{1+\ldots}{\ldots}}}{\frac{1+\ldots}{\ldots}}}{\dfrac{1+\frac{1+\ldots}{\ldots}}{\frac{1+\ldots}{\ldots}}}}{\dfrac{1+\dfrac{1+\frac{1+\ldots}{\ldots}}{\frac{1+\ldots}{\ldots}}}{\dfrac{1+\frac{1+\ldots}{\ldots}}{\frac{1+\ldots}{\ldots}}}} \qquad 1+\frac{2+\dfrac{4+\dfrac{8+\frac{\ldots}{\ldots}}{9+\frac{\ldots}{\ldots}}}{5+\dfrac{10+\frac{\ldots}{\ldots}}{11+\frac{\ldots}{\ldots}}}}{3+\dfrac{6+\dfrac{12+\frac{\ldots}{\ldots}}{13+\frac{\ldots}{\ldots}}}{7+\dfrac{14+\frac{\ldots}{\ldots}}{15+\frac{\ldots}{\ldots}}}}$$

The structure may be easier to discern by considering a more generic expression of the same form shown on the right of Fig. 1 (whose value is not at all evident due to the lack of recursive or unfolding structure).

In the left hand expression in Fig. 1, you can detect recursive presences of the whole as follows:

$$x = \frac{1+x}{x}\ , \quad x = \frac{1+\frac{1+x}{x}}{\frac{1+x}{x}} = \frac{2x+1}{x+1}\ , \quad x = 1 + \frac{1+\frac{1+\frac{1+x}{x}}{\frac{1+x}{x}}}{1+\frac{\frac{1+x}{x}}{\frac{1+x}{x}}} = \frac{3x+2}{2x+1}$$

This reproduces the Fibonacci coefficients. The DOTTY EG expression in the second column suggests a wide variety of fractions to investigate in which, like the expression shown, have different 1s replaced by different numbers. A Weierstrasse-Cauchy type question concerns characterising the conditions under which they converge; an Euler type question concerns what you might be able to do with them. If you also admit multipliers in front of the fraction signs, you can get special cases such as

$$1 + x\frac{1 + x\frac{1 + x\frac{1+\dots}{3}}{2}}{1}$$

which is the power series for e, and

$$1 + x\Big(1 + x\Big(1 + x\Big(1 + x\left(\dots\right)\Big)\Big)\Big) = \frac{1}{1-x}$$

which converges only when $|x| < 1$, but which represents a formal power series otherwise.

Using the iterated square roots idea, it is an interesting exercise to locate a value for

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \dots}}}}$$

as well as proving that it converges.

The point of these particular cases here, as well as offering something intriguing to think about, is to provide some instances of proceptuality. That is, to offer some situations in which there are opportunities to notice the interplay between process and object, between action and result, and how appreciation of both can sometimes emerge together rather than moving strictly from process to object or vice versa. As with young children’s pre-language ‘babble’, and with the over-generalised use of new words until they settle down to conventional use (Brown, 1973), it is often the case that new symbols and new words are used unconventionally and inappropriately at first, until thinking, speaking and symboling become coordinated and settle down into near-conventional form (Malara & Navarra, 2003; Berger, 2006). The unfolding process proceeds from an object via a process to another (infinite) object; an infinite process can sometimes be conceived of as a static object through recognising an unfolding process and expressing the object in terms of itself so as to locate and make sense of it.

Self-Referentiality

The notion of a *procept* is naturally and appropriately self-referentially itself a procept, for it is a label which signifies that a process (of reification) has itself become a concept. A process has been discerned and recognised *as* a process, which could apply to situations other than the immediate experience. The possibility that the immediate experience could be an example of a phenomenon, that there might be a family of similar situations is at least implicitly recognised. The process is isolated from its immediate context. As a rich web of associations grow around the seed-crystal of one or two experiences, the label triggers awareness (possibly below the level of consciousness) which sensitises discernment. Having the label come to mind can inform practice through alerting the teacher to a process of maturation which may take considerable time, as in Eddie and David's generic example, the technique of 'counting on'. It can also inform task and interaction design by alerting the teacher to supportive experiences such as accepting that there may be a period of word and symbol babble, and suggesting possible ways of fading the initial scaffolding so that learners are encouraged to become explicitly aware of, and to reify the process.

This raises what I consider to be an important question: what occasions this emergence? What initiates the transformation from a succession of experiences into an experience of the succession? What mediates that transformation, and on what prior experience does it act? In other words, how do pertinent and long lasting distinctions come into existence?

Concept Images

The notion of a *concept image* (Tall & Vinner, 1981) has been taken up by many generations of mathematics educators. The reason, I think, is because of the useful distinction it makes between formal definitions and the gradual development of a *sense* of what the term means to them. For example, the distinction speaks to many people's experience of learners displaying an apparently cavalier appreciation of the significance of details of formal definitions, and learners who seem stymied when expected to reason on the basis of those definitions. Thus the label identifies a phenomenon. As such, it can alert teachers to the underlying distinction, and thereby inform their choices as to what experiences and tasks to offer learners at each stage (see for example Mason & Johnston-Wilder, 2004, 2006).

Because it crystallizes features of a complex phenomenon based on a recognised distinction, the label has entered our vocabulary. The label helps us discern aspects of learner behaviour, and brings into existence a phenomenon to which we are more sensitised than previously. Through reification and the corresponding ontological commitment, it opens up the possibility for probing in more detail the constituents of concept images and the ways in which these arise and are integrated (to some degree or other) in individuals in and through their interaction with peers, with expert-others, and through reflection on their own experience. It is important to bear in mind that distinctions and associated phenomena are only a human social construction.

Advanced Mathematical Thinking

David's *magnum opus* (Tall, 1991) which arose from years of collaborative work with many colleagues through PME and elsewhere, alerts the reader to a distinction, but a distinction with a distinctive difference. In this case there seem to be neither a precise border nor a clear cleavage between ordinary or school-based mathematical thinking and advanced mathematical thinking. Every time you isolate some feature which might be thought of as advanced, such as axiomatisation, or multi-step reasoning, you find that it is or could be presaged by, and even found in, the thinking of young children. Every feature of mathematical thinking is within the reach and experience of virtually all learners, no matter how they are classified by the educational system. It is the tragedy of mathematics education research that as a community we have not managed to infuse and infect national curricula and national teaching standards with an ethos in which mathematical thinking, advanced or otherwise, is an integral part of every mathematics lesson.

For me the main contribution of this distinction is that it has supported and promoted the growth of attention to issues in teaching and learning mathematics in the tertiary phase of education. With the decline of interest in mathematics amongst learners, and attempts by many disciplines traditionally dependent on mathematics to develop curricula less dependent on mathematics, there are now good economic as well as socio-cultural-educational reasons for interest in and concern about the educational experience of learners of mathematics in universities.

One of the features, perhaps, of advanced mathematical thinking, at least within the domain of pure mathematics, is that learners need to distinguish between, and keep in touch with both the formal and the informal. The informal is the source of intuition, the domain of inductive and abductive reasoning. It informs choices for exploration or application. The formal is the domain of deductive reasoning. Concept images form a major component of what it means to appreciate and understand a mathematical topic (Mason & Johnston-Wilder, 2004). In my own work I am finding that some at least of the difficulties of learning mathematics and of reasoning within mathematics may come about because of a potential confusion between on the one hand perceiving properties of objects and reasoning about them, and on the other hand reasoning on the basis of selected and explicitly enunciated properties. These are quite different ways of perceiving and reflecting on the world of experiences and images. They signal for me distinctively different structures for attention (Mason, 2003, 2004). I mention this simply to try to give a flavour of how distinctions arise and develop.

INTERLUDE ON REFLECTIVE ABSTRACTION

Having reviewed just three of the distinctions that David and Eddie have offered our community and that have had a major and lasting influence, I am drawn now to engage in a process of *reflective abstraction*. Jean Piaget (1971) used this term to label the act of 'learning from experience', of making sense of situations and observations so that future practice is informed. It seems to apply not only to sense making arising from experience of process as a contribution to the achievement of a procept, but also to sense making that is not process based.

Reflective abstraction underpins a good deal of David and Eddie's work (e.g. Gray & Tall, 1994; Tall, 1995, 1999; Tall, Davis, Thomas, Gray, & Simpson, 2000 amongst many others). It involves much more than simply 'thinking back over what happened', because it is based not on individual actions but on coordinated actions amongst colleagues construing collectively. The import of *coordinated actions* has been explored and developed by Humberto Maturana and colleagues (Maturana & Varela, 1972, 1988; Maturana, 1988; Maturana & Poerksen, 2004) to show that language can be seen as the concensual coordination of the concensual coordination of action. It is through language that we weave the narratives which justify actions in our own eyes and in the eyes of others, and so coordinate our sense of self with the observed behaviour of others. Through narratives we construct a sense of ourselves as continuous entities rather than as bundles of fragmented memories. Coordination brings the process into the social domain, as a component of a complex action undertaken by individuals within and through their social milieu (Brousseau, 1997) to make sense of mathematics and to make mathematical sense.

Although it makes use of natural powers possessed by all learners, *reflective abstraction* certainly belongs in the Vygotskian category of *higher psychological processes* (Vygotsky, 1978), accessible to learners mainly through being in the presence of others for whom it has become an established practice. Reflective abstraction requires a standing back from participation in action, a shift of attention away from acting to becoming aware of that action. It is probable therefore, that reflective abstraction is an attempt to describe or capture the essence and core of all learning.

It is typical of many pertinent and robust distinctions that it is difficult if not impossible to formalise the distinction without losing its force. Furthermore it is non-trivial to introduce a distinction to others. It is, for example impossible to force someone into reflection, much less into reflective abstraction, yet the process can be displayed and demonstrated.

DISCERNING DISTINCTIONS

Of course the remarks in the previous interlude and in the first section have already begun the process of addressing the question of ontogeny and ontology of distinctions. Indeed the notions of *procept* and *reflective abstraction* already constitute a partial response, by drawing attention to the way in which processes become objects through reflective abstraction. But to be perceived *as* a process, processes depend on distinctions being perceived and acted upon. The process itself has to be distinguished from the surrounding environment and accompanying actions (scratching your foot, standing up or sitting down, etc.), and the process will necessarily involve the discerning of distinctions in order to act upon chosen elements in order to carry out the process. For example, to add two fractions requires distinguishing those fractions from other marks on the paper, and to discern and identify two numerators, two denominators, and the symbol which indicates division. If this seems trivial, then try adding the following fractions, while at the same time trying to catch what you do with your attention:

$$\frac{\frac{2+\sqrt{3}}{5}}{\frac{2-\sqrt{3}}{3}}+\frac{\frac{6}{\sqrt{5}-1}}{\frac{1}{\sqrt{5}+1}}$$

It takes a little bit of conscious work to identify and retain the numerators and denominators as objects while at the same time being aware of their substructure as fractions.

Distinctions are made as a result of experiencing a disturbance, because that is how our sensory apparatus works. Both animals and plants react or respond to perceived disturbance to the *status quo* within the ecology of their environment. Thus distinction-making is fundamental to all life forms.

Distinctions are in a sense vaccination doses based on previous disturbances, triggering reactions which were made in the past as well as offering possibility of less habitual responses. So a disturbance which attracts a need for a fresh response (accommodation rather than assimilation) generates what we call a distinction. When there is enough similarity to trigger a similar response, or as Marvin Minsky (1975) put it, if there are sufficient parameter values in addition to default parameters so as to fire a 'frame of mind', we detect relative invariance in the midst of other changes. We become aware of 'similarity': we become aware of aspects of a situation which it is possible to vary without changing the essence of the situation. This is what gives birth to our sense of a phenomenon: we perceive a potential property which generalises a relationship detected in a particular instance. Variation as the essence of learning has been developed particularly by Ference Marton and colleagues (Marton & Booth, 1997; Marton & Trigwell, 2000; Marton & Tsui, 2004). It is no surprise then that detecting and characterising relative invariance in the midst of other change is a fundamental theme of mathematics.

How do we detect or determine similarity rather than difference? In one sense, no two experiences are the same, or as Heraclitus put it, you cannot step into the same river twice. Yet if every instance were entirely novel, organisms could not function. Attention would be totally absorbed by constant novelty. Practices are only possible where there is repetition, and hence perceived sameness. Our perceptual apparatus does an enormous amount of classifying and rejecting before our cognitive functioning even gets a look in (Norretrander, 1998). This is perhaps the principal source of many philosophical conundra, for it is very easy to fall into the trap of believing that the perceptions of which we are aware somehow truly reflect or present some objective material world, rather than being the tip of an iceberg of sensory processing below or beyond the level of consciousness.

Minsky's notion of neural states which fire when sufficiently many parameters are instantiated is certainly attractive as an explanation of perceived sameness. Caleb Gattegno's notion of stressing and consequent ignoring (Gattegno, 1987) as the origin of generalisation serves a similar purpose in explaining the experience of sameness. So *phenomena* are what we call the experience of 'sense of sameness' of having 'been here before', and around which it is then possible to theorise, making use of distinctions to discern elements to which we are currently sensitised, amongst which we recognise relationships, which in turn become properties that might obtain in other situations in the future, and on the basis of which we can reason about the worlds we inhabit.

A Soviet Perspective

Lev Vygotsky and his colleagues (Vasilii Davidov, Alexei Leontiev) put forward a radical proposal based on Lenin's insights triggered by Hegel among others, to the effect that scientific concepts (as distinct from natural or ordinary concepts) are met first in the general and only then instantiated. They thus call into question the dominance of empirical or inductive generalisation which forms the basis of most western mathematics curricula. In these, learners are given lots of examples and somehow expected to see through the particulars to generality. Davydov (1972/1990) is unsparing in his criticism of empirical generalisation and the dependence on 'same and different' which marks such pedagogical approaches. He stresses the necessity of the general or the abstract as capturing essences and so being richer than experience of particulars. Seeing the general through the particular can be construed as contacting an essence of the particular as a property shared by many other particulars. The experience of each particular is enriched by relationship to the general through appreciation of an essence. This is in marked contrast to the subsuming of particulars in a general in such a way as to diminish appreciation of the rich variety and complexity of particulars which instantiate the general.

Reification

Numerous authors have addressed the question of how reification comes about, perhaps most notably (other than David and Eddie), Sfard (1991; 1992; 1994), Dubinsky (1986; 1991a; 1991b) and Lakoff & Nunez (2000), not to say Dienes, Piaget, Freudenthal, and of course Plato through his Socratic dialogues. In a sense all enquiry into learning of mathematics addresses this issue, however peripherally. Desire for actions which guarantee that reification will necessarily take place for any particular learner or learners at any particular time is in my view the origin of the failure to 'solve' problems in mathematics education. There is and can be no such guarantee. It is the lack of guarantee which makes me doubt the efficacy of cause-and-effect as a suitable mechanism for discussing learning or designing teaching. However, trust in the organic and developmental thrust of life in a challenging but not entirely hostile environment is usually rewarded. What we can do is seek favourable (I am not even confident about the possibility of optimal) conditions which support, for example, reification. We can devise labels for distinctions which can sensitise us to notice pertinent details, recognise relationships, perceive these as properties, and so support us in choosing to act in a principled, if not always an appropriate manner in situations we encounter as teachers and researchers.

Ference Marton offers an example of how we might probe beneath the surface of reflective abstraction and reification. He argues that exposure to limited variation over a short period of time is, if not necessary, particularly advantageous, especially in an educational context (see Marton & Booth, 1997; Marton & Tsui, 2004). Here Marton and colleagues depend on natural powers of learners which seem to be hardwired into the brains which 'we construct' as our organism grows and develops. Gattegno (1987) put it most challengingly as "I made my brain" during development in the womb. But not everyone uses these powers in the same way. Reflective abstraction as a label crystallises ancient wisdom that something more is required for learning than merely suffering experience, and whatever this is, it depends on a suitably supportive classroom rubric (Floyd, Burton, James, & Mason, 1981), or as it has come to be called, a *community of practice* (Lave & Wenger, 1991), or a *local community of practice* (Wimborne & Watson, 1988). It seems to me that the question of how to stimulate learners to engage in reflective abstraction is one of the abiding and core issues which requires more (and more detailed) research, in order to inform practices as well as policies.

Ontogeny

Distinctions which inform behaviour in a positive or acceptable manner reinforce the pathways which triggered them. For our conscious awareness to become involved and to participate in choosing, several elements are required. First, the distinction needs to be encrusted with a rich web of experiences. It is not just that individual experiences need to come to mind, because the pathways triggering action need to become robust and stable. Second, it helps conscious participation if there is a suitable label. What functions well is a few chosen words which have metonymic associations with making the distinction. It is much harder to make use of an unfamiliar word or sound as a label than a familiar one, but it is also difficult if there are contrary associations which trigger other actions and discernments. One of the problems with introducing labels into the mathematics education discourse is that unhelpful or extraneous associations are either already present or soon develop. One has only to think of the misuse of associations with *zone* in the zone of proximal development, or the ways in which *concrete* ties people's thinking to the physical, material world when discussing mediating tools.

The subtle move in the ontogeny of distinctions seems to be the reappearance of the same or similar distinction on a subsequent occasion. Sometimes this awareness is well below the surface of consciousness and so beyond the level of cognitive control. Sometimes the distinction emerges into conscious awareness. When our tendency to language throws up a label, a distinction comes into social existence, enabling, even promoting and prompting communication with others. We are already being socialised into recognition of and reaction to repeated or recurring instances of this phenomenon, according to characteristic samenesses of which we may or may not be consciously aware.

Communicating Insight & Jargon

The act of naming has consequences. The up-side is that naming makes it possible to communicate with others about what we discern, what relationships we are stressing, and hence what properties are salient for us. A label makes it possible to communicate efficiently and effectively with yourself and with others. The down-side is that naming seduces us into ontological commitment that the distinction being made is a natural cleavage in the world, rather than simply an indication of an instance of the coordination of our perceptual and languaging systems. Thus it is important to raise questions about distinctions which we try out. Do they inform future practice? Are they communicable to others who likewise find them informing their practice? What do they obscure and what do they illuminate?

A classic example of the negative effects of a distinction can be experienced in the terms *theory* and *practice.* This distinction pervades not only the mathematics education literature, but philosophic and psychological literature stretching back as far as Plato. The very fact that this distinction is so well established makes it difficult to generate an alternative frame of mind. One such attempt is through *enactivism* (Kieran, 1988; Varela, Thompson, & Rosch, 1991) in which knowledge and action are seen as synonymous. The power of an established distinction can be experienced by trying to encompass *enactivism* as a stance: it is remarkably difficult to accommodate yourself to a strongly enactivist perspective starting from a traditional perspective which distinguishes so essentially between knowledge and action.

Mathematics and Distinctions

One of the features of mathematics is that it works explicitly on similarity and difference through its pervasive theme of *invariance in the midst of change*: most mathematical theorems can be seen as statements about something which is (relatively) invariant, and aspects that are permitted to change and still retain that invariance. For example, the sum of the angles of a planar triangle is invariant, while the shape and dimensions of the triangle are permitted to change; a triangle has to have three 'angles' formed by three line segments meeting at three points, but those points can be anywhere in a plane, including lying on a single straight line, or even coincident; the product of the lengths of the two segments from a point to the two points of intersection of a line through that point with a given circle, is independent of the line, as long as it meets the circle!

Mathematics goes further, distinguishing and drawing attention to relationships which connect different features (e.g. where two straight lines cross, the pairs of opposite angles are always equal). Relationships in particular situations are generalised to properties, and a good deal of mathematics is about locating the logical interdependencies between properties. An extreme form of this is axiomatisation, in which certain properties are isolated and identified as axioms, and then other properties are derived from them. Although axiomatisation as a formal process is usually mentioned explicitly only at undergraduate or perhaps sixth form level as if it involved advanced mathematical thinking, the process itself is woven through the entire school curriculum. The fact that this is often overlooked or unnoticed by learners and by teachers may contribute to the way in which school mathematical experiences often seem to miss or overlook the essence of what mathematics is about.

Robustness

Distinctions which serve a purpose and inform behaviour naturally become more robust and stable with repeated use over time. However, after a while they may actually blunt perception and obscure rather than sharpen and inform, because their quick triggering inhibits different disturbances leading to fresh distinctions and further sensitivity.

Very often, the more a distinction is used, and the more people who use it, the fuzzier and less distinct it becomes. This might be happening to *advanced mathematical thinking*, and it often happens during the ontogeny of an individual's discernment of a distinction that the label encompasses rather too much before it narrows again and is used consistently with the way others use it. This is a natural feature of acquiring language (Brown, 1973). It is not always easy to stand aside and let others babble about and appear to mangle what for you are clear and useful distinctions, as they struggle to accommodate their interpretations into their own thinking.

Beyond Distinctions

Distinctions may be the origins of perception and fluent participation in thought and action, but they are far from the end. There is more to life than distinguishing this from that, foregrounding and consequently backgrounding, stressing and consequently ignoring (Gattegno, 1987). It is possible to discern much finer structure in attention than merely discerning boundaries and variation. Without becoming aware of relationships and the emergence of properties and reasoning based on those properties, distinctions would soon become an unmanageable collection. It would be impossible then to describe or explain how some distinctions dominate others, and how distinctions coagulate into larger structures which can be discerned in the complexity of experience.

Ockham's razor is an admirable tool, but every attempt to reduce complexity to simplicity leaves itself open to mechanisation. The present trend to record and count every action, and to value only that which can be explicitly observed, recorded and counted, will eventually have to give way to a resurgence of appreciation of subtlety of that which is left unsaid, tacit and implicit, of that which maintains complexity. In short, what is needed is a revivifying of trust in the organic. Just as we trust plants to grow given suitable conditions, we have to trust children to learn to read, count, and use their undoubted powers to make mathematical sense of the world, and to make sense of mathematical thinking given suitable conditions. As long as we try to ensure learning, to ensure proceptual development, to ensure the growth of concept images, I suspect we will fail with a significant proportion of the population: cause and effect is not a suitable mechanism for analysing and accounting for educational progress. I suspect however that the notions of procepts and concept images will continue to be an integral part of the awareness of mathematics teachers and researchers for a long time to come.

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Learning and Teaching Infinities: A never-ending story

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ABSTRACT: *We start this paper by providing a brief account of some of Tall's contribution to the study of the learning and teaching of infinities. Then, we describe an interview that Tall conducted with Nic, his youngest son, when he was seven years and one month. The interview reveals the thinking of a brilliant child about infinity, for instance, his grasp of the idea that aleph plus aleph equals aleph. We then describe another interview that we conducted with a bright seven year old girl, where she expressed her view that while infinity is the largest number, infinity plus one is larger than infinity. We conclude by discussing the contribution of such interviews to the learning and teaching of infinities.*

Key words: *Children, Formal infinities, Infinity, Learning, Natural infiniries, Teaching.*

In 2001, a special issue of *Educational Studies in Mathematics* was dedicated to the study of the learning and teaching of infinity. This issue was entitled: *Infinity – The never ending struggle*. In a way, this volume, which was edited by Tall and Tirosh, reflected a long standing shared interest of members of the mathematics educators at the university of Warwick (e.g., Tall, Monaghan) and several members of the mathematics education group at Tel Aviv University (e.g., Fischbein, Tirosh, Tsamir, and Dreyfus) in infinity. The volume included papers by Tall, Monaghan, Fischbein and Tsamir, as well as papers by Kleiner, Jahnke and Mamona-Downs.

In his paper Tall coined the terms *natural infinities* and *formal infinities*. He distinguished between "natural concepts of infinity, developed from experiences in the finite world, and formal concepts of infinity, built from formal definitions and deductions" (Tall, 2001a, pp. 235).

Tall explains that

> *By beginning with different properties of finite numbers, such as counting, ordering or arithmetic, different formal systems may be developed. Counting and ordering lead to cardinal and ordinal number theory and the properties of*

> *arithmetic lead to ordered fields that may contain infinite and infinitesimal quantities… while natural concepts of infinity may contain built-in contradictions, there are several different kinds of formal infinity, each with its own coherent properties, yet each system having properties that differ from the others.*
>
> (Tall, 2001a, pp. 199)

Tall further described different types of formal infinites, showing that a formal theory can lead to a *structure theorem*, and concluded:

> *The moral of this tale is that there is not a single concept of infinity, but several distinct ones, and that it is not appropriate to assign to any one of these concepts the title of 'the' concept of infinity. In particular, it is not appropriate to consider infinite cardinal number theory (with its lack of subtraction and division) as the only mathematical infinity. There are other infinite concepts, including the intuitive infinities and infinitesimals in the calculus, that can be given formal theories of their own, consistent in themselves and yet different from the notion of infinite cardinal.*
>
> (Tall, 2001a, pp. 236)

Indeed, in his extensive work, Tall explored a remarkable spectrum of infinities-related issues (e.g., Tall & Schwarzenberger, 1978; Tall, 1980a, 1980b, 1981a, 1981b, 1990, 1992, 1993, 2001a, 2001b, 2001c). He addressed, for instance, the learning and teaching of limits, cardinal infinity and ordinal infinity, and the notion of measuring infinity.

Our work addressed mainly the notion of cardinal infinity. We focused on the comparison of infinite sets, being attracted by the deep contradiction between the intuitive and the formal infinity. That is, between our intuitive reasoning, based on practical, real life experiences and the formal theorems. For example, we assumed that propositions like *the whole is equivalent to one of its proper parts* or $\aleph_0 = \aleph_0+1$ or $2\times\aleph_0 = \aleph_0$ may contradict our usual mental schemes, and consequently, provide a fruitful setting to examine pivot issues related to the learning and teaching of mathematics.

In the following sections we briefly describe two infinity-related interviews that touched upon the above mentioned surprising characteristics of cardinal infinity. The first interview was conducted by Tall (2001b), the second by Tirosh and Tsamir (Unpublished).

DAVID AND NIC TALL: TALKING ABOUT INFINITY

Tall opened his paper on *a child thinking about infinity*, by stating that: "young children's thinking about infinity can be fascinating stories of extrapolation and imagination. To capture the development of an individual's thinking requires being in the right place at the right time" (Tall, 2001b, pp. 7).

The paper, then, described several episodes in which David and Nic (his youngest son) talked about infinity. At the age of seven years and one month, Nic came to David with a comment that took him (David) "totally by surprise".

> Nic: I've invented a number bigger than infinity.

The result of the comment was an outstanding manuscript published in the *Journal of Mathematical Behavior* where David described the development of Nic's thinking about infinity. Nic started by treating infinity as a very large number.

> David: What is infinity?
>
> Nic: A very, very high number.
>
> David: How high is it?
>
> …
>
> David: What about a googol times a googol, is it bigger than that?"
>
> Nic [thinks briefly]: Well, … I think that equals just about infinity.
>
> …
>
> David: I see … Is a billion, billion, billion, billion, infinity?
>
> …
>
> Nic: Well, I think it just about is.

At that point David challenged Nic in an attempt to explore whether Nic believed infinity to be the largest number, discovering that (in the first episode), for Nic, infinity and one was larger than infinity.

> David: ... Do you know any numbers bigger than infinity?
>
> Nic [firmly]: Infinity and one.
>
> David: How much bigger than infinity is that?
>
> Nic [firmly, with a questioning tone]: One.
>
> …
>
> David: Tell me another number bigger than infinity.
>
> Nic [firmly]: Infinity and two.

This led David and Nic to discussing operations with infinity. David found out that Nic regarded infinity as a large number that can be added, subtracted and multiplied like any other number. Nic said, for example,

> Nic [with great conviction and force]: …if you take away infinity from infinity, you're left up with nothing, but if you take away two infinity from infinity, you get minus infinity.

At the end of the interview, after David challenged Nic's conceptions regarding the arithmetic of infinite numbers, and introducing Nic to $\aleph$, Nic concluded that $2\times\aleph = \aleph$.

David: … How many whole numbers are there?

Nic: Aleph.

David: Aleph. That's right! Well, how many even numbers are there then?

Nic: Aleph?

David: … and how many odd numbers are there?

Nic: Aleph.

[These questions were repeated to confirm the ideas.]

David: So what happens if we add aleph plus aleph, what's the answer?

Nic [immediately]: Aleph!

David [Feigning amazement]: Aleph plus aleph equals aleph? Why is that?

Nic: Well it's just … like so! [With sudden conviction]: Like nothing add nothing equals nothing!

…

David: So … what happens if I have two aleph, is that the same as aleph?

Nic: Yes!

This interview was conducted by a special interviewer (mathematician, math educator, and father) with a brilliant child. In his reflection on these interviews, Tall raised the question: "Did I do right to press on with infinite correspondences with a seven-year old?" He admitted: "I don't know". This issue is still open to research and discussion.

DINA TIROSH, PESSIA TSAMIR AND NOA: COMPARING INFINITE SETS

Our conversation with Noa, a seven year old Israeli girl, illustrates the fascinating nature of young children's ideas about infinity. Noa was identified as highly talented in mathematics. She was also very talkative and extremely communicative, volunteering her thoughts, hesitations and ideas in a very open manner. Noa knew that we were interested in infinity, and one day as we were working together and Noa was around, she approached us saying enthusiastically:

Noa: I can show you infinity.

Before we had a chance to react to her statement, she took a piece of paper and wrote: 100000000000000 (one and fourteen zeros, so that the entire row was filled with numerals and the last zero was somewhat squeezed to the edge. No room was left for another digit).

Then, Noa looked at us with great satisfaction and said:

Noa: That's infinity.

We realized that:

(1) The number "infinity" that Noa wrote was one, followed by 14 zeroes;

(2) The 14^{th} zero was squeezed into the row, touching the edge of the page. There was no room on that row, on the piece of paper, for another digit. It seemed that Noa attempted to completely fill an entire row.

This led to an unexpected conversation.

Dina: Yesterday, another child was here and wrote: 1000000000000000 (one followed by 15 zeroes, to complete an entire row). He said that this is infinity…

Noa [counting the zeroes and interrupting]: NO! This is not infinity! Infinity is one and fourteen zeroes.

Dina: So… what is this (one and fifteen zeroes)?

Noa: This is not a number… one and fourteen zeroes is infinity… the greatest number… the greatest! So…this (points to the one and fifteen zeroes) can not be a number…

At that point Noa went to her room. It seemed that for Noa, infinity was a number that could be presented in a finite manner with a fixed number of digits. She grasped infinity as the largest number, and so any representation that included more digits could not be a number.

After some time Noa returned to us, asked for her paper, and wrote below her previous item (100000000000000, i.e. one with fourteen zeroes), another number: 1000000000000000000, i.e., one with eighteen zeroes, in smaller digits, still completing the row:

Noa: You see? One and (counts) eighteen zeroes is also the number infinity…

Pessia: Why?

Noa [writes 100000000000000 and again 100000000000000 below it, and then, below both numbers she writes 200000000000000.] She said:

Noa: Look, if I write it like that, I get 2 infinities. But infinity is the biggest number…. So it's (pointing at 200000000000000)… infinity…

Pessia: So what is infinity plus infinity?

Noa [laughing]: Infinity!

Pessia: ...and… infinity plus infinity plus infinity plus infinity plus infinity?

Noa [laughing]: Also infinity!

Dina: And this (writing one with twenty seven zeroes)?

> Noa [confidently]: Also infinity! From 1 and fourteen zeroes all are infinity… infinity is the biggest number… nothing is bigger than infinity…

At that point Noa seemed to believe that:

(1) Infinity is a number;

(2) The number 100000000000000 (one with fourteen zeroes) is infinity;

(3) Any number beyond the number 100000000000000 (infinity) is infinity;

(4) Two times infinity and five times infinity are infinity (perhaps any number times infinity is infinity).

(5) There is nothing larger than infinity;

In an attempt to further explore Noa's ideas of infinity, we asked:

> Dina: What about infinity plus one?
>
> Noa [hesitating]: Infinity and one?...
>
> Pessia: What about it? Is it larger than (writing): 1000000000000000 (one and fourteen zeroes)?
>
> Noa: Yes. It's infinity and one.
>
> Dina: So???
>
> Noa: It's one more than infinity…

Interestingly, Noa kept thinking that infinity is *a number*, and that it can be presented by one with fourteen zeroes. These two ideas resisted various manipulations, suggested by us or by her. However, while "infinity times two" and "infinity plus infinity plus infinity plus infinity plus infinity" was grasped as "the same infinity", a seemingly simple manipulation (adding one) challenged her belief that this "infinity number" (100000000000000), is *the largest* number, and that no number is larger than the "infinity number". She regarded infinity plus one as a number, larger than the "infinity number".

SOME CONCLUDING REMARKS

What can be learnt from such conversations? What is the relevance of conversations with young, bright children to the learning and teaching of mathematics?

In both cases, at certain points of the interview the interviewees regarded (1) two times infinity as infinity, and (2) infinity and one as larger than infinity. Yet, these conclusions might reflect different reasoning. These seemingly small, yet interesting data guided us in our attempts to explore students', prospective teachers' and teachers' conceptions of cardinal infinity, and to assist them in constructing related secondary intuitions (e.g., Fischbein, 1987; Tsamir & Tirosh, 1999; Tsamir, 2001).

For instance a task that we used in our attempts to present ideas related to cardinal infinity, to prospective teachers and teachers is provided in figure 1:

$$\begin{array}{cccccc} \star & + & \star & = & \star \\ 2 & \times & \star & = & \star \\ \star & + & 1 & = & \star \end{array}$$

Figure 1. Task for prospective teachers.

Till now, all participants, in various groups responded that there is a solution to the first two items (i.e., zero), but no solution for the last item.

This leads us to a discussion of extensions of number systems (e.g., from positive natural numbers to non-negative natural numbers). The discussion of cardinal infinite numbers is then presented as an additional extension of the number system (à la Cantor).

Clearly, more research on the learning and teaching of infinities is needed. We are sure that David Tall will continue to play a major role in advancing this domain of research as well as other domains of advanced mathematics thinking.

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Procepts and Property-Based Thinking; to what extent can the two co-exist?

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ABSTRACT: *The paper examines the relationship between procept and property-based thinking. At the first sight, property-based thinking is different to thinking in terms of procepts, as the former assumes an a-priori status of mathematical objects, whereas for the procept the identity of the objects or conceptual input is negotiated through processes, and vice-versa. The paper will argue through some examples that the two can be reconciled to some degree, and made to act productively in tandem. The illustrations concern the limit of a real sequence, the Fundamental Theorem of Calculus and the prime decomposition of positive integers.*

Key words: *Procept, property-based thinking, modes of symbolism, role of definition, reformulation.*

INTRODUCTION

The seminal paper by Gray and Tall (1994) introduced the notion of procept. The term was conceived as a convenient word to employ when discussing the relationship between processes and concepts in mathematics in cognitive terms. The main feature to their framework is the role of symbolism that can evoke both a process and a concept. The concept is considered as the conceptual output of the associated process. For example, the symbol

$\Sigma\, a_i \quad i = 1,\ldots, n$ where n is a natural number

suggests both a process of addition and a completed sum concerning the integers a_i. (Many more examples are given in the literature, especially in Tall, Gray, Bin Ali, Crowley, De Marais, Mc Gowen, Pitta, Pinto, Thomas, & Yusof, 2000). The sum of course can be regarded simply as a particular real number, but the conceptual weight was on how the result was obtained from the process. The conceptual side can induce the student to negotiate certain ways to act on the process without affecting the outcome (ibid), so that the process can be thought of as a collection of different procedures. For example

$\Sigma\, (-1)^i \quad i = 1,\ldots, n$ where n is a natural number can be re-formulated as the summation of sums of pairs of two successive terms; these would cancel if n is even, or leave the result -1 if n is odd. Focusing on the symbolism would seem to encourage this processing. This illustrates that thinking about the kinds of actions induced by the conception, allied with symbolism, can be extremely powerful. Because of this, the notion of procept has very wide application, and it encourages flexibility in thinking. The symbolism is the key to how the student can compress the conceptual complexity that is implicit in the given situation. If the students cannot use symbolism as an effective channel to make a synthesis between their use of process and conceptual understanding then they will tend to perform less well than those who can. This phenomenon is referred to as the 'proceptual divide' (Gray, Pitta, Pinto, & Tall, 1999).

However, in the work of Tall and Gray it is acknowledged that procepts are not adequate to account for all conceptualization that happens in the doing of mathematics. Tall in particular has built a more general framework that he calls the "The Three Worlds of Mathematics" where thinking in terms of procepts constitutes the second 'world' (Tall, 2004). The first world is related to concept acquisition from direct perception of physical or geometric systems and to its interpretation. The third is called the "formal-axiomatic" world: here properties determine objects rather than the vice-versa. The resultant mathematical activity would tend to promote an investigation to determine or classify the objects satisfying the axioms, and in order to advance this, one is likely to seek out new properties from the ones given (ibid.). This third world has a strong claim to be taken as a platform to characterize 'advanced mathematical thinking', as suggested in Tall (1991). (One has to say here, though, that there have been many models put forward by the literature concerning 'advanced mathematical thinking' that are not particularly consistent with each other. For a recent example, see the collection of papers devoted to this theme in *Mathematical Thinking and Learning*).

As mathematics becomes more abstract, extracting meaning remains crucial not only in forming mathematical argument, but in deciding what are the mathematical issues that should be examined (Dubinsky, 2000). Thus, most educators agree that all mathematics is embodied, and this opinion is often stressed in Tall's work. Some confusion can arise about the cognitive issue of 'objects' in this respect, as the word object might suggest that exterior sources dictate our lines of mathematical thinking. However, the notion of (mental) object is crucial psychologically, as the actions that have to be performed in mathematics presuppose the idea that 'things' have to be operated on (e.g., Asiala, Brown, Devries, Dubinsky, Matthews, & Thomas, 1996), and there is no real clash here with embodied thought. But the status of object remains rather tenuous; for example, in both procept and property-based thinking, the 'entity' that is highlighted in the mathematical activity sometimes seems more appropriately termed an 'object', a 'concept' or perhaps even a 'system'.

There are other theories concerning conceptualization and the so-called 'objectification'. These tend to be (as indeed is the notion of procept) influenced by Piaget's well-known categorization of empirical, pseudo-empirical and reflective abstraction. The theoretical frameworks most often used in educational research on this issue is APOS (acronym for Action-Process-Object-Schema) espoused by Dubinsky and his colleagues (e.g., Asaila et al., 1996) and the notion of 'reification' mostly developed by Sfard (e.g., Sfard, 1994). There are many papers that include discussions comparing these frameworks (e.g., Gray et al., 1999). As these theories have become such a common currency in the literature, we feel that there is no point in summarizing them in this paper, apart from pointing out that the procept notion allows a more bi-directional stance between concept and process.

The aims of this paper will be to explore some issues concerning the interplay between procepts and property-based thinking. This is done by commenting on certain mathematical themes, discussed theoretically in the sense that we do not refer to specific fieldwork results. The issues that will be raised and illustrated are the following:

- Can a procept be unsound, but be made reliable through the intervention of property-based thinking?

- How symbolism can be made to change in response to a more abstract approach?

- Can there be cases where the 'result' of a process is understood as a property first, only determining conceptual weight by its utility after?

PARADIGM 1: THE CONCEPT OF LIMIT

Here we shall consider limits of real sequences. There is a 'process' already suggested in an infinite sequence: you take the first value, then the second and so on. One might consider whether in the long run the values 'settle down' to a particular value, and if so this value is regarded as the 'limit'. (Whether this issue occurs naturally or is raised in the context of teaching is in question.) Because of this, it has been observed in the literature that the limit is considered as the 'last entry' in the infinite sequence; sometimes students would denote it a_∞ if the sequence is denoted by (a_n). Other student behavior related to this belief is listed in Davis and Vinner (1986). The feeling that the limit is integrated with the sequence itself can be restrictive in cognitive terms; it can lead students to believe that the sequence acts to 'produce' a limit in certain circumstances. However, in the definition of limit, the situation is different in that the sequence is *acted on* when obtaining the limit. Given $\varepsilon>0$, one has to provide a positive integer N for which all $n \geq N$, a_n must satisfy a particular condition involving ε. This means that the concept of limit involves a second process beyond the one inherent in the sequence itself. Also the definition explicitly involves the symbol l for the limit (for all n 'sufficiently' large, $|a_n - l| < \varepsilon$), so it cannot be straightforwardly interpreted as a result of the second process (because it is already 'there' whilst the process is enacted). This leads to a somewhat complicated case concerning how processes are related to objectification/conceptualization. Further, because the second process is implicit in the definition, the limit can be regarded also as a property-based concept. We shall discuss these two issues further.

Initial intuitions addressed to what the limit is (if it exists) will naturally focus on what is given; that is, a sequence. Either the intuition can be expressed by descriptions in colloquial language that is too vague for mathematical modeling, or it is possible to consider the difference of sequence values in the 'long run'. (The latter can lead students to formulate some properties that would be consistent with the definition of limit even if the student was ignorant of the definition; the most sophisticated property would be the condition characterizing Cauchy sequences, though this would be a demanding task.) However, both a process and some interpretation of what the process 'yields' is present in the student's mind, and it is likely that the students have been exposed to the symbolism, $\lim_{n\to\infty} a_n$. Hence we have all the ingredients for a procept, yet it is not fully explicit and in some cases could even be dysfunctional. How can this situation be retrieved?

The crucial step, mentally, is to shift your attention primarily to the object, the limit, then subsequently to how the given sequence 'fits in' with this. Psychologically, this shift is very important. (For the topic of shifts of attention, see for example Mason, 1989). From a stance that regards a limit as being produced by some *a priori* process somehow integral to the sequence, we turn everything on its head and we say that the limit is just a number associated with the sequence through a process that is itself designed to delineate what a limit is. For the latter viewpoint, the underlying function between convergent sequences and real numbers ($(a_n) \to \lim a_n$) comes to the surface more; in this respect, the uniqueness of limits can be raised as an issue, whereas it simply would be assumed otherwise. However, the (ε, N)-process that we construct is complicated and has to be expressed with great care. This suggests that the precision that it requires has to be presented formally, and so the notion of limit needs a formal definition. At this stage, one formulates a suitable property on which the definition is based. A limit l, then, is determined by a certain property that the original sequence must satisfy. The exact form of the property has to be guided by first intuitions, by the reformulation of these (probably involving the resolution of epistemological conflicts), and by taking the primary status of a limit as a real number. (Note that there is no absolute basis to choosing the property taken, so the definition to some degree still depends on conventional decree.) Here we see that the consonance between intuition and the forming of the definition is a prerequisite for devising the (ε, N)-process. This process with the enriched understanding of how a limit l should be regarded could be thought of as a *grounded* procept, where property-based thinking has intervened.

Now the extracting of a process from the mathematical form of the definition may well require a 'new reading'. For example, the definition has a quantifier 'for all ε', whereas one might expect for the process to consider a particular ε, then (for example) $\varepsilon/2$, and so on, and this indeed is sufficient to check for the limit. Such differences of the definition with a natural process have been expounded in Mamona-Downs (2001). A study by Pinto and Tall (2002) features a case study that chronicles the stages that a particular mathematics undergraduate goes through in order to understand the concept of limit of a sequence. At the end of the study, he is able to state (without intermediate 'working') an acceptably rigorous statement about what it means for a sequence *not* to have a limit, a task with which he had no previous experience. Most of the other participants failed in this task. The successful student showed himself particularly able to reconcile the process and property aspects in the definition and it is likely that because he could use his sense of property, he understood also what it meant to contravene that property. The more general point illustrated here is that property-based thinking can comprise a conceptual core that can be allied with processes, in which case property-based thinking and procepts are closely related.

In the discussion above, we have referred to procepts but have largely left out the role of the symbolism that supports them. The situation here is interesting. The symbol $\lim_{n\to\infty} a_n$ suggests a process directly focused on the given sequence, the stark assignation of a letter, usually l, for the assumed limit encourages a formal and property-based frame of mind, whereas the symbolism $a_n \to l$ as $n \to \infty$ could be thought as a composite of the other two. (The latter can be effective if its reading is consistent with the enhanced procept but not otherwise). This theme of symbolism taking different forms for procept and abstract modes of thinking will be taken up further in the next section.

PARADIGM 2: INVOKING THE FUNDAMENTAL THEOREM OF CALCULUS

The definite integral is defined as a limit of Riemann sums. In being a limit, much of what we said for the first paradigm can be paralleled. However, we take a different tack, and consider the significance of the notion of procept to the so-called Fundamental Theorem of Calculus. We will not examine the cognitive difficulties for students in understanding the 'fabric' of the proof itself, though this theme seems rather under-represented in the AMT literature. Instead, we use it as an illustration of when a concept (the definite integral) is motivated by a process, yet its channel of calculation is through another concept (the anti-derivative). There is no essential problem in linking the limiting process of Riemann sums with the idea of 'area under the graph' of the given real (continuous) function between the given limits as a procept. However, the procept might be finally regarded as being incidental (functionally anyway) when a new process that produces explicit answers is introduced. This new process demands ways of determining which functions, when differentiated, give the original. It is difficult to consider such a program as having a conceptual character, but it fits in well with a property perspective. In effect, integration is linked with a process that is not directly in tandem with its *a priori* conceptualization, and so what started as a procept does not remain one, because of the re-structuring of the cognition due to the Fundamental Theorem. Very quickly, students relate the symbolism $\int_a^b f(x)dx$ with the task of finding the anti-derivative and applying the limits of integration. However, this symbolism is clearly motivated by the original definition, not by its formal equivalence, the anti-derivative. It is interesting how the retention of this symbolism can cause problems to students in proving more or less trivial identities of integrals, such as

$$\int_{a+c}^{b+c} f(x-c)dx = \int_a^b f(x)dx$$

$$\int_a^b g'(x)f(g(x))dx = \int_{g(a)}^{g(b)} f(x)dx$$

The task of validating the first identity is included in a list of problems for which it is claimed that the 'average' calculus student would fail to answer satisfactorily (Eisenberg, 1992), even though the identity is open to informal argumentation using visualization. Both identities are easily seen if you use an alternative form of symbolism, i.e., instead of $\int_a^b f(x)\, \mathrm{d}x$ we write $[F(x)]_a^b$ where F is understood as a differentiable function whose derivative is equal as f:

$$[F(x-c)]_{a+c}^{b+c} = F(b) - F(a) = [F(x)]_a^b$$

$$\int_{g(a)}^{g(b)} f(x)dx = [F(x)]_{g(a)}^{g(b)} = F(g(b)) - F(g(a)) = [F(g(x))]_a^b = \int_a^b (F \circ g)'(x)dx = \int_a^b g'(x)f(g(x))dx$$

The notation $[F(x)]_a^b$ stresses (at least) two aspects involved in obtaining the proofs. First, it emphasizes that an integral has a functional basis, where its value is determined by substituting the limits in the usual way. The rather fabricated symbolism $\int_a^b f(x)dx$ that reflects the limiting process underlying the concept of integration seems to interfere with handling this aspect; in particular the first proof is simply a special case of a property that a function has generally. Second, introducing the symbol F is done without any expectation to determine what it represents. The idea is we know that something, F, exists with some characterizing properties, and once F is introduced, its properties are exploited. After this, F as a symbol becomes redundant, but through its agency we have produced a new expression in the givens, thus obtaining an identity. Though we have argued that the role of the anti-derivative does not fit the idea of a procept, we do see something analogous happening here. The symbolism F represents both a result of definite integration and a 'process' of finding functions whose derivative is equal to f. The process involved, though, does not directly reflect the original concept. Even the word 'process' seems less appropriate, as the anti-derivative seems better characterized as a device to obtain an answer in a problem-solving context. Perhaps the way in which this situation most differs from a procept is that for a procept there is an assumption that the conceptual basis (of the objects involved) is to be determined explicitly through the process that produces the specific result. This is not the case in our example, where a device is used to address an issue motivated elsewhere. Such a difference might well affect students' cognition.

PARADIGM 3: THE PRIME DECOMPOSITION OF A POSITIVE INTEGER

This theme brings out the relativity between process and property well. Certainly, there is a process in determining the decomposition, and there is a property because there is a theorem that says that any natural number has a (unique) prime decomposition. But is it appropriate to call prime decomposition a procept? In this issue, the question of whether the property qualifies as a concept is central. At the first sight the case seems poor; a 'literal' reading is that the decomposition gives information about the highest power of a prime that divides the given natural number *n*. Such information would seem 'neutral' in conceptual terms. However, when the decomposition is read in another way, i.e., it characterizes the divisors of *n*, its conceptual significance is suddenly enhanced. Some intriguing results can be formulated very easily, such as that the square integers are characterized by their having an odd number of divisors. In the framework of a procept, one has a process, pivotal symbolism and conceptual *output*. In the present case, the first and third exist and there is standard symbolism

$$n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$$

which can raise suggestive linkage between them, hence finally there is a good case to regard prime decomposition indeed as a procept. However, there is a need for a certain mental reformulation for students to acquire the procept; the process of finding the highest powers of different primes that divide *n* is not necessarily recognized immediately in the symbolism representing the prime decomposition of *n*. Because of this, some students may not integrate the process and conceptual factors well, resulting in a proceptual divide (for this topic, in any case). Such a need for reformulation via structural considerations means that some procepts will require considerable reflection to obtain the requisite new interpretations for the procept to be complete and effective.

CONCLUSION

The notion of procept is a very wide and useful tool to describe how much of the doing of mathematics is influenced by associating processes with consonant conceptual output via symbolism. However, quite often meaning has to be extracted from a formal definition, where no *a priori* sense of process accompanies the entities satisfying the definition. This difference can be regarded as a candidate for characterizing advanced mathematics thinking. Nonetheless, in this paper we illustrate several ways in which procept and property-based thinking can be reconciled, as in our first and third example. The second example, though, illustrates how easily the consonance and directness that is expected between the process and concept aspects of a procept can be disturbed in the face of a property-based reformulation. All the examples presented in this paper have commonalties in the sense that there was some reformulation of the original conception towards a more workable or utilitarian meaning. In the first example, a procept had to be negotiated from realizing the weaknesses of another procept. The second and third both involved influences from utility: the difference was that in the third, the aspect of utility was open to spontaneous re-interpretation, whereas the second was not, explaining why the third was regarded as a procept, the second not. The importance of symbolism has always been stressed in the theory of procept, but our examples illustrate how important it is to adopt different forms in the face of switching between procepts and property-based thinking.

Students passing from the secondary to the tertiary level of mathematics learning often sense a vast difference in the 'mode' of argumentation, that is both abrupt and difficult to cope with. This phenomenon is perhaps the most fundamental issue concerning the branch of mathematics education commonly known as 'advanced mathematical thinking'. The argument that procept and property-based thinking interact naturally in some cases, but not in others, may help in deciding what material should be presented to students to alleviate the ‘transition’ between school and university mathematics.

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The Importance of Compression in Children's Learning of Mathematics and Teacher's Learning to Teach Mathematics

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ABSTRACT**:** *This paper examines the pragmatic and theoretical basis of the Connecting Mathematics for Elementary Teachers (CMET) Project. In particular we attempt to answer the following question: "Why should we connect research on how children learn mathematics with preservice teachers' learning of mathematics"? We answered this question easily at a pragmatic level but found that the notion of compression (i.e., for complex understanding to occur one must compress previous ideas into compact, more precise mathematical objects) was invaluable in justifying our theoretical basis. Using compression as a lens provided preservice teachers with insights into how children learn mathematics and also helped them with their own learning of mathematics. Focusing on compression also enhanced preservice teachers' motivation to learn mathematics. Our analysis indicated that the CMET approach, explaining how mathematical knowledge is compressed, was essential because some knowledge is not 'decompressable' without examining research on children's learning of mathematics.*

Key words: *Compression, Mathematical Knowledge, Mathematical Thinking, Mathematics Content Courses, Preservice Teacher Education, Teacher Education, Teacher Learning.*

The Connecting Mathematics for Elementary Teachers (CMET) Project attempts to help preservice elementary teachers connect the mathematics they are learning in content courses with how children learn and think about mathematics; to tie research on children's learning of mathematics with practice. To this end we have developed a supplement that parallels the typical mathematics content course topics. Our intent in helping preservice teachers make these types of connections is that they will both improve their own understanding of mathematics and eventually improve their future teaching of mathematics to children.

The CMET materials primarily consist of descriptions, written for prospective elementary teachers, on how children think about, misunderstand, and come to understand mathematics. These descriptions are based on current research. Some of the connections include: how children come to know number, addition as a counting activity (Gray & Tall, 1994), how manipulatives may 'embody' mathematical activity (Tall, 2004), and how concept image (Tall & Vinner, 1981) relates to understanding in geometry. For example, we discuss how linking cubes may embody the concept ten in understanding place value and at a more sophisticated level of mathematical thinking Base Ten Blocks (Dienes Blocks) may be a better embodiment of the standard algorithms for addition and subtraction.

In addition to these descriptions the CMET materials contain:

- problems and data from the Third International Mathematics and Science Study (TIMSS) and the National Assessment of Educational Progress (NAEP)
- our own data from problems given to elementary school children
- questions for discussion by preservice teachers

It is important to note that the CMET supplement is not a methods textbook. It is designed as a supplement for traditional mathematical content courses for elementary teachers; hence it is not a mathematics textbook either.

CMET has also undergone extension evaluation with pre/post, treatment control using both qualitative and quantitative methods. Surveys and interviews were conducted to determine preservice teachers' beliefs about children's learning, their self-efficacy, and knowledge of how children learn and understand mathematics. One aspect of the analysis focused on preservice teachers' knowledge of the compression of mathematical ideas in the learning of mathematics. Specifically the analysis focused differences between preservice teachers using CMET and those that did not in their decompression of the mathematical knowledge typically taught to children in elementary school.

WHY CONNECT LEARNING

In analyzing both the theoretical and practical basis for our project, a significant question became: "Why should connections to research on how children learn mathematics be useful and necessary in preservice teachers' learning of mathematics?" This question can be addressed on several different levels. At the most basic and obvious level, making connections with how children learn mathematics helps preservice teachers see the usefulness of the mathematics they are learning and how they will apply and use this mathematics in their future teaching. As Bransford, Brown, and Cocking (1999) suggest, learners are more motivated to learn and develop richer understandings when they see the contexts of applicability. This explanation is one that both teacher educators and students might give for the benefits of such an approach. Indeed, in our preliminary evaluations, students did provide similar reasons for why they thought CMET materials were useful. Many indicated that they were learning how children think and were reflecting on their future teaching. A few even mentioned that they had used the ideas presented in the supplement in opportunities they had to work with children. Hence, from a pragmatic and motivational perspective this approach was beneficial.

Another goal of the CMET project was for preservice elementary teachers to improve their mathematical understanding. We maintain that by understanding how children understand mathematics, prospective teachers will enhance their own understanding of mathematics. Preservice teachers often have an instrumental (Skemp, 1987) view of mathematics because the mathematics prospective elementary teachers learn in content courses is disconnected from what they will be teaching. Typically they have only their own, often negative experiences of learning mathematics, to relate to the mathematics they are learning. As a consequence, mathematics learning becomes a disassociated group of facts and procedures without meaning; mathematics is not learned as a sense-making activity. In post surveys some students indicated that the CMET materials helped them understand mathematics better. Making these connections then also helps preservice teachers understand mathematics relationally (Skemp, 1987). This understanding should likewise help them in their future teaching of mathematics to children.

Yet these practical justifications do not fully explain why such an approach is necessary. These justifications suggest that this approach will improve preservice teachers' future teaching and perhaps their motivation to learn mathematics. But we further maintain that these connections will help prospective teachers in their development of mathematical understanding and, more particularly, help them achieve the mathematical knowledge that is necessary for teaching that goes beyond basic mathematical content knowledge (Hill & Ball, 2004).

COMPRESSION

A deeper analysis is necessary to address our overarching question as to why this type of approach is viable. A theoretical justification for the CMET project, and the primary focus of this paper, is the notion of *compression* (Tall, 2004; Thurston, 1990). Compression is the idea that in order for complex mathematical thinking to occur one must compress previous ideas into compact, more precise mathematical objects. The hugely complex organ, the brain, can only function by focusing on a few essential aspects of any given situation at a time to make decisions. Consequently, other aspects and information must be suppressed. In order for the brain to convert information into manageable pieces, it must *compress* that information into a useable form. In order to achieve this feat, it is helpful for the mind to compress mathematical processes so that they can be thought of as mental mathematical objects. The mind can then operate with and on the object without recreating the process each time.

In order to understand the concept of compression, it is useful to consider David Tall's work on how children learn to count and to do arithmetic. Tall writes:

> [W]e need to look at how we human beings develop from a single cell and by successive cell-division which depends on our genetic structure, the embryo develops into a complex individual. As the cells divide, they specialize according to the genetic code where this cell is part of a finger-nail and that cell is a neuron in the brain. The child is born with a huge number of connections between the various parts of the brain linked to the rest of our body. There are myriads of connections between the brain and the eye, the brain and the ear, the brain and the hand, which require organising by experience. This making of connections in the brain is how the child learns. As the child moves its arm around, its hand may touch an object it sees and over time this action is remembered by strengthening the connections in the brain and the child begins to build up a sequence of actions known as 'see'-'grasp'-'suck' whereby it coordinates sight and touch to grasp an object to bring it to its mouth to taste.
>
> Arithmetic develops in exactly the same way. The parent talks to the child and sings nursery rhymes such as 'One, two, three, four five, once I caught a fish alive' and other activities which involve the first words of the number sequence; these lead to the learning of a sequence of numbers and the act of pointing successively to objects in a collection, one at a time, saying the numbers as we go, to build up the sequential organisation of the counting scheme. The child may count several toys 'one, two, three' and, no matter which order they are counted in (provided each one is counted once and once only), the count always ends with 'three'. The realisation that the number we reach when counting a set is always the same is called 'conservation of number'. It is technical language to say that the number found from counting a particular set is always the same. The focus of attention changes from the action of changing to the concept of number.
>
> The learning of the counting sequence and the conservation of number take many months, perhaps years to learn. It brings the first example of what Thurston (1990) calls 'compression'. In this case the counting schema, including all the pointing and saying of the sequence of numbers, is compressed into a single concept, the concept of *number*. While counting occurs in time and we have to *do* it, a number can be represented by a single number word and we can *think about* it. This allows the child to begin to use numbers themselves as thinkable things to operate with and to do arithmetic. However, before going on to do arithmetic, the child has a range of increasingly subtle ways of counting to learn to be able to do the operations more efficiently and more powerfully.

This notion of compression can also be applied at a more advanced mathematical level by considering the natural progression in the ways that children solve 4 + 5. At first children will need actual objects to count such as pencils.

The child initially counts each pencil in the first group, *touching* (and later in the child's development, pointing at) each pencil as he or she counts, then the child counts the second group in a similar manner, and then to the amazement of most adults combines the two groups and counts the total, again touching each pencil as the child counts. At a later stage the child may recognize that there are 4 pencils in the first group and simply 'count on' from four saying, "5, 6, 7, 8, 9". Here the child has *compressed* the concept of number rather than having to go through the procedure to count four objects in order to know that there are four; the child has compressed the counting procedure into a single number concept. The child no longer needs to count each group to know how many are in it, he or she can focus on the aspect of the number without having to make it.

By building on experience, the child develops more sophisticated and more compressed methods of doing arithmetic – the more compressed, the more powerful the technique. Later in the child's development when asked what is 4 + 5, he or she may respond, "4 + 4 = 8 and since 5 is one more than 4, 4 + 5 = 9." Similarly, using the knowledge that three and two makes five may lead the child to the idea that twenty-three and two makes twenty-five. Finally, the child, like the preservice teacher, instantaneously knows that 4 + 5 is 9. When asked why, he or she says, "I just know it" or "I have it memorized." When a learner has this kind of "instantaneous" mathematical knowledge, we call that knowledge "derived facts." What is interesting about derived facts is that they often represent multiple levels of compression.

In the last two examples, the child has compressed the process of counting to find a sum to a "mathematical fact." The child will eventually compress counting and known facts into standard algorithms. As a final example of compression, consider the ways that a child learns multiplication. To solve 3 x 6, a child will initially think of multiplication as repeated addition and that 3 x 6 means 6 + 6 + 6. When asked, the child might respond, "6 + 6 = 12" and then may count on: "13, 14, 15, 16, 17, 18." In the example the child uses his compressed knowledge (e.g., number, derived facts) to make sense of multiplication. The average child will eventually compress the notion of multiplication and continue the process of using compressed knowledge with more sophisticated mathematical thinking such as division. Layer upon layer of compressed knowledge is constructed.

Compression is a lens through which to view mathematical learning. In our case, the CMET Project, is looking at the learning of mathematics after the fact. We are not trying to help children learn mathematics but rather describe how they learned mathematics. We are attempting to help preservice teachers by looking at how children, and they themselves, have compressed mathematical knowledge.

DECOMPRESSING MATHEMATICAL KNOWLEDGE

Our intent as mathematics educators is that preservice teachers gain a rich mastery and understanding of mathematics in an efficient, compressed format. This level of knowledge gives the teacher a powerful way for *them* to think about mathematics. It is often what is termed basic content knowledge. While content knowledge is necessary for teaching, it is not sufficient. To encourage children to learn mathematics in a powerful way it is necessary to have insight into how children go through the process of compressing knowledge from carrying out mathematical actions to compressing them into fluent processes and even further into mental objects that they can manipulate flexibly in the mind.

Our project materials describe the natural processes that children may use in doing mathematics and how children develop more sophisticated mathematical thinking through compression. From our perspective teachers should be able to decompress, or "decompose and recompose at will" (Gray & Tall, 1994), their mathematical knowledge in order to teach children mathematics.

Elaborating on prior characterizations of compression, we make the following distinction: there are two types of compressed knowledge important for teachers. The first type is compressed knowledge that can be decompressed through reflective thought or by focusing on the learning process. For example, most adults, without too much thought, can explain that multiplication is repeated addition e.g., 3 x 6 is 6 + 6 + 6. However, the second type of compressed knowledge is so far buried in our learning that our conscious mind cannot recreate it. The only way to develop this knowledge is to study how children learn mathematics.

For instance, a closer analysis of how children might actually add to solve this problem reveals that how children might add is not how most adults add. A child who does not have his multiplication facts committed to memory might solve this problem this way, "I know that 6 + 6 = 12, and 12, 13, 14, 15, 16, 17, 18." Those who work with children on a regular basis will recognize this solution as a reasonable response. Significantly, this solution strategy is not readily apparent or decompressable to most adults. An adult that does not work with children on a regular basis when asked how a child might solve 3 x 6 may indicate that the child would add or even count, but he or she probably would not arrive at the solution strategy previously described. Perhaps somewhat surprising is the fact that, as children, we probably solved multiplication problems in the same manner as the described child, but most of us do not remember this process. We have compressed knowledge that we have forgotten how to decompress. As a final illustration of the ways that adults forget how to decompress knowledge, consider how many adults actually remember having to *touch* objects in order to count them.

EVIDENCE OF STUDENTS DECOMPRESSION OF MATHEMATICAL KNOWLEDGE

Evaluation of the CMET project is ongoing and considers multiple areas besides the focus of this paper, including: teacher self-efficacy, beliefs, and parents' use of these materials. The results presented here are from Likert questionnaires given to treatment and control groups. For the study, 168 Likert survey questions were developed to correspond with the mathematical content courses for elementary teachers. An analysis of the most popular textbooks for these courses was done to determine the content of CMET and the survey questions. The Likert questions were developed by two team members, and two different members suggested revisions and verified the reverse-worded questions. The control group consisted of 301 students who were given the survey the semester prior to the use of the CMET materials. The same survey was given to 249 students who used CMET materials. Instructors completed on-line surveys and students were interviewed. Because each university in the study organized their content courses differently, a separate questionnaire from the 168 beginning questions was developed for each course at each institution. The surveys ranged from 33 to 44 questions depending upon the content being taught in each course. The CMET materials were piloted at five sites, all in the Midwest region of the US. Control group data was obtained from three sites; all sites provided treatment data.

This paper only examines and compares treatment groups, which were taught by the two primary authors of the CMET materials, with control groups at a comparable institution. Both institutions are primarily commuter, regional campuses of Midwest nationally known universities. The campus student populations were 7,500 for the control group and 3,500 for the treatment group.

RESULTS

The following analysis compares the data from the treatment courses, which used CMET and focused on knowledge of children's mathematical thinking with control courses, which were taught without using the CMET materials or an emphasis on children mathematical thinking. Some significant differences in the beliefs and knowledge of the preservice teachers were found in comparing the control and treatment courses at a single site which did not include the CMET authors (Feikes, Pratt, & Hough, 2006). Using CMET for the typical mathematics content courses did make a difference in preservice teachers' beliefs and knowledge.

The Likert items possible responses ranged from Strongly Agree to Strongly Disagree, responses were given numerical values accordingly from 5 to 1. The **negatively worded questions are in bold**. Positively worded questions with higher scores up to 5 and negatively worded questions with scores closer to 1 are the closest to our theoretical position or an indication of an adherence to the beliefs and knowledge we believe are most important.

Significant differences between the control and treatment groups were evident on a wide range of questions. As might be expected since the focus was on how children learn and think about mathematics, preservice elementary teachers' scores were more in line with our theoretical expectations in knowledge and beliefs of Children's Mathematical Thinking. The following tables contain some of the questions that have significant differences between the control and treatment groups using a simple t-test with p-values all less than 0.01.

Table 1

Question	Control			Treatment		
	Mean	SD	(n)	Mean	SD	(n)
Initially addition is a counting activity for children.	4.07	0.741	(55)	4.66	0.479	(58)
Children who can count, say the numbers in order, understand the concept of number.	2.84	1.058	(56)	2.27	0.997	(59)
The concept of ten is the basis for place value.	3.79	.680	(56)	4.17	0.699	(59)

The significant differences reported here illustrate how preservice teachers' knowledge of how children understand mathematics is a decompressing or unpacking of their own mathematical knowledge. For example, these three questions indicate that CMET has helped preservice teachers gain better understandings of how addition is a counting activity for children and how the concept of ten is essential to understanding place value. We suggest that this knowledge is the result of decompressing their own mathematical knowledge to gain better understanding of how children think about mathematics.

Students also possess a greater knowledge of the methods that children use to successfully and not successfully solve problems.

Table 2

Question	Control			Treatment		
	Mean	SD	(n)	Mean	SD	(n)
Children are likely to cross multiply to solve ratio and proportion problems.	3.88	0.715	(56)	2.88	1.076	(24)

The CMET book explains in details the ways in which children solve ratio and proportion problems before they learn to cross multiply and pointing out that children initially perform less well on these problems when first taught cross multiply. Most high school-aged students simply cross multiply without considering these methods. Here preservice teachers demonstrated an understanding of children's capabilities which we maintain is a result of their decompression of how children initially solve ratio and proportion problems and their understanding that in children's first attempts they are applying a compressed method, cross multiplication, often without understanding. Decompressing the knowledge of how children solve ratio and proportion problems should help preservice teachers in their future teaching of mathematics.

Similar results were found for students' understanding of how children learn concepts in geometry where concept images are a compression of geometric thinking. CMET provided opportunities for preservice teachers to become aware of the notion of concept image and how children construct geometric concepts through the physical and mental manipulation of both physical and mental objects.

Table 3

Question	Control			Treatment		
	Mean	SD	(n)	Mean	SD	(n)
Children first understand shapes as a 'concept image', i.e., a shape is a rectangle because it looks like a door.	4.14	0.787	(37)	4.57	0.502	(37)
Children frequently look at the lengths of the rays or the distance between the arrows to determine which angle is larger or smaller.	3.70	0.661	(37)	4.35	0.633	(37)
Children best learn geometry by actively manipulating the physical world around them.	4.08	0.894	(37)	4.57	0.502	(37)
Mental imagery is essential to learning geometry.	3.78	0.947	(37)	4.43	0.728	(37)

The CMET supplement is helping preservice teachers understand how mathematical knowledge is compressed. In one sense the supplement asks these college students to relearn the mathematics they have already compressed and constructed. We find that the teaching of content mathematics to undergraduates is a conducive to introducing them to research on how children learn and think about mathematics. As they are studying undergraduate mathematical content, they are also learning how this knowledge is compressed by children and how they can decompress their own knowledge. Work like ours, whether with preservice or practicing teachers, is essential to participants' learning and development as teachers. Certain mathematical knowledge can be easily decompressed through activities that focus on how children learn. In many ways teachers probably do this intuitively in their teaching. Likely asking themselves, "How did I learn this mathematical concept?" or "How do I understand this concept?" However, there is mathematical knowledge that is not easily decompressed and there are subtle aspects of decompressable knowledge that are not readily apparent. Research on children's mathematical thinking provides an avenue for helping teachers understand how children learn mathematical knowledge that is not readily decompressable.

CONCLUSIONS

The concept of the compression of mathematical knowledge is one reason that connections to how children learn mathematics are *useful* in preservice teachers' learning of mathematics. At the pragmatic level, understanding how children think mathematically is useful because it is more motivating for preservice teachers in their own learning of mathematics. They may see how what they are learning is applicable to their future teaching. The notion of compression may also provide motivation for preservice teachers because they may gain a better understanding of mathematics as they come to understand how mathematical knowledge is compressed. They may see how mathematical knowledge is compressed by children and themselves. They are able to examine mathematics from children's perspectives, not just from a mathematical perspective, which is sometimes alien to them. Understanding how children think is essential to enhancing teaching. However, one cannot understand how children think without an awareness of how children compress mathematical knowledge.

It is important that prospective teachers gain both types of compressed knowledge – knowledge that can be decompressed through reflection and knowledge that cannot be decompressed without examining research on how children learn. The CMET curriculum provides tools to help preservice teachers reflect on their own learning and consider how child learn mathematics. CMET is essential in helping these preservice teachers gain both types of compressed knowledge as the project emphasizes self-reflection and presents research on children's learning of mathematics.

The process of unpacking compressed knowledge is an integral part of preservice teachers' development as teachers. Understanding what mathematics children compress and how they compress that mathematics provides powerful insights into the teaching and learning of mathematics. Not only do preservice teachers learn how children think but they also develop deeper mathematical understandings by looking at children's mathematical thinking and at compression in particular. This knowledge is different from mathematical content knowledge. Likewise it is different from pedagogical content knowledge. Yet, this knowledge is reflexively related to both mathematical and pedagogical knowledge and has a profound influence on both. In a broader sense we are attempting to incorporate a theoretical framework which uses research on children's learning of mathematics applied to teaching practices.

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Developing Versatility in Mathematical Thinking

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ABSTRACT: *This paper documents some of the historical steps in the development of the concept of procept. It then describes how these ideas have led to the construction of an emerging theoretical perspective on what constitutes versatile mathematical thinking and learning. Three parts of this theory are presented: process/object versatility; visuo/analytic versatility and representational versatility. Each of these aspects is illuminated by the use of data and examples gleaned from research projects in areas as diverse as algebra, calculus, linear algebra and statistics. It is hoped that it will be possible to infer how this developing conception of versatile mathematical thinking can be fostered and promoted in students.*

Key words: *APOS, advanced mathematical thinking, procept, representation, versatile, visualization.*

BACKGROUND

Way back in 1969, as a first year Warwick University undergraduate, I was taught a course called Foundations of Mathematics by a young lecturer called David Tall (see Figure 1). I never imagined then that our paths would later become so intertwined. Two years later I took a course on Mathematics Education that he was teaching, and this was to mark my first encounter with Piaget, problem solving (e.g., in two dimensions what shape has the greatest area that can be pushed around a right-angled corner of width 1 unit?), and knots, which have been an interest ever since. Following graduation I was very happy teaching mathematics in secondary schools for the next 10 years or so, until I felt a need to engage with ideas that would again challenge me.

Figure 1. A young Dr. Tall.

During a chance meeting at Warwick University my undergraduate tutor, Professor Rolph Schwarzenberger, suggested that I talk with David about some mathematics education research he was engaged in. Thus in 1983 I was accepted by David as a part-time master's student, while continuing to teach mathematics at Bablake School, Coventry. This heralded the beginning of my foray into mathematics education research, learning from someone who has displayed tremendous insight into many of the major issues involved in mathematical thinking and learning. This paper describes some of our joint work as well as developments arising from research that has built on those ideas.

HISTORICAL OBSERVATIONS—AN EMERGING CONCEPT

While studying at Warwick I was fortunate to be involved at the start of a period of great activity, including the construction of concepts that were to have a considerable international impact. One of these, of course, is the idea of a *procept* (Gray & Tall, 1991; Gray & Tall, 1994). In many ways I am uniquely placed to chart some historical details of this emerging concept. It must have been around 1986 that I stumbled across a response from one of the 14 year-old students in my doctoral research who made what I thought to be an interesting comment about a fraction. I followed this up in a questionnaire by asking students to explain whether $\frac{6}{7}$ is the same as 6÷7, or not. The actual question was:

> *a) A girl wrote the following in a Mathematics test at her school. Write underneath each part in the space provided whether she was right or not and explain why you so answer. …*
>
> *b)* $\frac{6}{7}$ *is the same as 6÷7*

The results were quite surprising to me at the time, with a number of students, such as those whose work is shown in Figure 2, not seeing these as the same, explaining that one is a 'sum' and the other a 'fraction'.

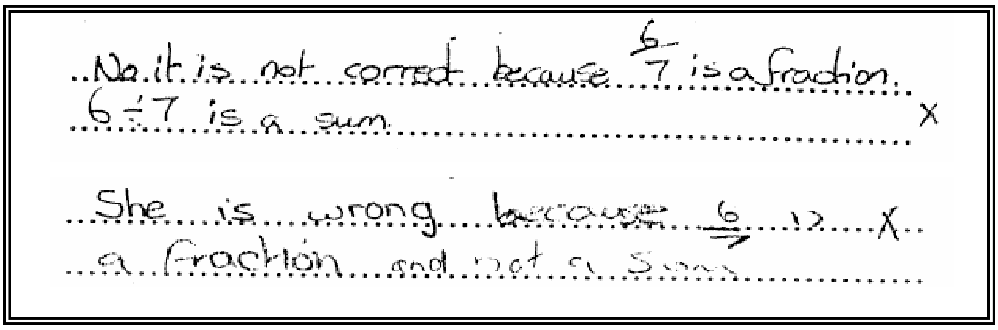


Figure 2. Student work showing differentiation between process 'sum' and object 'fraction'.

Commenting on this differentiated perspective on the symbols, I wrote in my PhD thesis (Thomas, 1988, p. 467) the beginnings of a foray into the world of versatility.

> *Of the 147 pupils given the questionnaire, 22.4% of them did not consider them to be equivalent, because, as they explained it, the former is 'a fraction' or '6 sevenths' but the latter is 'a sum' or 'a divided by' (see Figure 10.12). Thus a high proportion of children at this age see $\frac{6}{7}$ as an indivisible whole, an entity, propagated by the holistic mode of processing which very strongly evokes the concept of fraction for this image. However, for the imagery 6÷7, many pupils process this sequentially, since in their schemas the image is strongly linked to the concept of a process, or a sum rather than an entity, and they view the symbolism in three distinct parts, as 6 (value), divided by (process or operation) and 7 (second value). What these pupils lack is the versatility, described in Chapter 3, which would enable them to view either or both of these symbolisms or images in a global/holistic or a serialistic/sequential way.*

As early as 1989 David and I were writing about this idea in a PME paper (Thomas & Tall, 1989) that also referred to the encapsulation of this division process as an entity (or an object as it is now usually described, e.g., Dubinsky, 1991; Dubinsky & McDonald, 2001).

> *An interesting example of this, although arithmetic rather than algebraic, is the first question in table 1, where many of the controls did not consider the two notations as the [sic] equivalent because*
>
> *"$\frac{6}{7}$ is a fraction, 6÷7 is a sum".*
>
> *This is a good example of a response which is based on sound conceptual reasoning, but one that is limited because it implies the inability to encapsulate the process 6÷7, as a single conceptual entity. The encapsulation occurred far*

> *more often amongst the computer group, again underlying what we believe is a more flexible global view.*
>
> (Thomas & Tall, 1989, p. 218)

In our later paper in Educational Studies in Mathematics, written shortly afterwards but not appearing until two years later, in 1991, we refer to the *process-product* obstacle, where students are unable to see a symbolisation as standing for both the process and the object, which "requires the encapsulation of the process as an object". Here can be seen the genesis of the idea of procept, applied to algebraic thinking, but without the semiotic signifier.

> *Another closely related dilemma is the process-product obstacle, caused by the fact that an algebraic expression such as 2+3a represents both the process by which the computation is carried out and also the product of that process. To a child who thinks only in terms of process, the symbols 3(a+b) and 3a+3b (even if they are understood) are quite different, because the first requires the addition of a and b before multiplication of the result by 3, but the second requires each of a and b to be multiplied by 3 and then the results added. Yet such a child is asked to understand that the two expressions are essentially the same, because they always give the same product. Such a child must face the problem of realizing that the symbol 3a+6 represents the implied product of any process whereby one takes a number, multiplies it by 3 and then adds 6 to the result. This requires the encapsulation of the process as an object so that one can talk about it without the need to carry out the process with particular values for the variable. When the encapsulation has been performed, two different encapsulated objects must then be coordinated and regarded as the "same" object if they always give the same product – a task of considerable complexity.*
>
> (Tall & Thomas, 1991, p. 126)

Later in the same paper (*ibid*, p. 144) we applied these ideas to the $\frac{6}{7}$ versus 6÷7 dilemma, saying "This reveals the perception of 6÷7 as a *process* involving value-operation-value rather than as a global entity – the single number – produced by this process." Sadly, it turns out that many students have not progressed to the point where they have encapsulated the division of integers as fractions, but instead their fraction object is essentially a *pseudo-encapsulation* (Thomas, 2002—after Vinner's, 1997, *pseudo-conceptual*), often based on a sharing conception, or a dividing of a whole quantity into equal parts.

Finally, it was the addition of strong evidence from Eddie Gray's PhD on process-object occurrences in arithmetic learning that led to the generalisation of the idea, and cemented in place the now familiar concept of *procept* (Gray & Tall, 1994). More recently we have enlarged on some of our ideas on process-object thinking (Tall *et al.*, 2000; Tall, Thomas, Davis, Gray, & Simpson, 2000). The latter paper considers the nature of mathematical objects, and in particular how encapsulation leads to a mathematical object. In recent years, for me, as may be seen from the quotation above from page 467 of my thesis, it has been an examination of the differences in thinking the examples highlighted, and analysing the versatility of thought necessary to cope with them, that has occupied much of my research over ten or more years. These two areas will be considered below.

DEVELOPING VERSATILITY

Examining the role of proceptual thinking

In a recent paper (Thomas, in print) I have described *versatile mathematical thinking* as comprising three aspects, and I would like to consider the development of my thinking about each of these, in turn, below. The first flows from the work I did with David, described above, and is what I describe as

> *process/object versatility—the ability to switch at will in any given representational system between a perception of particular mathematical entities that may be seen as a process or an object.*

This could also be called proceptual versatility. One of the areas of the curriculum where the idea of *process/object versatility* was examined was in the relationship between the process of integration and the concept of integral. Working with a PhD student Ye Yoon Hong (Thomas & Hong, 1996; Hong & Thomas, 1997), we found that students generally had a process view of the integral symbol, and hence could not deal with an integral where the process could not be carried out. Two of the questions we used to examine this thinking were:

If $\int_1^3 f(t)dt = 8.6$*, then write down the value of* $\int_2^4 f(t-1)dt$.

If $\int_1^5 f(x)dx = 10$*, then write down the value of* $\int_1^5 (f(x)+2)dx$.

Our results showed that only 17.0% of 47 Form 7 (age 18 years) high school students were able to answer the first of these questions correctly, and 12.8% the second, while 59.6% were unable to make any response at all. The situation was similar at university, with 49.1% of 161 first year university calculus class students (age 17~22 years) able to use a standard procedure to find $\int_2^4 (x-1)^2 dx$ (surprisingly low) but only 27.3% and 22.4% respectively could answer the two questions above. Some students were rather resourceful in their efforts to circumvent the problem of considering the integral as representing an (area) object. For example, to introduce procedures, one student wrote

Let $f(t) = ax + b$ *[interestingly, but not surprisingly, a function in x not t]*

Then $\int_1^3 f(t)dt = 3a + b - a - b = 8.6$ *so 2a = 8.6 and a = 4.3. [no integration done]*

Thus $\int_2^4 f(t-1)dt = 4a + b - 1 - 2a - b - 1 = 2a - 2 = 8.6 - 2 = 6.6$. *[using f(t–1)=f(t)–1]*

A second example of this, a favourite of mine, is seen in Figure 3. Here the *process-oriented* student (Thomas, 1994) is so focussed on carrying out a process that he identifies the t as something he can integrate to get $\frac{t^2}{2}$, and so he does, taking the rather troublesome f as a constant whose value is to be found (f=2.15). Combining this with the expansion of $f(t-1)$ as $f(t)-1$ and integrating as before we get…the correct answer!

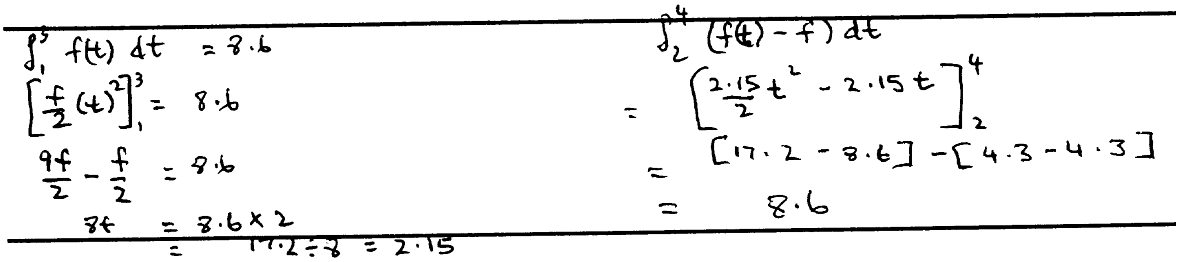


Figure 3. The influence of a strong process-oriented view of integral.

These two examples demonstrate how far students with a procedural perspective will go to introduce known algorithms into a question where they lack the necessary conceptual understanding. Our belief was that investigation of the processes lying behind the concepts of integration would enable improvement of conceptual understanding, assisting with encapsulation. The research showed that it was possible to design curriculum materials using technology to do this and give an improved cognitive base for a flexible proceptual understanding of integral and other concepts (Hong & Thomas, 1998).

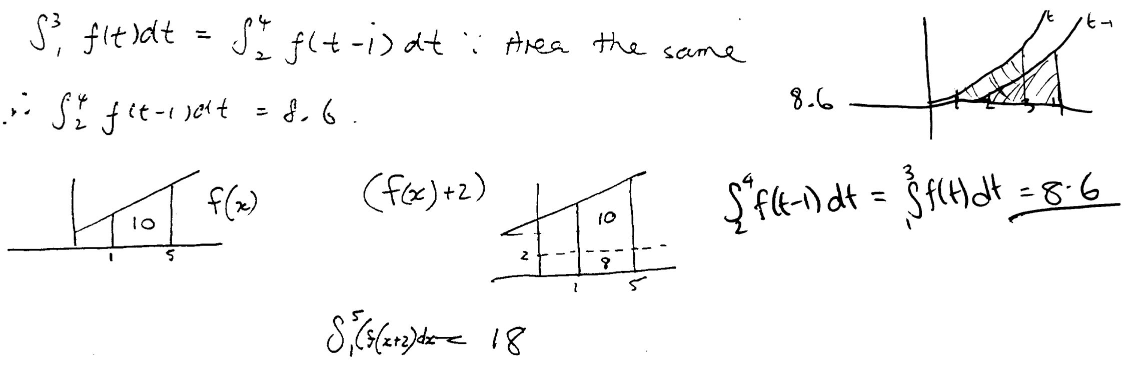


Figure 4. Understanding an integral procept as an object.

Figure 4 shows some of the work of students on the questions after such a programme. Here we see that the students were able to perceive the relationship between the first integration symbolism and the area object and to then operate on this area according to the change in the integration symbols. This is much different from process-oriented students, who need to carry out a procedure.

I have also spent some time investigating student understanding of differentiation procepts such as $\frac{dy}{dx}$ (Delos Santos & Thomas, 2001; Delos Santos & Thomas, 2003; Delos Santos & Thomas, 2005). Among our results, we found (Delos Santos & Thomas, 2001) that only 45% of 22 final year school students could make any interpretation of $\frac{dy}{dx}$ in $z=\frac{d(\frac{dy}{dx})}{dx}$, and only 1 thought that it had anything to do with rate of change or gradient of a tangent. A possible explanation of this problem (Thomas, 2002) can be expressed in terms of the differences in the manner students perceive $\frac{d^2y}{dx^2}$ and $\frac{d(\frac{dy}{dx})}{dx}$. The first is often seen as a repeated application of the differentiation process, but there are problems interpreting $\frac{d(\frac{dy}{dx})}{dx}$ in this way because it requires one to operate on $\frac{dy}{dx}$ as an object (in the differentiation process), and hence students with only a process view of $\frac{dy}{dx}$ meet a cognitive obstacle. One of the students in the study, Steven, epitomized the lack of an object perspective; when asked about $\frac{dy}{dx}$ he immediately responded in a process-oriented way, saying "I must differentiate." (Delos Santos & Thomas, 2003).

In this same study we probed our students' process/object versatility further by taking them into uncharted waters, presenting them with unfamiliar function constructions that required object perspectives. While the students were familiar with the composite function form *f*(*g*(*x*)), we employed the unusual notations $f(f'(x))$ and $f'(f'(x))$ to access their thinking. For the first of these Steven responded "the original function times the differential of the original function", and proceeded to illustrate this by multiplying the function $f(x)=2x^2+1$ by its derivative 4*x*. He followed through consistently for the second, describing it as "the differential times the differential" Thus when faced with an unfamiliar representational form Steven's recourse was to interpret the juxtaposition of f and f' (and later of f' and f') as a known operation or process, namely multiplication. Hence he operated *with* the result of a process $f'(x)$, but not *on* it as an object, as the composite function requires. James on the other hand displayed the ability to think of the symbol $f'(x)$ as an object, what he called the derivative function. He tried to interpret the symbols using specific functions, of the form $f(x)=x^n$, and wrote:

$$f(x)=x^2 \qquad f'(x)=2x \qquad\qquad f(x)=x^3 \qquad f'(x)=3x^2$$
$$f(f'(x))=\ (2x)^2\ =\ 4x^2 \qquad \text{and} \qquad f(f'(x))=\ (3x^2)^3\ =\ 27x^3$$

Using a graphic calculator he was able to generalise this, getting $n^n x^{(n-1)n}$, and recognise that the power would always be even. Moreover, he provided a graphical interpretation saying that "it's always gonna be steeper than this original function …it's also gonna be concave up". Interestingly, when asked to describe $f'(f'(x))$ he responded "that does imply second derivative". Hence instead of applying the same composite function thinking he had used seconds before, he saw this as the second derivative $f''(x)$. This could be the result of a strictly linguistic interpretation of the symbolism. Reading $f'(x)$ as *f*-dashed of *x*, may cause one to read $f'(f'(x))$ as *f*-dashed of *f*-dashed of *x*. This in turn leads to James' statement that "It's the derived function of the first derived function.", and a parallel with $\frac{d(\frac{dy}{dx})}{dx}$, the second derivative, takes over. Whatever the reason this appears to be a common initial reaction to this unfamiliar proceptual symbolism.

The ideas surrounding process/object versatility, that started with my conversations with David years ago, continue to occupy my time, and more recently I have been considering their application to the learning of linear algebra (Stewart & Thomas, 2006a; Stewart & Thomas, 2006b). One problem we have identified here is in the definition of eigenvalue and eigenvector, often a form of:

A non-zero vector x is called an eigenvector of a square matrix A if and only if there exists a scalar λ such that $Ax = \lambda x$.

Analysing this equation from a proceptual perspective we see that the right hand side involves a process in which a vector is multiplied by a scalar, resulting in a vector object. However, the left hand side of the equation has a quite different process, with a vector multiplied by a square matrix. The potential difficulty for students in understanding the definition is to see that both processes can be encapsulated as the same vector object. We have found that students with a process perspective of the equation's procepts do not see this. This lack has implications for understanding of the standard algorithm for finding the eigenvalues. Figure 5 shows a version of how this is often described in coursebooks or textbooks.

Definition 4.1. *Given a square $n \times n$ matrix A, we can sometimes find a non-zero vector $\mathbf{v} \in \mathbb{R}^n$ and a corresponding scalar λ, such that*

$$\boxed{A\mathbf{v} = \lambda\mathbf{v}.} \tag{4.1}$$

We call any non-zero vector $\mathbf{v}$ *which satisfies (4.1) an* **eigenvector** *of A, and the corresponding scalar λ an* **eigenvalue**.

The matrix equation (4.1) can be rewritten

$$(A - \lambda I)\mathbf{v} = \mathbf{0}. \tag{4.2}$$

Figure 5. An 'explanation' of the move from $Ax = \lambda x$ to $(A - \lambda I)x = 0$.

Asked to reproduce and explain the missing steps in this transformation students had problems bringing the two processes together. Rather than multiplying either just the *x*, or both sides by the matrix identity I_n, as the student whose work is shown in Figure 6 did, others ran into process difficulties such as needing to subtract a scalar from a matrix, as seen in Figure 7. We concluded that this apparently simple step is not straightforward for many students, and the one-step jump tends to cause a focus of attention on λI rather than on Ix.

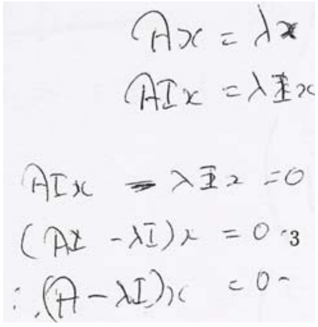


Figure 6. One way of understanding the transition from $Ax = \lambda x$ to $(A - \lambda I)x = 0$.

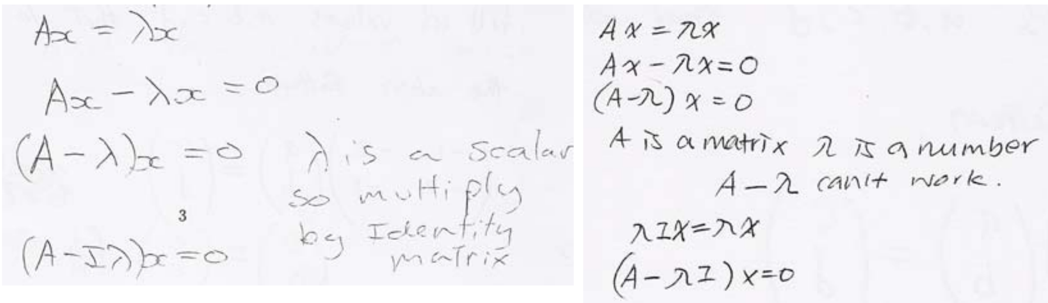


Figure 7. Failure to understand the transition from $Ax = \lambda x$ to $(A - \lambda I)x = 0$.

The research description above gives a brief indication of the fruitfulness of an investigation of the role of process/object versatility in mathematical thinking. No doubt there are still many other areas of mathematical thinking where a similar analysis will repay further dividends.

The role of visualization in versatile mathematical thinking

The versatility of thinking required to switch from a process view of symbol to an object view is only one example of the considerable flexibility needed in mathematical thinking. A second, which I have been thinking about since I wrote my PhD thesis (Thomas, 1988), is also described in a recent paper (Thomas, in print):

> *visuo/analytic versatility—the ability to exploit the power of visual schemas by linking them to relevant logico/analytic schemas.*

In my thesis, and in papers since (e.g., Tall & Thomas, 1991; Thomas, 1995; Thomas, 2002; Booth & Thomas, 2000) I have described a model of *cognitive integration* that seeks to incorporate visual thinking into a description of versatility. This model (see Figure 8) comprises a *first-degree* knowledge structure, the elements of which are primarily mental images of various forms, 'existing in' the brain’s minor hemisphere.

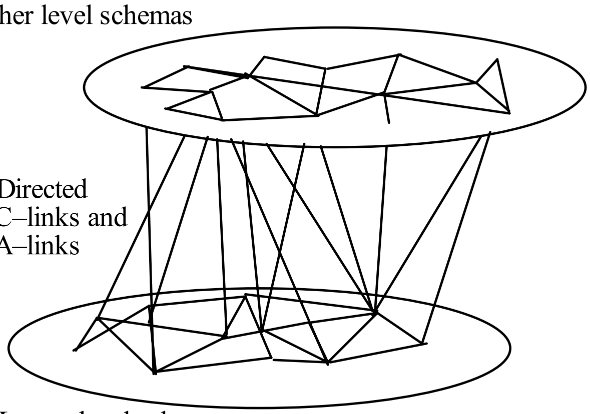


Figure 8. A simplified model of cognitive integration.

These elements or states are connected to form schemas in a way similar to those of the largely language based, serialist/analytic *second-degree* structures of the left hemisphere, and are also constructed by Piagetian assimilation and accommodation, although not consciously. It is these *first-degree knowledge structures,* I contend, that give rise to global/holistic mental abilities. Piaget and Inhelder (1971, p. 366) have described evidence for the schematic nature of the mental imagery generated by these abilities, and say that

> *If we use the word 'scheme' (scheme) to designate a generalization instrument …there are perceptual schemes, sensori-motor schemes, operational schemes, and so on. And in this senses there also exist imaginal schemes enabling the subject to construct analagous schemes in comparable situations…Imaginal figuration, on the other hand, is 'schematized' precisely in the 'schema' sense, though at the same time it may entail 'schemes'.*

We may glimpse this ourselves by thinking of the name of a town that we are very familiar with. Mentally naming the town and then concentrating on the rapid sequence of images which may appear and disappear we may conclude that those connected to the initial image of the chosen town are linked in a structure such that each image evokes others. With such a structured sequence of images we may even take a tour around the town, though it may be half a world away. Such a structure constitutes, I maintain, an example of the schemas of mental imagery. However, if these *first-degree knowledge structures* are to have value in the goal-directed mental activity of the individual, and their processing is to be acted upon, then a means of accessing the results of the goal-directed activity of this schematic thinking from the qualitatively different, conscious, logical/analytic schemas of the second degree must be included in a psychological model. This is achieved by introducing links between the concepts in each level of the model, facilitated by the flow of data between each hemisphere, across the corpus callosum. Hence it includes, not two separate knowledge structures, but two connected, distinct modes of operation, working in parallel, within an integrated whole. It is by means of the 'vertical' links between the levels, I contend, that the mental imagery schemas influence the higher-level cognitive functions of the mind. In the context of this model, *visuo/analytic versatility* may be defined as attaining the construction of meaningful schemas at both the higher and lower cognitive levels, as well as appropriate two-way, inter-level—that is, inter-hemispheric—links. Thus, such a learner is able to use the conscious higher-level relational schemas (with their serialist/analytic processing) in parallel with the unconscious lower level relational schemas (with their global/holistic processing), and, most importantly, to switch easily, although often unconsciously, between the two as and when appropriate.

I have applied the principles behind this theory of the power of visualization in a number of projects arising from my PhD work. For example, we tried to promote a *versatile* view of equation using a *Dynamic Algebra* computer environment (see Figure 9), which encourages students to construct equations in terms of variable and expression objects that can be simultaneously evaluated (Thomas & Hall, 1998). This approach emphasised the visual aspects of variable as a location or store and an accompanying label, and equation (as two equal expression boxes) in an environment where a number of processes for equation solving, including trial and error substitution and balancing can be investigated. After the visual module of work the 11 and 12 year-old students improved significantly in their ability to solve linear algebraic equations, including using a method of solution where most of them applied the same operation to both sides of the equation and 'cancelled' terms, for equations such as $5n + 12 = 3n + 24$. We felt able to conclude that they had developed a more versatile view of equation.

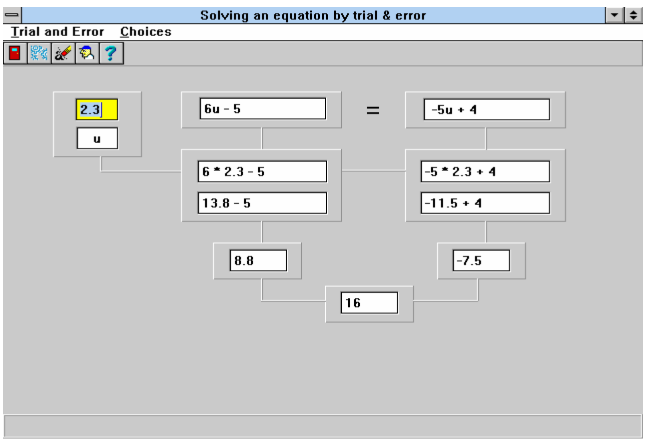


Figure 9. A screen from the *Dynamic Algebra* program.

A similar approach was employed (Graham & Thomas, 2000) to give students an improved understanding of generalised number or variable, using graphic calculators. The controlled experimental study showed that the visual mental model, using a graphic calculator, assisted in significantly improving student understanding of symbolic literals, regardless of their ability level, although the gains were particularly noticeable for the weakest students. The value of visualisation can be seen in the remarks of one student:

I think the STORE button really helped, when we stored the numbers in the calculator. I think it helped and made me understand how to do it and the way the screen showed all the numbers coming up I found it much easier than all the other calculators which don't even show the numbers.

It appears that the way the graphic calculator screen preserves several computations on variables in view, along with the mental model, had assisted this student to think mathematically and make powerful connections. The flexibility to link visual and embodied thinking with analytical reasoning is an important part of versatile thinking.

The role of representation in versatile mathematical thinking

Versatility of mathematical thinking involves more than the two types of flexibility described above. Engaging in further discussion of these ideas with David Tall led me to consider further the role of representation in mathematical thinking. In my recent paper (Thomas, in print) I have described this third aspect of *versatile mathematical thinking* as:

> *representational versatility—the ability to work seamlessly within and between representations, and to engage in procedural and conceptual interactions with representations.*

Following the leading of the field of semiotics introduced by Peirce (1902) and others, signs (icons, indexes, or symbols) are objects of thought that bring to mind their referent. While there are icons and indexes in mathematics, it predominantly comprises symbols, signs that have become associated with their meaning by usage (Peirce, MS404, 1894); becoming significant simply by virtue of the fact that they will be so interpreted. However, the signs are not usually isolated but are grouped together into families (Saussure, 1966), such as logic signs, algebraic symbols, matrices, or graphs. These groupings can be called *representation systems*, and the individual signs representations, making a representation a sign associated with a given system of signs.

Kaput (1989) drew a distinction between the way some *representation systems* (he also called them *notation systems*) are used mainly to display information and relationships (*display notations*) while others support a variety of transformations and other actions on their objects (*action notations*). An important class of mathematical activity involves manipulation of mathematical concepts both within and between these different representations, or "translations between notation systems, including the coordination of action across notation systems." (Kaput, 1992, p. 524). While the ability to establish meaningful links between and among representational forms and to translate meaning from one representation to another has been recognised, and referred to as *representational fluency* (Lesh, 1999), I introduced the concept of *representational versatility* to include both this fluency of translation between representations, and the ability to interact procedurally and conceptually with individual representations (more details below).

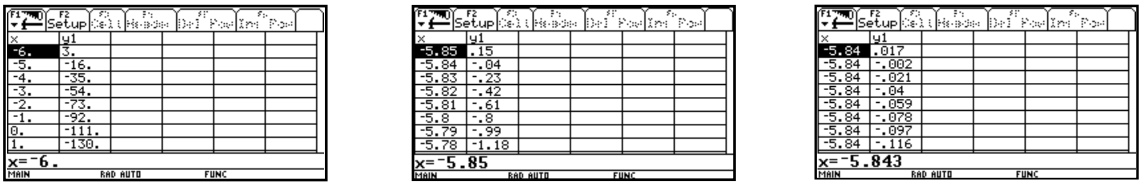
Interacting with mathematical signs or representations can be a complex multi-stage process. One may interact with them by *looking at* the images or *looking through* them (Mason, 1992, 1995) depending on whether the focus of attention is surface or deep. For example, Laborde (1993a, b) described how one may see a geometric icon in two different ways. She talks about how a "*Drawing* refers to the material entity while *figure* refers to the theoretical object" (Laborde, 1993b, p. 49). Likewise, Fischbein (1993, p. 141) refers to how "successful geometric reasoning can be achieved when we stop considering only two distinct categories of mental entities (images and concepts) and we deal apart from them with a third type of mental object, the figural concept". In both cases looking through, or deep observation, requires linking of an image with a conceptual base, or schema. This process is part of the *cognitive integration* described above. Thus a surface observation of an icon may lead one to think that it may be a representation of a mathematical object, but in order to move to seeing it as a figure, referring directly to the mathematical object, requires interpretation. The perceived object needs to change role from an icon to a symbol. This interpretation involves the use of a link to an appropriate, existing second-degree mathematical schema to ascertain the properties of a rectangle that may be overlaid in memory on the first-degree drawing or representation. Thus the mathematical concept of rectangle, is a combination of a perceived icon, its object referent and data (properties) from the mathematical 'rectangle' schema. With many signs this process may have more than one stage and in order to produce a mathematical figure from a picture we need to pay attention to the essential property-revealing details of the picture in two steps. First we mentally or physically produce a diagram or figure from the picture, and secondly we need to overlay its conceptual properties in order to see the figure as representing the theoretical object (Booth & Thomas, 2000).

It has struck me as interesting that in all the analysis of process/object versatility the question of how representations other than those of the symbolic algebra representation system relate to the process-object conceptualisation of mathematics had not really been addressed. Since mathematical concepts can clearly be perceived in this dual manner one would expect graphical, tabular, ordered pair, and other representation systems common in secondary mathematics, to be amenable to a corresponding analysis.

When the interaction with a sign or representation progresses from observation to performance of an action on the representation, and learning from it, *doing* and *construing* in the sense of Mason (in print), then I describe the representation as becoming a *cognitive tool* (Thomas & Hong, 2001; under review). I propose (Thomas & Hong, 2001; Thomas, in print) that a crucial difference between a process and object tool interaction is that the former comprises a discrete, elementwise approach in terms of its parts (e.g pointwise for a function) while the latter requires a holistic perspective.

Technology, such as graphic calculators, makes available some novel interactions such as the ability to solve equations by numerical processes by zooming in on solutions using tables of values (see Figure 10). This kind of dynamic interaction is not as simple as it appears. It is based on the continuity of the function and the Intermediate Value

Theorem, and requires not a discrete, point-by-point view of the table of values, but a holistic focus on at least the part of the function revealed by the representation.



$-6 < x < -5$ $\quad$ $-5.85 < x < -5.84$ $\quad$ $-5.843 < x < -5.842$

Figure 10. Dynamic interactions with a tabular representation using a calculator.

We have also tried to promote representational versatility by using a geometrical approach linked to algebraic representations with CAS calculators for the Newton-Raphson method for approximating zeros of functions (Hong & Thomas, 2002). We noted that local examiners' reports over a number of years, had stated that 'many candidates had no geometrical appreciation of the Newton Raphson method.' and so such students were reduced to calculation processes based on the algebraic formula $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$. We employed a method based on the graphical representation and linked it to the algebraic representation by rearranging the equation of the tangent: $y - f(x_1) = f'(x_1)\,(x - x_1)$, with $y = 0$ when $x = x_2$, to get $f'(x_1) = \frac{f(x_1)}{(x_1 - x_2)}$. The algorithm used was to draw the graph of the function under consideration, choose a suitable first estimate, and then use the CAS to draw the tangent at the point and to find and display its equation. The representation is then acted upon in a conceptual way by using the symbolic manipulator of the CAS to solve $f(x)$=0, and find where the tangent crosses the x-axis. This method can then be repeated until the zero is found to the accuracy required, while zooming in on the graph to see what is happening.

We found that this conceptual interaction with two CAS representations helped students to understand why the Newton-Raphson method works, and to form conceptual links between the graphical and algebraic representations. For example, Figure 11 shows how one student understood how the sign of f and f' (ie the gradient of the tangent) affect whether the second estimate is greater than or less than the first estimate (Hong & Thomas, 2002).

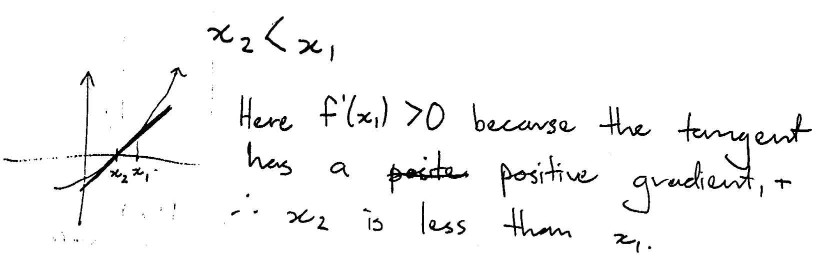


Figure 11. Interacting with a geometric representation in a conceptual way.

Figure 12, shows another student interacting with a graphical representation and appreciating that the first estimate has to be sufficiently close to *a* for the gradient to be large enough for the tangent to cut between *x*=*a* and *x*=*b*.

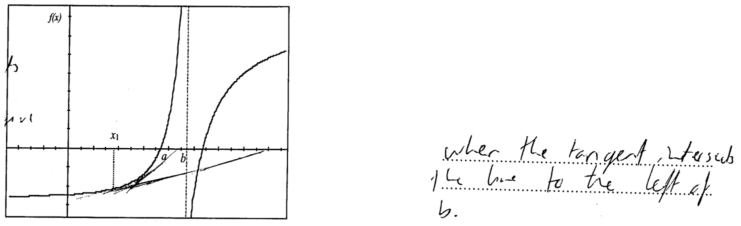


Figure 12. Interacting with a geometric representation in a conceptual way.

Our students also showed an understanding of how the choice of the first approximation to a root influenced the result, especially how the gradient at the point could be too small or be zero, with consequent problems:

> *Student A: You can't choose a max or min point or else you won't cut the x-axis. Also the tangent could go towards the wrong root.*

> *Student B: It must be close to the root so the tangent gives you the nearest value. Also you can't choose a stationary point as a first value.*

As part of the study students were asked to construct a graph of a function to put the first and second estimates on opposite sides of the zero. This involves a consideration of the concavity required, and Figure 13 shows the result of the graphical thinking of one student who managed this task.

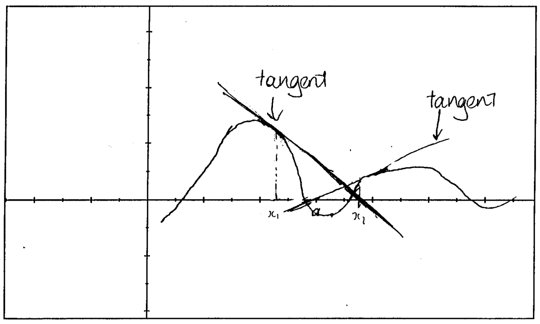


Figure 13. Interacting with a graphical representation in a conceptual way.

The idea of *representational versatility* has also found application in other areas of mathematics. Recently we have applied it to the learning of statistics (Graham & Thomas, 2005). A key part of promoting *statistical thinking* (Wild & Pfannkuch, 1999) is for students to develop a "curiosity for other ways of examining and thinking about the data and problem at hand" (Chance, 2002, p. 4). It seems that promoting an examination of a number of representations would form a crucial part of such thinking.

To illustrate, a student might interact with a statistical data representation in a procedural or a conceptual manner. In the former they might perceive that what is required of them is to carry out a series of steps or actions on their part, while in the latter they would focus, not on a procedural algorithm such as calculating a summary value, but on an idea or concept; on statistical thinking. We suggest (Graham & Thomas, 2005, p. 9) that this is "more likely to create in the student an 'object' perspective of the data representation where they can consider the data and its statistical properties as a whole."

For example, when reflecting conceptually on the regression properties of a set of paired data the student may examine the graphical representation to gain an awareness of the overall pattern of the points. The switch of focus from process to object for such a data set is subtle, but in this case students may be encouraged to draw a line of best fit by eye and then consider each point in turn, calculating the vertical deviations from their line along with the sum of the squares of these deviations. This process approach can then progress into a conceptual interaction where the student is able to manipulate a property of the whole of the data as a single object. It is only when thus viewed holistically that questions about minimising the sums of the squares of the deviations and whether a straight line is the best model to fit, can take on meaning for the whole data set. They may then link to a symbolic algebraic representation to consider a suitable regression model, and once this has been found from working within this representation, the data set may no longer perceived as a set of discrete points (ie in a process manner) but rather as a single entity that may be represented by its regression function. It seems important to stress that in statistics the student's interaction is sometimes with a representation of the original data, and sometimes with a mathematical model of the data. They need to appreciate when the representation they are dealing with depicts the original data and when it is one step removed from the data, being a theoretical model fitted to it. This is a key part of representational versatility in statistics.

In the above discussion I have tried to give a flavour of the growth of what I still see very much as a developing conception of versatile mathematical thinking. It is still clearly not complete and is very much a work in progress. Regardless of the extent to which the picture becomes clearer in the future, the exhilaration of the enquiry is assured. One key question that immediately follows from a description of qualities deemed favourable for learning is, rightly, "How can we promote and encourage such thinking in our students?" I am currently giving this some thought and hope that others may also infer from the thoughts presented how this may be done.

THE ROAD AHEAD

Over the past 23 years I have been able to work with David Tall in various capacities, writing some 14 papers together, and producing an edited book on Richard Skemp's ideas (Tall & Thomas, 2002). Throughout this whole period I have been conscious of the debt I owe to him for the friendship that has helped refine many of the ideas we have tried to bring to fruition. I am hopeful that our productive collaboration will continue for some time to come, even though David will be ‘retired’. At present we are working together on a paper describing what we know, and can know, about mathematical thinking from brain studies. We are also just beginning to look ourselves at fMRI brain scans and what they may (or may not!) tell us about such thinking; so there is much more still to accomplish.

I certainly appreciate the truth of what another of the world’s leading mathematics educators, the late Jim Kaput, wrote in a reference for me, when he said “He was fortunate to study with a pioneer in the field, Professor David Tall.” While this is most certainly true, to me (and I know Jim would have agreed) David has been much more than just a pioneer in mathematics education; I would suggest that his research has been, like Jim Kaput’s, of the highest possible order (see Figure 14).

Figure 14. Jim Kaput and David Tall: Two of the best.

His careful and thoughtful elucidation of theoretical concepts, coupled with his innovative uses of technology have often had, and will continue to have, a profound impact on teaching and learning. The advances he has invited us to follow also demonstrate that David Tall is not merely an excellent theoretician but is someone who cares deeply about the practice of mathematics education in the classroom and the lecture theatre. I know I am not the only person who has valued David’s keen insights and while I’m sure we all wish him a long and happy retirement, we are eagerly awaiting the publishing of ‘the book’.

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Becoming Mathematical

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ABSTRACT: *This paper explores a perspective on becoming mathematical built by David Tall and other researchers who collaborate with, or were supervised by him; in particular, his colleague Eddie Gray. Tall´s perspective moves from an initial focus on mental representations to the different modes of operation or mathematical practices with representations, thence to the variations on the language used to carry out actions and to communicate results. My intention is to feed the debate on his account of the nature of mathematical knowledge as consisting of the categories called 'the embodied world', 'the symbolic-proceptual world' and 'the formal axiomatic world'. Supported by empirical research, I argue that the theory still has explanatory power even if only referring to three modes of operating in mathematics. It suggests a diversity of modes in which individuals may be engaged in mathematical activity.*

Key words: *Mathematical knowledge, visualization, procepts, natural and formal learners, mathematical language.*

INTRODUCTION

In this paper I discuss a framework which supports theories and teaching projects developed by David Tall during his academic life. Some of the empirical research conducted by several authors who collaborate with or were supervised by him are organized in order to compose a coherent whole, rather than a chronological report on his publications and research. I hope I have, in part, captured some of his main ideas and I hope I have offered a useful interpretation of some elements of his important and extensive research. My intention is to feed a debate on some ideas developed as a result of that work.

At the centre of his academic life and research projects is his attraction for new technologies, side by side with his appreciation of the use of visualization and visual software to represent mathematical ideas. His interest in visualisation began with his desire to teach calculus in a meaningful way, assisted by new technology. Despite the power of technological tools for educational purposes, Tall acknowledges that an understanding of the mathematical activity does not simply follow from the use of new technology. From his perspective, there is still a need to help students to construct meanings for mathematical concepts that not only work mathematically but are also personally meaningful, while allowing further routes into mathematical experiences. In his projects, he implicitly uses mathematical notions which would support formal mathematical ideas, devising grounds for abstractions as cognitive roots.

Building on these orientations, Tall started his research in mathematics education, which has influenced many researchers in his country and around the world. His initial agenda was gradually broadened out to include the role of visuospatial and dynamical ideas within the whole mathematical experience. Communication and the role of language and gestures in mathematical growth were explored empirically through investigating the relationships between iconic-enactive information, verbal descriptions, and concept construction.

Here, I present my own reading of some features of Tall´s recent research on the mathematical growth of an individual. First, I introduce his theoretical perspective by referring to a framework which, in my view, supports the development of his ideas. Second, I describe the debate already initiated on some of its core notions. Feeding the debate, I argue in favor of the explanatory power of that theoretical framework, which improves our understanding of some dimensions of the mathematical activity and knowledge construction. Finally, I discuss some crucial reconstructions that Tall made on earlier theoretical views, and I also argue about the relevance of language and communication within his long time research.

ON BECOMING MATHEMATICAL: A PERSPECTIVE

I understood that Tall´s theoretical perspective on mathematical growth draws on Bruner’s (1966) three modes of mental representation – the sensori-motor, the iconic and the symbolic – which he considers as a sequence in the cognitive growth of the individual (see also Tall, 1994). In his theory, Bruner was broadly referring to a person’s effort to “translate experience into a model of world” (Bruner, 1966, p.10). His educational hypothesis was that “any idea or problem or body of knowledge can be presented in a form simple enough so that any particular learner can understand it in a recognizable form” (ibid, p.44). He suggests three ways of doing it: through action, followed by the use of “summarising images” (p. 10-11) and finally in words or language. These ways correspond to enactive, iconic and symbolic modes of representing an experience.

Referring specifically to mathematical representations, Tall (2003) reinterprets these categories and proposes three fundamentally distinct modes of operation with them:

> *Embodied: based on human perceptions and actions in a real-world context including but not limited to enactive and visual aspects.*
>
> *Symbolic-proceptual: combining the role of symbols in arithmetic, algebra and symbolic calculus, based on the theory that these symbols act dually as both process and concept (procept)* (see Tall, Gray, Bin Ali, Crowley, DeMarois,

McGowen, Pitta, Pinto, Thomas, & Yusof, 2001).

Formal-axiomatic: a formal approach starting from selected axioms and making logical deductions to prove theorems.

As I see it, Tall (2003) appears as grouping Bruner's two first modes of mental representation – the sensory motor and iconic, when conceiving a mode of operation in mathematics which he named *embodied*.

In fact his actual use of the term *embodied* does not correspond to its use in some other recent theories on the nature of human thought (see for example, Maturana & Varela, 1987; Lakoff & Johnson, 1999; Lakoff & Nunez, 2000)[i]. In the case of Tall´s theory, the author gives the term a more focused meaning in mathematical thinking, using it "to refer to thought built fundamentally on sensory perception as opposed to symbolic operation and logical deduction." (Tall, 2003). Comparing the different theories, Mejia (2004) observes that, in essence, they aim at different descriptions. For instance, Tall uses the term to refer to conscious activities in a theory of mathematical growth, while, for example, Lakoff and Nunez (2000) mainly concentrate on unconscious thought, and reveal (their) conceptions of today´s mathematical knowledge. Tall´s notion does not refer to the general idea that all knowledge, let alone all mathematics, is embodied, as those other authors do.[ii]

Bruner´s third mode of representation- the symbolic, seems to be split in two: the symbolic-proceptual and the formal-axiomatic.

The term *proceptual* in the '*symbolic-proceptual* mode of operation' is a reference to mathematical processes which are symbolised in such a way that the symbols dually stand for both the process to be carried out and the concept to be thought (Gray & Tall, 1991).

Gray and Tall developed empirical research showing that mathematical practice involves evoking a symbol as a process or as an object, whichever it is appropriate to the specific situation. For the authors, most of the time this is uncounsciously done by individuals during a mathematical activity which is familiar to them. To account for this idea, the researchers named the term *procept* to mean:

…the amalgam of process and concept in which process and product is represented by the same symbolism. (Gray & Tall, 1991)

In these terms, grasping the mathematical activity leads individuals to think of a *procept* in a way which makes explicit duality (as process or as concept), flexibility (adapting its use to the context) and ambiguity (not always making it explicit which is being used). Empirical research at different school levels has focused on the construction of mathematical objects from mathematical processes, pointing out such dichotomy or duality of mathematical objects. In general, theories emerging from such a perspective claim that, first, it is necessary to acquire the ability to carry out a procedure, which after practice is then compressed mentally into a mathematical object. Gray and Tall (1991) are among those who stress that the use of symbolism enables the objects and their relationships to be on the focus of attention at a more abstract level. Therefore conquering a procedure would be the initial task. Research by Gray (1991) indicates that encapsulating procedures and the use of appropriate symbolism seems to result in deriving facts and new knowledge from old. In the route described by Gray and Tall, the recognition of many procedures to carry out a mathematical task is identified as a *process* level, a different and higher stage from the initial *procedure* one. Reflecting on processes, their encapsulation into objects represented by symbols could result in flexibility of thought and *proceptual* thinking[iii]. It represents a higher stage of symbolical mathematical thinking.

At last, the *formal-axiomatic* mode recalls the mathematical activity as performed by professional mathematicians. The transition from an elementary to the formal-axiomatic mode of operating in mathematics is characterised by a change in focus from the discovery of *properties* in elementary mathematics that are found through exploration and operation, to the *specification* of properties as a basis for formal *definition and deduction*.

> *To move from elementary to advanced mathematical thinking involves a significant transition: from describing to defining, from convincing to proving in a logical manner based on those definitions. … It is the transition from the coherence of elementary mathematics to the consequence of advanced mathematics, based on abstract entities which the individual must construct through deductions from formal definitions. (Tall, 1991, p.20)*

Research on how different individuals come to terms with formal axiomatic theory has shown different strategies used by students in their attempts to build up concepts given by formal definitions (Pinto, 1998). In the latter study, students were followed over time and we observed two different styles of learning. In the lights of Tall´s distinction between the three modes of operating in mathematics, these two styles may be distinguished according to whether or not students regularly construct knowledge by combining the three modes of operation - embodied, proceptual-symbolic and formal axiomatic. Those who use a formal mode of learning, mainly operate within the boundaries of the symbolic-proceptual and formal-axiomatic modes. There seems to be an act of construction within the formal world: learners seem to concentrate on interiorizing the formal definition through familiarisation with their use and from the technicalities of building up formal proofs. Iconic or graphical representations are used by them to sustain their verbal ideas and propositional constructions, as an aid to the compression of pieces of information into cognitive units. Those learners seem to build the formal theory through acting on cognitive units, in a manner captured by Dubinsky Elterman and Gong (1988).

Meanwhile, there are those learners who follow a natural mode of learning (Duffin & Simpson, 1993) by operating within the boundaries of the embodied and formal-axiomatic modes. For those students, the process-object encapsulation models do not seem to explain their process of concept construction and abstraction. In fact, such theories seem to be excessively restricted to the symbolic-proceptual and formal axiomatic modes of experiencing mathematics. They appear to ignore those who to reorganize their earlier embodied experiences to fit new knowledge. Reconstruction of earlier experiences seems to be essential for these learners, as they explore their embodied representations making connections and discovering properties.

Pinto (1998) showed there is a spectrum of performance in each of these styles of learning, which indicates that neither of them necessarily lead to success or failure. Such results help explain the different demands from the formal theory which are related to learners' different strategies of learning. Gaining insight into these strategies would have great potential in designing the curriculum of Real Analysis for future students.

From an epistemological point of view, Tall attempts to relate each of these three modes of operation to different *worlds of mathematics*, "each with its own world of meaning and distinct methods of justification.". (Tall, 2003).

> *The* embodied *world is the (world of) fundamental human mode of operation based on perception and action. The* symbolic-proceptual *world is a world of mathematical symbol processing, and the* formal-axiomatic *world involves the further shift into formalism…* (Tall, 2003)

However, I personally find useful to consider the above notions by just referring to the specific modes of operation (meaning the quite different forms of reasoning and establishing truth in each case). These modes of operating may be combined during the mathematical experience. From this perspective, I understand that the 'three worlds' notion opens up not only three, but a universe of possibilities for mathematical experience which are useful to consider, once we are interested in the teaching and learning of mathematics. Further, our responses within each of those contexts (or worlds) entail our acknowledgement of each of them as the perspective taken by who is speaking; and more importantly, the world of meaning and communication of who we are talking to.

Tall (2003) notes such singular power of the theory in referring to its three dimensions for mathematical experience:

> There are some 'mathematical truths' such as the commutative law or associative law for addition which we 'know' are true from our experience in the embodied world. When we operate on numbers, the results we get confirm these as facts. In general, we naturally assume these results hold in algebra. Other truths such as the formula for the difference of two squares ($a^2 - b^2 = (a-b)(a+b)$) are shown to be true by carrying out the algebraic manipulations. In the formal world, axioms will support our arguments. (Tall, 2003)

As an overall comment, I would say that by focusing on actions and modes of operation the above notion of the three worlds of mathematics is not restricted in its explanatory power to explore the mathematical experience. In doing that, one may avoid the investigation into the nature of mathematical objects emerging from each of these modes of operation (Watson, Spyrou & Tall, 2002; Gray & Tall, 2001; Tall, Thomas, Davis, Gray & Simpson, 1999). From a teaching and learning perspective, such discussion could be indecisive; in particular if the observer takes into account the point of view of learners (see, for instance, Tall & Thomas, 1989; Tall, 1994; Gray, Pitta, Pinto & Tall, 1997; Pinto, 1998; Pinto & Tall, 1996, 1999, 2001; Gray & Tall, 2001; Tall, 2002; Inglis, 2003; Tall, 2004). In the next section, follow some remarks on the debate initiated by the three worlds of mathematics ideas.

THE DEBATE: ON THE NATURE OF MATHEMATICAL KNOWLEDGE AND THE THREE WORLDS OF MATHEMATICS

Inglis's (2003) example of Lambert´s imaginary sphere is a case where the object may be thought of as arising out of the symbolic-proceptual world in which the algebra of spheres $x^2 + y^2 = r^2$ is applied to the case where $r^2 = -1$. Inventing his result by analogy and exploring it symbolically, Lambert formulates his conjecture which could later be explored in embodied mode through the use of Poincare´s disk model of hyperbolic spaces, and could also be made axiomatic by Riemann in 1980.

Our empirical research has also shown that individuals are capable of successfully experiencing even advanced levels of formal mathematics through different routes (Pinto, 1998; Pinto & Tall, 1996, 1999, 2001), which are supported by more than a single mode of operation. We have data showing that routes for the construction of formal mathematical knowledge may include thought experiments and the continual refinement of imagery (Pinto, 1998; Pinto & Tall, 2001). Our findings also suggest that the expected shift from visual and enactive actions to the efficiency of symbol manipulation in arithmetic may be done successfully by students who retain links with perceptual notions and embodied modes of operation. (Gray et al., 1997). Therefore, in my view, it would be unfruitful to expect that mathematical objects we talk about have a permanent status, as if confined within one of those three worlds taken as classical categories. The boundaries between the three worlds are fuzzy, and a *versatile approach* to mathematics means a constant cross crossing within them through using "whichever method is more appropriate to support the required thinking at a given time."(Tall, 1994, p.27)

For example, Gray and Tall (2001) observe that a symbolic representation such as 2/3, which might have emerged from embodied modes of operation such as cutting a cake into three pieces and taking two, may also be reinterpreted in formal set theory suggesting modes of operation such as dividing a set into three equal pieces and two of these being selected.

Tall (2004) recalls that Cayley's theorem encourages us to think of an axiomatic group as a subgroup of a group of permutations, returning operations on formal group theory to the embodied mode of permuting elements of a set. He observes that results from symbolic manipulation and also from formal definitions and formal proof can produce *structure theorems* stating the structural properties of formal or symbolic systems (Tall, 2002). Such theorems yield new journeys into the embodied or symbolic worlds, through speculations in terms of thought experiments or calculations[iv].

In summary, I believe that the complex task of describing the objects in each world does not seem adequate to explain the diversity of modes they may actually be worked out. On the other hand, the understanding of the notion of modes of operation to characterise worlds or contexts for mathematical activity represents a simple, though powerful, idea to explain mathematical experience.

The *embodied world* is a world of sensory meaning, where “warrant for truth is that things behave in an expected way” (Inglis, 2003; Tall, 2004). It refers to thought which is fundamentally built on sensory perception, encompassing visual and spatial imagery which includes Bruner’s enactive and iconic modes. The *symbolic-proceptual world* refers to mathematical symbol-processing. Symbols allow calculations and manipulations to establish truth. The *formal-axiomatic world* relies on formal definitions for concepts, from which deductions are made. The notion of truth is then established through formal deduction.

CRUCIAL REFINEMENTS AND A NEW PERSPECTIVE ON A SYMBOLIC MODE OF OPERATION

As in most developmental theories, Bruner accounts for an individual’s growth in stages, from dependence on sensory perception through physical interaction to sophisticated modes of thought through the use of language and symbols. Solo Taxonomy (Biggs & Collis, 1991) has offered a perspective that when more sophisticated modes of operation are attained by the learner, earlier modes remain available to be used when appropriate.

The three worlds of mathematics operate similarly. As each mode becomes available, it remains available. Later on all three may be used in any appropriate sequence. Tall (2003) observes that the child first encounters embodiment, and through actions on objects builds symbolic ideas in arithmetic. Later the individual focuses on the properties and their relationships that lead to proof. As the person becomes mature, the three worlds can support each other with the boundaries between them not clearly defined. Other than our empirical research, the history of mathematics offers examples indicating the convenience of assuming this perspective. For instance, complex numbers had arisen from symbol manipulation – representations in the symbolic-proceptual world, which are later explored in the complex plane through representations in the embodied world, which remained available and provided a context for new explorations and thought experiments.

Regarding the splitting of Brunner´s last mode of mental representation – the symbolic, into two other modes or categories – the symbolic-proceptual and the formal axiomatic, it appears that Bruner (1966, pp. 18 - 19) had approached on such a formulation. He explained that symbolism would include both “language in its natural form” and “the two artificial languages of number and logic”. Tall (2003) reconstructs these last two categories to include algebraic and functional symbolism, other than being restricted to just number, and the language of axiomatic mathematics.

In doing that, Tall´s perspective represents a shift from Brunner´s initial focus on mental representations to the actual different mathematical practices with these representations, or modes of operating, and to the variations on the language used to carry out actions and to communicate results.

LANGUAGE, GESTURES AND MATHEMATICAL DEVELOPMENT

Language in mathematics, and mathematical language, is often identified with mathematical symbols and the logic used to perform mathematical calculations and to deduce theorems. In contrast, Tall´s research has come from a perspective that symbols and logic alone cannot provide a complete environment for mathematical thinking. Following Skemp (1971), a formal presentation of mathematics expressed in the language of symbols and logic – “the end product of mathematical discovery” – is to be considered as the final product of an extensive process of mathematical thinking, which is pervaded by the use of natural language.

Most research conducted by Tall and aimed at exploring the individual cognitive processes during concept acquisition are oriented by notions such as those of *concept image* and *concept definition* (Vinner & Hershkowitz, 1980; Tall & Vinner, 1981; Vinner, 1983, 1991). From such a perspective, understanding a mathematical concept requires building a *concept image* for it; which means developing a collection of impressions and experiences related to the concept. The authors refer to the *concept definition* as “…a form of words used to specify that concept” (Tall & Vinner, 1981, p.152). In our own research (Pinto, 1998; Pinto & Tall, 1999, 2001, 2002) we came to consider a distinction between the *concept definition*, which we take as personal (given that at times it includes peculiar ways of expressing the mathematical notion), and the *formal definition* as conceived by the mathematicians and presented in scientific texts and texts books. Such a distinction allowed us to follow students during the process of concept formation by focusing on whether they construct mathematical theory from formal definitions or their concept images, or a combination of the two. Throughout the process, learners occasionally evoke aspects related to the concept name which are not coherent with the *formal definition*, or with their own *concept definition*. Such aspects are in many cases carried by experiences in everyday life associated with the concept name, as it may be used in natural language (Tall & Vinner, 1981; Vinner, 1991). From this perspective, an investigation on doing mathematics in any of the three worlds might not eliminate the role of the mother language, in particular, of pieces of the mother language in use in mathematics - such as the word ‘limit’, with its technical meaning when referring to the mathematical notion (see, for instance, Tall & Vinner, 1981; Vinner, 1991; Monaghan, 1991).

Another important dimension in the relationship which is set up by Tall´s research between language and mathematical experience is given by his great interest in the use visualisation during the process of mathematical concept formation. Tall considers visualisation as an “intermediate representation which enriches the mathematical experience with additional concrete exploration of the object". For just a few examples, see Tall (1986 a, 1986b, 1989, 1991b), Tall and Thomas (1989), Tall, Blockland and Kock (1990). Looking back from the perspective of this new theory, the mathematical activities which are presented in those papers essentially relate different modes of operation within each of the three worlds of mathematics, bringing, in fact, embodied modes to the actual activity. At the time, the research was, in part, suggested by cognitive psychologists who had identified different styles of learning, named *serialist* (or *analytic*) and *global* (or *holistic* or *intuitive*)[v], and by the mathematics educators who had indicated that the current approaches to mathematics lead to a narrow symbolic interpretation of the content, being *serialist* in style[vi] (Robert & Boschet, 1984; Tall, 1986; Blacket, 1987; Thomas, 1988).

For Tall and Thomas (1989), "activities to encourage the development of *holistic* thinking patterns, linking them to *sequential*, deductive thinking, may be of benefit in aiding students to obtain a better overall performance in mathematics"[vii]. Going further, Tall and Thomas saw the computer as a suitable tool to encourage *versatile learning* and to develop a "dynamic approach" to algebra. They refer to a combination of activities emphasising conceptualization and the use of mental images rather than just skill acquisition. From that perspective, mental images are not restricted only to pictures in the traditional sense and may include dynamical representations. Such images may also account for symbolic concepts, "provided that they are capable of being visualised and mentally manipulated".

Supported by these ideas, Tall and Thomas´s first module of work is designed to encourage the learner's ability to visualise the concept of an algebraic variable. In essence, they provide experiments within the cross-boundaries of the embodied and the symbolic-proceptual worlds, relating the embodied activities with the symbolic-proceptual mathematical mode. In addition to the iconic representations, enaction is brought to the activities through dynamical experiences using the computer.

More recent research also brought the embodied modes of operation through gestures and bodily experiences. For instance, experiments described in Watson and Tall (2002) focus on the relationship between embodied and symbolic modes of operation with the concept of vectors. With special reference to the embodied mode of operation, Watson and Tall observed that the range of physical experiences gives very different meaning to the concepts, and play a role in the process of knowledge construction. I perceive an analogy among these results and those explored on the proceptual-symbolic mode: we could recall the situation where learners distinguish processes based on different procedures to carry them out. This is also the case for vector addition, since 'vector as a journey' leads more naturally to the use of the triangle law for addition and 'vector as a force' leads more naturally to the parallelogram law. Watson and Tall found that a natural interpretation of vector addition by using the parallelogram law may be experienced by our bodies being pulled forward by our two arms, which causes the individual to move in a forward direction representing the combination of the two forces. The two rules are theoretically identical, but students often appear to conceive them as different. In more general terms, a flexible idea of vectors as a free vector with given magnitude and direction seems to be an obstacle for learners. From interviewing students the researchers came to the notion that a set of actions in the embodied world could be identified if they have the same effect. From this perspective, the triangle and the parallelogram law for addition of vectors would be identical. An experiment based on such reconstruction of the notion of 'identity' within the embodied world was carried out to help build a notion of free vector.

Watson and Tall developed the experiment physically translating an object (for example a triangle) on a table, pushing it without turning it. Encouraging students to imagine the beginning and the end position of each moving point, focusing on the hand as a whole or on a single finger that moves during the action, they agreed that the shift could be represented by arrows. The notion of free vector emerged from the various possibilities of enactive representations which could be identified by the recall of the effect. In a step further, Watson and Tall realized that the idea of a set of actions having the same effect in the embodied world could correspond to the notion of different procedures having the same output at the process level within the symbolic-procedural world. This observation led not only to the design of many experiments relating the worlds of mathematics but also to the reinterpretation of earlier experiments.

Different hypotheses emerging from psycholinguistics have been feeding a debate on the relation between gestures, language and conceptual processing. Amongst those, Kita (2000) recognises gestures as involved in the conceptual planning of the message to be verbalized. This view gives to gesture a role not only in speaking, but also in the thinking processes and, therefore, confer on it a role in the learning context. Kita understands gesture and speech as related to two complementary modes of thinking he named spatio-motor thinking and analytic thinking.

In mathematics education, research by Edwards (2003) draw upon McNeill's (1992, 2000) ideas, who acknowledges a distinction between the two complementary modes of thinking but sees them as integrated into a multiple representation of a unified task. Edwards has adopted McNeill´s identification of *deictic* gestures as those pointing to an object, *metaphoric* gestures which represent an abstract idea with no physical form and *iconic* gestures which recall a semantic context. Further analysis in the mathematical learning context led Edwards (2003) to refer to *iconic-physical gestures* and *iconic symbolic*-gestures when distinguishing between those gestures associated with symbolic inscriptions or with procedures on these inscriptions. From the viewpoint of those researchers, we could say that gestures come into play in the mathematical development by revealing to a certain extent a 'knowledge in construction'. Reflecting on Tall´s experiments described above, we must say they are grounding language and mathematical experience within the embodied mode of operating on mathematical notions, which is brought into all three worlds of mathematics.

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[1] In those theories, the term refers to the assumption of a biological and cultural interdependent relationship between animal and environment (Maturana & Varela, 1987). Human thoughts are shaped by sensori-motor experiences, through the use of metaphors (Lakoff & Johnson, 1999).

[2]In its turn, once concentrating on unconscious thoughts with no mention to the conscious ways in which we do and think about mathematics, Lakoff and Nunez misses historicity and also important cultural and contextual aspects which interfere in knowledge construction. For instance, the roots of today´s complex number theory as retraced by Lakoff and Nunez (2000, p.420) presents them as we are nowadays

presented with complex numbers – as points in the plane. Neverthless, in the sixteenth century, Tartaglia and Cardano (in his Ars Magna, 1545) performed calculations when solving cubic equations which led to the square roots of negative numbers. They happened to cancel and, in the end, give a genuine real solution. In mathematicians' practices at that time, such 'numbers' were not initially linked to the individual's geometric representation. This shows that the use of symbol manipulation can lead, in the end, to a conceptual metaphor (in Lakoff and Nunez´s terminology); and not just the only other way round (Mejia, 2004).

[3] Research has shown that students may have more than one procedure to carry out a process but may not have a flexible concept of the process itself (Ali and Tall, 1996).

[4] Structure theorems may suggest results that turn out not to be true. For instance, the 'completion' of the rational numbers to add the irrationals to give the real number line is often conceived as being the ultimate destination, with the real numbers filling out the whole line, banishing the possibility of infinitesimal quantities on the line. Yet this conceptualization limits our imagination and is simply untrue in the formal world of mathematics. It is very simple, mathematically, to place the ordered field R in a larger ordered field (e.g. the field of rational functions consisting of quotients of polynomials in an indeterminate x) which can be mentally imagined as a more sophisticated line that can be magnified to 'see' infinitesimal quantities (Tall, 2002a).

[5] The first is characterized by "immediately breaking a problem or task into its component parts, and studying them step by step, as discrete entities, in isolation from each other and their surroundings." The second is characterized by "an overall view, or seeing the topic/task as a whole, integrating and relating its various subcomponents, and seeing them in the context of their surroundings." (Brumby, 1982, p.244)

Brumby identified three distinct groups of students: those who consistently used only serialist/analytic strategies, those who used only global/holistic strategies, and those who used a combination of both, who she described as *versatile learners.* Overall 42% of her sample maintained a serialist/analytic style, 8% were global/holistic and 50% were versatile.

[6] Results from empirical research have shown that the most successful students in advanced mathematics are those who have more than one representation available. Those who are limited to one representation, which is generally symbolic or numeric are also limited in their ability to solve a wide range of problems.

[7] Tall and Thomas see as vital "the ability to switch one's viewpoint of a problem from a local analytical one to a global one, in order to be able to place the details as part of a structured whole".

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Theoretical and Methodological Implications of a Broader Perspective on Mathematical Argumentation

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ABSTRACT: *The analysis of student-generated proofs and purported proofs has become an important part of mathematics education research. In this paper we argue that rather than concentrating on analysing only purported proofs, there are advantages to analysing student generated arguments of all kinds: those which reduce uncertainty in a conjecture, and those which remove uncertainty in a conjecture. In this paper we discuss the rationale for this approach and argue that adopting such a stance has both theoretical and methodological implications for mathematics education researchers. We particularly focus on what implications our perspective has for Tall's (2004) theory of three worlds of mathematics.*

Key words: *Argumentation, Methodology, Proof, Qualifier, Three Worlds of Mathematics, Toulmin.*

TOULMIN'S ARGUMENTATION STRUCTURE

Toulmin (1958) advocated an approach to analysing arguments that dramatically departed from traditional approaches to formal logic. He was concerned with the semantic content and structure in which an argument is located, and downplayed the importance of its formal logical coherence. This manner of analysing argumentation has become known as 'informal logic' in order to emphasise its differences from formal logic.

Toulmin's scheme has six basic types of statement, each of which plays a different role in an argument. The *conclusion* (C) is the statement of which the arguer wishes to convince their audience. The *data* (D) is the foundations on which the argument is based, the relevant evidence for the claim. The *warrant* (W) justifies the connection between data and conclusion by, for example, appealing to a rule, a definition or by making an analogy.[i] The warrant is supported by the *backing* (B) which presents further evidence. *The modal qualifier* (Q) qualifies the conclusion by expressing degrees of confidence; and the *rebuttal* (R) potentially refutes the conclusion by stating the conditions under which it might not hold. Importantly, in any given argument, not all of these aspects will necessarily be explicitly verbalised. These six components of an argument are linked together in the structure shown in Figure 1.

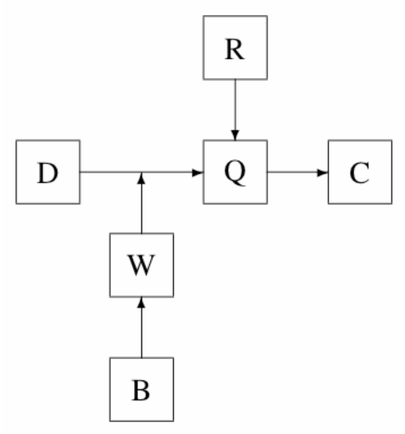

Figure 1. Toulmin's Scheme.

In the field of mathematics education, many researchers have applied Toulmin's scheme to analyse arguments constructed by students. However, it has become commonplace to use a reduced version of the scheme by omitting the modal qualifier and rebuttal. Such a stance has been adopted by, among others, researchers studying classroom interaction (Krummheuer, 1995), basic number skills (Evens & Houssart, 2004), logical deduction (Hoyles & Küchemann, 2002; Weber & Alcock, 2005), geometry (Cabassut, 2005; Pedemonte, 2007), and general proof (Yackel, 2001).

Inglis, Mejia-Ramos and Simpson (2007) argued that without using Toulmin's full scheme it is impossible to accurately model the full range of mathematical argumentation. They gave research active mathematicians a series of conjectures and asked them to decide whether or not the conjectures were true, and to prove or disprove these claims. It was found that these mathematicians regularly constructed arguments with non-deductive warrants in order to *reduce* rather than *remove* their doubts about a conjecture's truth value. Inglis et al. pointed out that it would be impossible to accurately model such arguments without incorporating the modal qualifier part of Toulmin's (1958) scheme.

MODELLING MATHEMATICS WITH TOULMIN'S SCHEME

Consider the following extract, taken from Inglis et al. (2007). Chris, a postgraduate research student, was investigating the conjecture "if p_1 and p_2 are primes then p_1p_2 is not abundant" [ii]. Chris tried a couple of examples (2 and 3; 5 and 97) and said

Chris: Since the smallest numbers I could find to put in this equation showed it was perfect and in the larger limit it showed p_1p_2 was deficient. So it's possible it holds for all p_1, p_2.

Interviewer:	Do you think it does?
Chris:	I think it probably does. But I'm not sure why [laughs]. Yeah, the fact that this is, in some ways, sort of monotonic. In other words, I know that this statement is true for large p_1, p_2; I know it's true for small p_1, p_2; so I feel therefore that it should be true for p_1, p_2 in the middle. Umm, but I might have to do some work to show that.

This argument is modelled using Toulmin's (1958) scheme in Figure 2. Crucial to Chris's argument was a non-absolute modal qualifier: he was not deducing his conclusion with certainty, but instead offered plausible evidence using a non-deductive warrant.

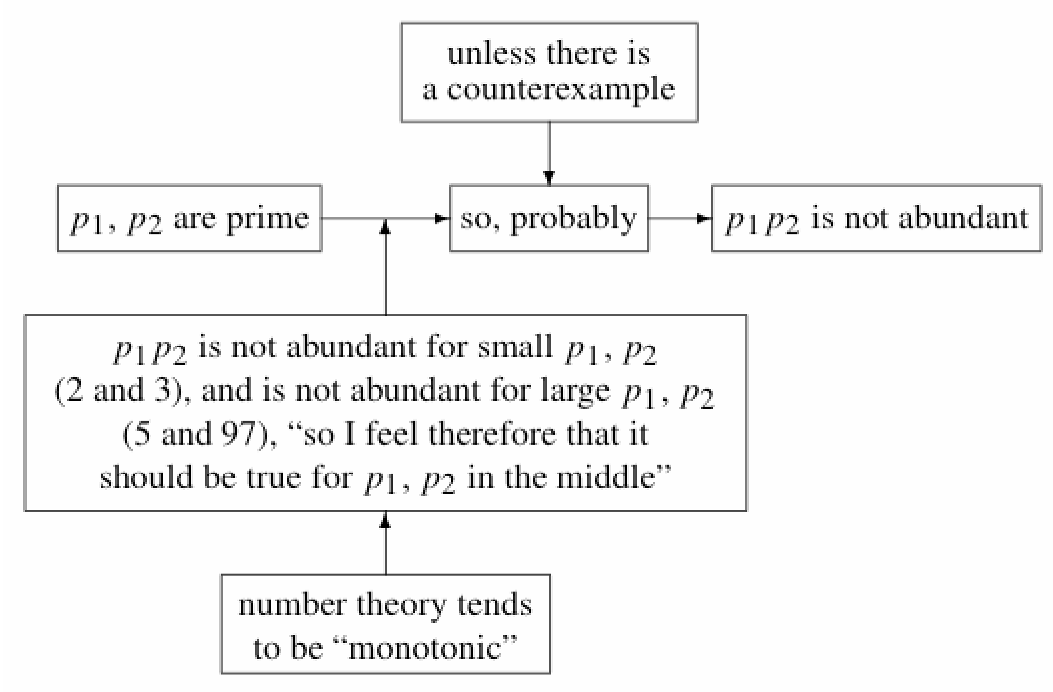

Figure 2. A model of Chris's argument.

Inglis et al. (2007) found that mathematics research students frequently used arguments where conclusions, as in the example of Chris, were drawn with plausibility rather than certainty. They suggested that earlier researchers, who omitted the qualifier and rebuttal components of Toulmin's scheme would not have been able to successfully model such arguments, and arguably would have rejected them as being non-mathematical.

The goal of this paper is to explore the methodological and theoretical implications of adopting a broader framework which recognises that arguments are frequently constructed where plausibility and high confidence are used instead of certainty and absolute confidence.

THEORETICAL IMPLICATIONS OF A BROADER FRAMEWORK

Several researchers have proposed frameworks for discussing the evidence students use to persuade others and convince themselves of the truth of mathematical statements. For example, Harel and Sowder (1998) put forward a typology of students' so-called proof schemes, the types of arguments which allow a student to "remove her or his own doubts about the truth of an assertion", and which they would use to persuade a third party of the truth of the assertion. A proof scheme, then, is about *removing* uncertainty, whereas we have shown that arguments in mathematics may be given in an attempt to *reduce* uncertainty; this broader range of argumentation is not accounted for in Harel and Sowder's theoretical framework. Tall's (2004) recent theoretical framework of warrants for truth in the three worlds of mathematics is also affected by this focus.

Tall's Three Worlds of Mathematics

Gray and Tall (2001) put forward the suggestion that there are three distinct types of concept in mathematics; and the idea is appealing. Recognising the dangers of claiming that all mathematical objects are encapsulated processes (Dubinsky, 1991; Sfard, 1991), Gray and Tall proposed that there are three "fundamentally different" types of concept, the ontological origins of which should be considered separately:

> "For several years now [...] we have been homing in on three [...] distinct types of concept in mathematics. One is the embodied object [...] Another is the symbolic procept [...] The third is an axiomatic concept in advanced mathematical thinking."
>
> (Gray & Tall, 2001, p. 70)

Tall has since expanded this threefold categorisation into the notion of "three worlds of mathematics", moving from the initial characterisation of different types of mathematical concepts, towards a distinction between three ways of thinking about and operating with these concepts,

> "three distinct but interrelated ways of mathematical thinking each with its own sequence of development of sophistication […] that in total spans the range of growth from the mathematics of new-born babies to the mathematics of research mathematicians [and outline] how different individuals may develop substantially different paths on their own cognitive journey of personal mathematical growth".
>
> (Tall, 2004, p.158)

Tall (2006) described these three worlds of mathematics as follows:

> "– an object-based *conceptual-embodied world* reflecting on the senses to observe, describe, define and deduce properties developing from thought experiment to Euclidean proof;

− an action-based *proceptual-symbolic world* that compresses action schemas into thinkable concepts operating dually as process and concept (procept);

− a property-based *formal-axiomatic world* focused to build axiomatic systems based on formal definitions and set-theoretic proof." (p. 197)

More generally, these worlds categorise conceptions of mathematical notions (visuospatial in the embodied world, proceptual-symbolic in the proceptual world, and formally defined in the formal world), ways of conceptualising these notions (through perception, action, and reflection in the embodied world; through encapsulation in the proceptual world; and through the specification and deduction of properties in the formal world), and ways of operating with these notions (physical manipulation and thought experiment in the embodied world, symbol manipulation in the proceptual world, and deductive, axiomatic structuring in the formal world). Regarding the particular case of mathematical argumentation, and following Rodd (2000), Tall (2004) used the term *warrant for truth* to denote that which secures someone's personal belief and understanding in what is claimed to be known. He noted that each of the worlds of mathematics can be further characterised with a particular kind of *warrant for truth*:

- In the conceptual-embodied world something is initially true if one knows it is true through perception and action in the physical world. Later on, thought experiments and deduction on generic examples become the source of personal conviction.
- In the proceptual-symbolic world something is true if it is warranted by a correct calculation or symbolic manipulation. In this world, "algebra gives us a warrant for truth, based on implicit use of 'rules of arithmetic', by manipulation of symbols" (Tall, 2002, p.103).
- In the formal-axiomatic world something is true if it can be deduced from a set of axioms and basic definitions. In this world, formal mathematical proof is the warrant for truth.

The notion of *warrant for truth* is mainly used to contrast mathematical proof to that which engenders understanding and personally convinces someone of the truth of a given statement. Indeed, as Rodd (2000) attractively summarised in the title of her paper, a proof may fail to personally convince someone of the truth of the proved statement; and people often understand through, and are personally convinced by, arguments that would not be considered as mathematical proofs. Initially, it becomes clear that one difficulty of using Toulmin's scheme alongside Tall's worlds of mathematics lies in the differing use of the term warrant in their descriptive accounts of argumentation. While Toulmin allows warrants to be qualified, rebutted, and backed in a particular way, Tall's warrants for truth "secure knowledge", implying that the conclusion surely follows from the data, and the argument allows no rebuttal. Therefore, like the notion of proof scheme, warrant for truth is defined to refer only to arguments that carry full conviction, leaving out the broader range of mathematical argumentation that is constructed to *reduce* doubts.

To summarise, Tall (2004) suggested that students' mathematical development involves following (potentially different) routes through three distinct worlds of mathematics. This growth involves changing, depending on the world of mathematics students operate on, the kind of warrants/backings that support fully-qualified conclusions.

Warrants for truth, or warrant-qualifier pairings?

When seen within a broader framework of argumentation and proof, however, Tall's (2004) assertion may require modification and expansion. We suggest that while students' mathematical development certainly involves a change in the types of warrants/backings that support fully-qualified conclusions, it also involves learning to properly match warrants and qualifiers (in the manner practiced by experts). For instance, as students' mathematical thinking develops, they should grow somewhat sceptical of non-generic pictures, which often lead to partially qualified conclusions. The same is true about the numerical evaluation of a conjecture in a small sample of cases, which would not support a fully qualified conclusion for an expert. However, as shown by Inglis et al. (2007), successful mathematicians, purportedly used to operating in the formal world of mathematics, often use these types of warrants in their arguments. They do not use them as warrants *for truth*; they use them to *reduce* their doubts regarding a particular conjecture, which may help them decide whether to try to prove or disprove the conjecture, and choose a strategy for doing it.

For mathematicians, the key difference between the modes of operation used in the three worlds identified by Tall is related to the types of warrant-qualifier pairings they construct. All three modes of operation seem to be useful to research mathematicians; they simply use a wider spectrum of qualifiers to accompany conclusions reached with different types of warrants.

In the following section we argue that working with theoretical constructs that focus only on extreme levels of conviction (like *proof schemes* and *warrants for truth*) and disregarding the appropriate use of non-deductive warrants to reduce doubt, may bias empirical researchers into not only interpreting a whole range of qualifiers as absolute, but also leading them to misjudge the appropriateness of the arguments they analyse.

METHODOLOGICAL IMPLICATIONS OF A BROADER FRAMEWORK

Many researchers have studied the types of arguments which students find convincing and persuasive, both in terms of the arguments they themselves construct, and of the way they evaluate externally produced arguments (e.g. Harel & Sowder, 1998; Mejia-Ramos & Tall, 2005; Raman, 2002; Segal, 2000). Many researchers have found that students are happy to use examples to convince themselves of a statement's truth. The use of this sort of empirical evidence has been found across a wide range of mathematical topics, being used by secondary school pupils (Porteous, 1990; Coe & Ruthven, 1994; Edwards, 1998; Healy & Hoyles, 2000; Küchemann & Hoyles, 2004), secondary school teachers (Knuth, 2002) and undergraduates (Moore, 1994; Goetting, 1995; Recio & Godino, 2001).

Recio and Godino (2001) give several illustrations of the phenomena of using examples to gain conviction. For example, when asked to prove that the difference between the squares of two consecutive natural numbers is always an odd number, and that it is equal to the sum of those consecutive numbers, one student wrote:

> "36 – 25 = 6 + 5
>
> 11 = 11
>
> We have
>
> 49 – 36 = 7 + 6
>
> 13 = 13
>
> We have $A^2 - B^2 = A + B$" (Recio & Godino, 2001, p.86).

This student's response was described by Recio and Godino in these terms: "the student checks the proposition with examples, and asserts its general validity".

When working within the broader framework of argumentation and proof advocated by Inglis et al. (2007), this student's response can be interrogated more deeply. The response can be analysed with respect to the six components which make up Toulmin's scheme:

- Data: 36 – 25 = 6 + 5 and 49 – 36 = 7 + 6.
- Warrant: not explicated.
- Backing: not explicated.
- Qualifier: not explicated.
- Rebuttal: not explicated.
- Conclusion: $A^2 - B^2 = A + B$.

The student, based on empirical data of two examples, concludes that $A^2 - B^2 = A + B$. However, every other part of the argument – including, crucially, the modal qualifier – has not been verbalised. During their data analysis Recio and Godino *inferred* that the student's argument used an absolute qualifier, and categorised the argument as an empirical proof. Compare this student's response to the behaviour of another of the students interviewed by Inglis et al.. Andrew, when faced with a request to investigate the conjecture that divisors of deficient numbers are deficient, looked at the case of 10, found that 2 and 5 are deficient and concluded

> "Yeah, so apparently it works here. Yeah ok, so apparently, it seems to me that it's true." (Inglis et al., 2007).

Here Andrew argued that the statement was true and, using a non-absolute qualifier, he said that it "seems" to be true. At the time that this interview took place Andrew was in the final year of a PhD in functional analysis at a top ranked UK university: Andrew cannot be said to be mathematically naïve. He did, however, like Recio and Godino's student, appear to use a version of Balacheff's naïve empiricism to gain conviction. It is quite possible that Recio and Godino's participant was using a qualifier similar to that used by Andrew. We do not know, and neither do Recio and Godino, as the qualifier was not made explicit.

Sometimes even explicitly verbalised non-absolute qualifiers have been ignored by mathematics education researchers. Coe and Ruthven (1994) disregarded a participant's fully verbalised qualifier when analysing the use of examples in proving situations. Bill, a 17 year old student, checked a statement was true for six cases and said that it was "safe to make a conjecture". When pressed by the interviewer he estimated that he had a "percentage certainty" in the "high nineties". Despite this explicit use of a non-absolute qualifier, Coe and Ruthven wrote that Bill's "*certainty* appears to be gained just by checking a relatively small number of cases" (p.50, our emphasis).

The methodological dangers of adopting a narrow view of argumentation and proof, where either somebody has 100% conviction or 0% conviction in their conclusion, can be easily seen from Coe and Ruthven's (1994) analysis. When working within such a narrow framework, Bill's high, but not absolute, qualifier was analysed as if it were absolute. Consequently, his apparently normative argument – in the sense that it matched the practice of the experts studied by Inglis et al. (2007) – was presented as if it were in some sense non-mathematical.

Furthermore, a lack of awareness of the varying degrees and types of conviction may conceal important aspects of a student's argument. The following example illustrates this point. Linvoy is a participant in an ongoing longitudinal study concerning undergraduate mathematics students' production and validation of proofs. In his second one-to-one interview with the second author, Linvoy was asked to work on the following task (based on a problem by Raman, 2002):

> Determine whether the next statement is true or false (explain your answer by proving or disproving the statement): The derivative of a differentiable even function is odd.

After working for a couple of minutes with the definitions of even/odd function and that of the derivative of a function, Linvoy said:

> "What I'm thinking is that because this does not look like it's going anywhere, although it might go somewhere, what I'm thinking is perhaps if I think of a few functions maybe I might think of a counterexample, although if it's true obviously I won't. But let's see.
>
> Perhaps if I think of it in a bit of a less formal way, if I just think of it as the derivative of a function being the gradient at a particular point… and… um… [drew the graph of an even sinusoidal function] I think of some graph like this which happens to be [inaudible] because it's an even function, and then… yes, I suppose one way of looking at this is that at any point here, like say you take this

> point [picked a point of the function in the first quadrant], you've got this gradient going like that, if you compare the exact other part, you've got the gradient going in the opposite direction because it's exactly, ummm, it's like a mirror image, so... and that is, that is odd, because that gradient would be exactly the negative of that gradient.
>
> So, yeah, I suppose, just from that basic example I suppose that intuitively does, does seem like it would make sense, but what about… maybe it's just the example of the function I've chosen, but that can't be right, because, what I'm thinking is… if you take, I mean, any [drew another set of axis]… this can do whatever it likes, but say we're interested at some point where it's doing that [drew a small portion of the graph of a generic function in the first quadrant], then it's going to have that gradient and then if we transfer it it's going to be like that, so it's going to have that gradient, which would be the exact opposite of that… yeah, thinking of it like this, it does seem true, just thinking of it in those terms, ummm… like before I'd be happier if I could think of some way to prove it…
>
> [Pointed at the graph of the generic example] That, that's convinced me, so I suppose if I go back to look at this [pointed to his symbolic approach]."

Here, Linvoy used a variety of qualifiers like "[it] does seem like it would make sense", "it does seem true", and "that's convinced me". These qualifiers hint a growing level of conviction regarding the truth of the statement, and indicate to the researcher different factors that can be accountable for this growth.

After working with the definitions, Linvoy claimed not to know whether the statement is true or false. However, after studying one particular example of an even function, he concluded that the statement did seem to make sense. At this stage, he did not appear to believe the statement is definitely true; instead this experiment suggested to him that his initial search for a counterexample was not going to be fruitful, and that the statement was actually plausible. Immediately after this, Linvoy voiced the possibility of a rebuttal by saying: "maybe it's just the example of the function I've chosen". This led him to conduct another experiment, this time by exploring a generic example represented by an imaginary graph that could "do whatever it likes", while drawing and focusing only on a small portion of that graph. With this new warrant, Linvoy qualified the conclusion with a higher, but still non-absolute "it does seem true". Finally, after expressing his appeal for a proof and studying his generic example for a couple of seconds, Linvoy claimed to be convinced and returned to his symbolic approach. Although Linvoy said he was not fully satisfied with his generic example, he claimed to be convinced by it (for a wider discussion of different types of persuasion see Inglis & Mejia-Ramos, 2005). Failing to notice the different kind of qualifiers used by Linvoy would have blurred the complexity of his reasoning and the crucial role that particular and generic examples played in it.

Simply put, our position is this: when studying students' argumentation practices researchers need to be aware of all components of an argument, not merely the data, conclusion and warrant. Of special importance is the modal qualifier. If a student does not verbalise his/her qualifier then the researcher has two choices: to infer it, or to probe the student with the hope of encouraging them to verbalise it. Earlier researchers have primarily concentrated on the first option, inferring an absolute qualifier. In some cases this may be legitimate, but in many cases it may not be. We suggest that, when probing the student to encourage them to verbalise their argument's qualifier proves impossible or unsuccessful, researchers should only infer it from non-verbal clues if they can provide *explicit* convincing reasons for the validity of their inference.

CONCLUSIONS

In this paper we have attempted to explore the theoretical and methodological implications of adopting a broader framework on proof and argumentation when studying students' approaches to mathematics. Following Inglis et al. (2007), we believe that it is necessary to model students' argumentation using all components of Toulmin's (1958) scheme, in contrast to the established practice of earlier researchers. By adopting this strategy a broader spectrum of mathematical argumentation can be analysed and discussed.

Adopting this broader framework, however, brings with it some new and added requirements. Focusing on Tall's theory of the three world of mathematics, we have argued that relying solely on theoretical constructs like warrants *for truth,* which focus on a extreme level of conviction, to model human belief in mathematics, limits the theory's power to describe the broader range of actual mathematical argumentation. We have therefore suggested that the theory may benefit from shifting the focus on warrants for truth towards a more encompassing focus on warrant-qualifier pairings as described by Inglis et al. (2007).

Another consequence of adopting the framework we advocate is methodological. When participants in research studies did not explicitly verbalise the qualifiers they were constructing in their arguments, previous researchers have tended to infer an absolute qualifier (or, worse, replace an explicitly verbalised non-absolute qualifier with an absolute one). This approach has serious problems. In this paper we have argued that there are two tenable ways for researchers to proceed in this situation: either they must prompt their participants, through the use of probing questions, to verbalise their qualifiers; or they must provide explicit, convincing and detailed reasons for why the inferences they have made about their participants' qualifiers are valid.

Notwithstanding these theoretical and methodological implications, we believe that utilising the full Toulmin argumentation scheme in mathematics education research is an important improvement on the approach adopted by earlier researchers. Using the full scheme allows researchers to study a much larger range of students' purported and generated proofs, in a more legitimate fashion than has been allowed by existing frameworks.

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[i] Toulmin's (1958) use of the word 'warrant' is not identical to how the term has been used by some of the mathematics education literature. Rodd (2000), for example, saw a warrant as *removing uncertainty*, whereas Toulmin was more flexible, accepting that a warrant can be qualified with a modal qualifier to merely *reduce uncertainty*.

[ii] A number is abundant if the sum of all its factors (excluding itself) is larger than the number, deficient if this is smaller than the number and perfect if this is equal to the number.

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Mathematical Proof as Formal Procept in Advanced Mathematical Thinking

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ABSTRACT**:** *In this paper the notion of "procept" (in the sense of Gray & Tall, 1994) is extended to advanced mathematics by considering mathematical proof as "formal procept". The statement of a theorem (as a symbol) may evoke the sequence of deductions which form the proof (as a process). That proof may contain sequential procedures and require the synthesis of distinct cognitive units. Alternatively, the symbol may evoke the general notion of the theorem as an object like a manipulable entity to be used as inputs to other theorems. Therefore, a theorem could act as a pivot between a process (method of proof) and the concept (general notion of the theorem). The paper hypothesises that a mature theorem-based understanding (in the sense of Chin & Tall, 2000) should possess the ability to consider a theorem as a "formal procept", and that it takes time to develop this ability. Some empirical evidence reveals that only a minority of the first year students, studying in one of the top five ranked mathematics departments in the UK, could recognise a relevant theorem as a "concept" and none had the theorem with the notion of its proof as a "formal procept". A year later some more successful students showed their conception of the theorem as a "formal procept" and the capability of manipulating the theorem flexibly.*

Key words: *advanced mathematical thinking, equivalence relation & partition, formal procept, mathematical proof, procept, theorem-based understanding.*

INTRODUCTION

Mathematical proof is one of the most important aspects of formal mathematics. From most mathematics textbooks at the university level, it can be seen that mathematical proof can be considered as a sequence of statements using only definitions and preceding results, axioms, or theorems. Theoretically, the *process* of a mathematical proof occurs when the proof is developed and subsequently seen as the deduction of the statement of the theorem from definitions and the specified assumptions. A proof becomes a *concept* when its associated theorem can be used as an established result in future proofs without the need to unpack it down to the individual steps of its proof.

This paper focuses on this sequence of proof as a process of deduction becoming encapsulated as a concept of proof in a manner that would seem natural to most mathematicians.

It should be noted that there are some competing theories in the literature related to this encapsulation. For example, Dubinsky, Elterman and Gong (1988) conducted a research study on the student's construction of quantifier, focusing on the use of quantified statement as a process becoming a mental object by applying the quantifiers. Dubinsky and his colleagues suggested that the process occurs through reflective abstraction. During the abstraction, a predicate with variables can be conceived as a mental process that can be encapsulated into a mental object by the process of quantification. However, Pinto and Tall (2002), in contrast, report a case study of a student who constructs mathematical formalism not through the process of quantification, but through his own visuo-spatial imagery. Rather than constructing new mental objects through cognitive processes, Pinto and Tall (2002) showed how some students are capable of building formal proofs by the reconstruction of prototypical imagery used in thought experiments.

ORIGINAL NOTION OF PROCEPT

The original notion of "*procept*" was developed by Gray and Tall (1994). It is taken to be characteristic of symbolism in many areas of mathematics, and defined in the following terms:

> *An elementary procept is the amalgam of three components: a process which produces a mathematical object, and a symbol which is used to represent either process or object.*
>
> *A procept consists of a collection of elementary procepts which have the same object.*
>
> *(Gray & Tall, 1994, p.120)*

This can be seen as a "descriptive definition", in the sense of a definition in a dictionary, rather than a "prescriptive definition", in the sense of an axiomatic theory. However, if we consider the definition of "procept" from a prescriptive perspective, it seems possible to extend the original notion of "procept" to the notion of formal proof, which can be called "*formal procept*", by adding the following analysis.

EXTENDED NOTION OF PROCEPT: FORMAL PROCEPT

It should be noticed that there are three components of an elementary procept: *process*, *object*, and *symbol*. Now we can put the frame of Gray and Tall's "procept", particularly in the form of an "elementary procept", on the notion of formal mathematical proof. The *symbol* is the statement of what is going to be proved (which can be a theorem). The *process* is the deduction of the whole proof. And the *object* is the concept of the general notion of proof. Therefore, a theorem, for example, which is considered as a formal procept could act as a pivot between a process (method of proof) and the concept (general notion of the theorem).

It suggests that the individual is not considered as having conceived the real meaning of a theorem until the theorem has become a formal procept in one's understanding. With this interpretation we could see the role of a symbol as being pivotal not only in elementary mathematical thinking but also in this area of advanced mathematical thinking in allowing the flexible movement between using a symbol as a concept to reflect on and link to other concepts, and as a process formed from the detailed steps to deduce a proof. However, an immediate argument arises. One might suggest that this analysis does not always hold because even mathematicians sometimes use theorems without fully understanding their proofs. This may, however, be an *advantage* to this analysis, for it simply shows that such individuals are *not* using theorems as formal procepts; they only have *part* of the structure, usually the statement of the theorem which they then use as an ingredient in another proof without fully understanding the *totality* of the structure. In this analysis, the whole notion of a theorem is only fully grasped when the notion of proof of the theorem is also assimilated in the individual's understanding.

MEAN VALUE THEOREM AS AN EXAMPLE OF "FORMAL PROCEPT"

While there are many examples of the formal precept, the Mean Value Theorem [MVT] is a suitable example to illustrate this analysis. One proof of MVT applies Rolle's Theorem. To prove Rolle's Theorem, we refer to the Maximum-Minimum Theorem and the theorem which states that c is a critical point of f if $f'(c) = 0$ where c is an interior point of an interval I. Further, to prove the Maximum- Minimum Theorem, we apply the Least Upper Bound Axiom and the Heine-Borel Theorem. If the students understand this nested sequence of theorems, which should themselves be considered as individual "formal procepts", when proving MVT, it seems clear that the students have developed the whole notion of MVT as a "formal procept" since they could integrate the relevant ideas together. However, if the students only recite the products (the statements of the theorems) without understanding the ideas of the proofs, they may not be able to apply these theorems flexibly. When more and more formulae and theorems are to be learned, the less able students will become trapped in reciting all these products which increase the burden upon an already stressed cognitive structure.

HIERARCHY OF THE DEVELOPMENT OF SYSTEMATIC PROOF

Chin and Tall (2000) postulate a hierarchy running through the development of systematic proof, in stages consisting of *concept image-based*, *definition-based*, *theorem-based*, and *compressed concept-based*. These stages show successive compressions of knowledge in the sense suggested by Thurston (1990).

The first stage, which is concept image-based, sees the student having a concept image of a particular concept built from experience, but at an intuitive stage of development. The transition to the definition-based stage involves the first compression. From amongst the many properties of the concept-image, a number of generative ideas are selected and refined down to give the concept-definition. During the definition-based stage, the definitions are used to make deductions, all of which are intended to be based explicitly on the definitions. Many students, however, remain in the concept-image based stage, basing their arguments not on definitions and deductions, but on thought experiments using concept images (Tall & Vinner, 1981; Vinner, 1991). Bills and Tall (1998) introduce the term 'formally operable' to describe a definition (or theorem), proposing that:

> *A (mathematical) definition or theorem is said to be formally operable for a given individual if that individual is able to use it in creating or (meaningfully) reproducing a formal argument.*
>
> *(Bills & Tall, 1998, p.104)*

Tracing the development of five individuals over two terms in an analysis course, focusing on the definition of "least upper bound", Bills and Tall found that many students lack operable definitions, relying only on earlier experiences and concept images. The further noted that it was also possible for a student to use a concept without an operable definition in a proof, using imagery that happens to give the necessary information required. This suggests that the development from the concept-image based stage to the compressed notion of operable definition is a difficult one for many students. Even so, they are then expected to move on to the next, theorem-based stage, when theorems that have been proved (through the process of proof) are now regarded as being compressed into *concepts* of proof, to be used as entities in the process of proving new theorems. For this to be fully successful, it is suggested that students who have developed a mature theorem-based understanding should possess the ability to consider a theorem as a "formal procept". It is further hypothesised that individuals with this capacity to use theorems flexibly as processes or concepts are developing a compressed concept level of mathematical thinking that enables them to think with great flexibility and conceptual power.

The evidence outlined below shows that only a few students initially understand the notion of proof as a formal procept, but this empirical research also shows that, over time, more students grasp the subtlety of the idea.

EMPIRICAL STUDY

Research design

The purpose of the empirical study, which included a cross-sectional probe and a longitudinal probe, was to obtain a comprehensive perspective of the novice mathematics students' general understanding of some relevant theorems, then to investigate whether and how the students' understanding improves. This research was a qualitative study and adopted a "between methods" triangulation (Denzin, 1970) as the methodological structure. Two main methods of data collection adopted in the study are: (1) a questionnaire containing "open-form" questions (Slavin, 1984) to develop a *broader* perspective on the students' general understanding; and (2) in-depth interviews to obtain deeper insight into the students' concept images and to comprehend their cognitive structures.

In the cross-sectional probe, 277 first year mathematics students, following a course in one of the top five ranked mathematics departments in the UK, responded to an open questionnaire on "equivalence relations & partitions". This topic had been rated as the most difficult topic of the Foundations module in the student annual evaluation report for several years. 36 out of these 277 students were interviewed to develop a more insightful perspective of these students' understanding of the topic.

In the longitudinal probe, 15 selected students answered the same questionnaire and were tracked by interviews every week during the first term in their second academic year. These students' marks for their first year study are widely distributed — three are over 80, four between 70 to 79, four between 60 to 69, one between 50 to 59, and three between 40 to 49. All 15 subjects were also interviewees in the cross-sectional probe. Thus the qualitative data of these research subjects collected from the cross-sectional probe to the longitudinal probe offered a more complete and authentic picture of the conceptual development of this topic.

Scheme of questionnaire

The open questionnaire was the main research instrument in this study. There were six research questions taken into consideration while designing this questionnaire as follows:

(1) What are the students' concept images and concept definitions of "equivalence relations & partitions"?

(2) Are the students able to apply the formal definitions to make deductions and, if so, how?

(3) Are the students able to apply some relevant theorems to make deductions and, if so, how?

(4) How do the students relate these relevant notions in their concept images?

(5) How do the students understand the quantifiers in the formal definition?

(6) How do the students deal with mathematical symbolism and formalism?

Based on the above considerations, two major parts including eight problems in total were designed in the questionnaire. The first part contained two questions for getting a preliminary idea of students' concept definitions and concept images of the two notions and the second part contained six questions for gaining deeper insight into students' concept images of this topic.

This paper focuses on two questions of the second part in the questionnaire which were designed to examine how the students manage to apply a relevant theorem to make their deductions.

"Equivalence relation" at the theorem-based level

The following question was designed to examine if the students improve their understanding from the definition-based level to theorem-based level:

> *A relation on a set of sets is obtained by saying that a set X is related to a set Y if there is a bijection f: X→ Y. Is this relation an* ***equivalence relation****?*

Note specially that the following 3-part theorem, which can be directly applied to this question, had been taught before the topic of "relations" was introduced in the course:

> *(1) The identity map is a bijection.*
>
> *(2) The composition of bijections is a bijection.*
>
> *(3) The inverse of a bijection is a bijection.*

There is a need to compress the proofs of (1), (2), and (3) (as processes) into useable concepts (theorems).

In the cross-sectional probe, nearly a half of the students (132 out of 277) went back to examine the definitions step by step to answer this question (and were categorised as "definition-based"). Only a small percentage of the students (13%, 36 out of 277) tried to apply the above theorem to make their deductions. Less than a half of these 36 students (14 out of 36) briefly referred to the theorem without giving more detailed interpretation. That is, these 14 students could only state the theorem but seemed unable to unpack its meaning. For these 14 students, the notion of proof cannot be considered as a formal procept yet because they did not seem to know the process (method of proof) but only the brief concept (statement of theorem). In addition, it should be noted that, within the thirty-six interviewees (of whom six were categorised as "theorem-based"), thirty-three indicated a knowledge of the relevant theorem.

In the longitudinal probe, 12 out of 15 were able to apply the theorem in the second year, whilst only three were categorised as "theorem-based" in the first year. Although most of the students seemed to know the relevant theorem, they did not really have a clear idea how to apply it to this question. JULSON (68% for his first year study) was an example of such a student, offering a definition-based response despite vividly stating in one of the first year interviews — "*I remember I learned it [the theorem] in the lecture a couple weeks ago, but I'm sorry I haven't put it in my head yet.*"

Always bij. $f: X \to X$, $\therefore$ set $X \sim$ set X
if bij $f: X \to Y$, then bij $f': Y \to X$
if bij $f: X \to Y$ and bij $f': Y \to Z$, then bij $f'': X \to Z$

(JULSON (68%), 1st year response)

After being given a period of time to digest what they had learned, the twelve who were categorised as "definition-based" were able to upgrade their understanding to the theorem-based level by the second year. In addition, compared with their former responses, the quality of these fifteen students' deductions seems to indicate that the notion of the theorem had become more operable in their concept images. JULSON's response for the second year (classified as "theorem-based") offers us and example:

- id = bij : $X \to X$
- if f = bij : $X \to Y$ then f^{-1} = bij : $Y \to X$
- if f = bij : $X \to Y$ and g = bij : $Y \to Z$
  then gf = bij : $X \to Z$

(JULSON (68%), 2nd year response)

In the second year, JULSON not only stated the theorem but also explained in the interview how the theorem can be proved. Thus he clearly showed that the notion of proof of this theorem had become a "formal procept" in his concept image as he knew both the statement of theorem (as general concept) and the method of proof (as process).

The following written responses, with the corresponding excerpts from the interviews with DIAHUM, might offer us a more delicate insight into how the successive moves — from informal to definition-based, then on to theorem-based conceptions — happened with an individual.

DIAHUM (48% for his first year study) gave the following response (classified as "informal definition-based") in the first year:

$a \to a$
If $a \to b$ $b \to a$.
If $a \to b$ $b \to c$ have same no. of elements
$a \to c$.

(DIAHUM (48%), 1st year response)

He cleared up what he meant in his response as follows:

> *I was trying to apply the definition of equivalence relation to make the answer more formal. But I don't think my answer was formal enough because I didn't really know how to apply the definition even though I can remember it. And another problem is I can't recall the definition of bijection. What I can remember is a bijection is one-to-one and onto. That means the two sets have the same number of elements* [he explained later that this idea was from what he learned at A-level].

He also said that he knew the bijections theorem which is directly relevant to this question. But the theorem did not appear to be something which he could freely refer to at any time.

In the second year, he responded in terms of the relevant theorems as follows:

reflexive: yes since $\exists$ a bijection $X \to X$.
symmetric. yes since if $\theta: X \to Y$ then $\theta^{-1}: Y \to X$.
transitive: yes since if $\theta: X \to Y$ $\phi: Y \to Z$
then $(\phi\theta)(X \to Z)$.

(DIAHUM (48%), 2nd year response)

Although he did not use the term "identity" to indicate a bijection mapping from the set X to itself, he could precisely write down the composition of two bijections, even though some others gave it in the wrong order. In addition, he could explain the idea of how to prove the theorem in the interview. When being asked why he answered in this way in the second year, DIAHUM gave the following interpretation:

> *Well, I think it's fairly natural for me to make the deduction like this. When I faced the question, the theorems burst upon my head and I just wrote down the proof.*

DIAHUM's case seems to suggest that he cannot freely apply a formal conception until it is assimilated in his concept image as an embodiment. When DIAHUM could only recite the formal definition of equivalence relation but was still struggling with the implication of it, it was natural for him to consult the relevant ideas he learned at school to make his first deduction because they were more embodied and secure in his concept image. Having a year to digest all these notions, the theorem had been assimilated into his concept image as a formal procept that he could recall intuitively in the second test.

In the students' (written or oral) responses, we saw that most seemed to apply the relevant theorem directly to the question in the second year whilst most of them only gave a definition-based response in the previous year. This kind of result is consistent with the successive move from definition-based conceptions to theorem-based conceptions over time, during which the ideas are being used formally (Chin & Tall, 2000). From the improved quality of the students' deductions, we might consider that, the notion of proof of the theorem seemed to have become a "formal procept" in the concept images of some students. Initially, they only appeared to know the general concept (statement of the theorem) but not the process (method of proof) of the notion of proof of the theorem. But, a year later, some students seemed to be able to unpack the notion of the theorem to the proof process and to apply the theorem to the question more flexibly.

Linkage between "equivalence relations" and "partitions" (at the compressed concept-based level)

The notion of "equivalence relations" is linked to the notion of "partitions" since an equivalence relation determines a partition of a set and vice versa.

The following question was asked in order to examine whether the students appreciated this idea:

> *Write down <u>two</u> different **partitions** of the set with four elements, X={a,b,c,d}. For the first of these, please write down the **equivalence relation** that it determines.*

In the cross-sectional probe, the students' reponses to this apparently easy question reveal that only few students (61 out of 277) show a workable linkage between the two notions in their concept images. The others gave two correct partitions lacking, or with incorrect equivalence relations; incorrect partitions lacking, or with corresponding equivalence relations or gave completely incorrect answers. However, all 36 interviewees said that they knew of a theorem linking the two notions together. It seems fairly clear that being aware of the statement of a theorem does not mean that the theorem is operable in one's concept image. In this case it appears that the notion of proof has not become a "formal procept" yet, since the students could only remember the statement of the theorem as general concept but did not have the access to proof as process (the method of proof). Thus they could still not apply the theorem to this question in the first year.

In the longitudinal probe, there were only five out of the fifteen subjects able to apply the idea of the relevant theorem by successfully giving two correct partitions with a correct corresponding equivalence relation in the first year. The number increased to eleven in the second test. As to the other four students, three gave two correct partitions without a corresponding equivalence relation and one even failed to offer correct partitions. Note that all fifteen stated that they remembered they had seen the theorem which links the two notions together in the lecture course.

HELTON, getting 61% for his first year study, represents those who failed to offer a correct response in the first test but solved the question successfully in the second. HELTON seemed to try to define an exact relation as follows in the first year:

$\{a\}, \{b, c, d\}$ – $x \sim y$ iff x and y are in the same partition

$\{a, c\}, \{b, d\}$

(HELTON (61%), 1st year response)

In the interviews, he confessed that he merely remember the theorem without really understanding the meaning of it. He did not recognise the mistake he made in the above response, saying "in the same *partition*". In the second year, HELTON simply gave the following response but carelessly missed the two pairs (*b*,*d*), (*d*,*b*) in his equivalence relation:

$X = \{a, b, c, d\} = \{a\} \cup \{b, c, d\} = \{a, b\} \cup \{c, d\}$

$R = \{(a,a), (b,b), (c,c), (b,c), (c,b), (c,d), (d,c), (d,d)\}$

(HELTON (61%), 2nd year response)

When asked why he was able to offer an almost correct answer the second time, HELTON stated that when preparing for the examination he studied how the theorem can be proved and then grasped the idea of the theorem; thus he could simply solve the problem in the second year.

MAUHAM (71% for her first year study) is someone who failed to offer a correct answer to this question twice. She was able to give two correct partitions followed by a big question mark in the first test.

$\{a,\}\ \{b, c, d\} \qquad \{ab\}\ \{c, d\}$

?

(MAUHAM (71%), 1st year response)

She gave the following response in the second test:

$[a], [b]$

$[a] = \{x \in X \mid x \sim a\}$

(MAUHAM (71%), 2nd year response)

In the interviews, MAUHAM correctly outlined the content of the relevant theorem, but she confessed that she only recited the statement of the theorem and had no idea how the theorem could be proved.

The result of this question appears to parallel the former question in many instances. Most of the students showed an awareness of the relevant theorem linking the two notions together but only a few could apply it to the question in the first year. A year later, some students' understanding had progressed to a more mature theorem-based level. The theorem was no longer a "general concept" only but also a "process" which suggests the development of a "formal procept" in their concept images. However, merely trying to recite the statement of a theorem without understanding the notion of proof of the theorem was not helpful for improving the student's understanding.

DISCUSSIONS AND CONCLUSIONS

The proceptual encapsulation in this area of advanced mathematics seems to be slightly different from that in simple arithmetic, in which pupils appear to build up the notion of proceptual structure from encapsulating various processes, to obtaining the concept, then on to forming the procept of a symbol (Gray & Tall, 1994). The empirical data of this paper reveal that most students, at the university level, seem to have the product (the statement of a theorem) first, then to develop the notion of proof if possible. There is evidence that being stuck in processes of calculation seemingly prevents pupils from obtaining the concept (e.g. Gray & Pitta, 1997; Lin & Chin, 2005). Some similar evidence also reveals that the learning of the ideas of trigonometry is fraught with difficulty as it requires the student to link pictures to numerical relationships with the manipulation of the symbols involved in such relationships as well as coping with various formulae which seem rather complicated (Blackett, 1990; Blackett & Tall, 1991; Chin, Chen, Liu, & Lin, 2005). However, the use of the computer to carry out the process, and so enable the learner to concentrate on the product, significantly improves the learning experience (Blackett, 1990; Blackett & Tall, 1991; Gray & Pitta, 1997; Lin & Chin, 2005). This kind of evidence suggests that by concentrating on the product first developing the notion of procept is possible and may also be beneficial for improving the student's learning.

The empirical evidence presented in this paper gives us confidence to draw a conclusion that the notion of procept of Gray & Tall can also be extended to this area of advanced mathematics. Most of the novice university students appear only to have the product (the brief notion of the theorem) in their concept images. But they cannot grasp the essence of the theorem and develop more flexible thinking until they perceive the notion of the proof of the theorem. Therefore, the ambiguity of process and product represented by the notion of *formal procept* also provides a more natural cognitive development at the university level which gives the students enormous power to develop more flexible mathematical thinking.

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Conceptual Maps and Equations: What is the meaning of this?

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ABSTRACT: *In this paper we present data collected from a questionnaire and conceptual maps designed for 15-16 year old Brazilian students in an attempt to better understand the meaning they give to equations. We analyse data, using a theory of long term learning, that suggest three different kinds of cognitive mathematical development that inhabit three different Worlds of Mathematics (Tall, 2004). Analysis of both instruments shows that these students do not relate equations to either conceptual embodied aspects of equations or to symbolical meanings. Equations seem to be a calculation-like addition or multiplication, maybe with the need to search for an unknown value. The absence of embodied meaning is reflected in their work as a lack of symbolic meaning. We hypothesise that these students see an equation as a process of obtaining its solution but not as a concept. In addition, it is possible that excessive emphasis on procedures may guide students to the use of meaningless rules to solve equations.*

Key words: *Conceptual Map, Equations, Embodied, Procedures, Proceptual Thinking, Symbols.*

INTRODUCTION

Research on the learning of algebra has reported students' difficulties with this subject. For instance, Kieran (1981) presents evidences that students do not see the equals sign as an equivalence relation, but as a "do something signal", that is related to an arithmetic use in numerical expressions. This view of the equals sign may bring difficulties when equations are in focus, because in these cases the sign may represent an equality between two expressions, and not necessarily a specific calculation to be performed. Specifically in the case of equations, Linchevski and Sfard (1991) reported that students do not recognize the equivalence between two equations in terms of their roots. In this research, the evidence is that students agree that two equations are equivalent if one can be transformed into the other. However, this equivalence is frequently not taken into account. For instance, Freitas (2002) shows that students use procedures that may guide them to different kinds of errors related to phrases such as

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"change side, change sign", which are not necessarily connected to the algebraic principle of performing the same operation on each side, and therefore those procedures do not always result in equivalent equations during the solving process. Cortés and Pfaff (2000) claim that such mistakes may be explained by the fact that students may not understand such algebraic principles as an invariant mathematical property, and use techniques to get a result.

In an attempt to surpass these difficulties, teaching experiments using a concrete approach, such as the balance model (Vlassis, 2002) or the geometric model (Filloy & Rojano, 1989), have been tested. Both of them proved to be useful to help students give meaning to the equals sign as an equivalence sign (and therefore influencing the "do something" meaning reported by Kieran, 1981), but they have their own constraints as well. For instance, it is not obvious how to represent negative numbers by means of weights or areas.

In the face of all these difficulties, we have developed a research study in order to raise possible reasons why students make errors when attempting to solve equations. For this, we have worked with six Brazilian mathematics teachers that agreed to collaborate with us in order to design instruments and collect data from their students.

Data is analysed with a theory of long term learning, which considers three different kinds of cognitive mathematical development and the experiences related to each one of them that students have met before.

We claim that the absence in learning of aspects from at least two of those kinds of development may prevent students from giving meaning to equations and from having flexible thinking to solve them.

THE RESEARCH

As part of a broader project on the teaching and learning of mathematics, we discussed pedagogical and mathematical aspects related to equations and their solution methods in weekly sessions with six Brazilian mathematics teachers. Those discussions resulted in the teachers' agreement to collaborate with us to design instruments to collect data that could shed some light at what conceptions about equations those teachers' students would hold. Researchers and teachers together designed three instruments: a conceptual map, a questionnaire and interviews. In this paper, we wish to relate data from conceptual maps and the first four questions of the questionnaire in order to raise those students' conceptions of equation.

Data was collected from three classes: one with 32 fifteen-year old first graders (denoted as C15) and another with 28 sixteen-year old second graders (C16), both from a state school, and a third class consisted of 18 sixteen-year old second graders from a private school (P16). Both schools are located in the Greater São Paulo area.

The conceptual map (Novak, 1998) consists of the design by the students of a scheme which is based on the words from a brainstorming session: students were asked by their teacher to say a word that comes to their mind when they see the word EQUATION and the teacher wrote all the words on the blackboard. Figure 1 shows how a blackboard would appear after the brainstorming. After that, working in groups of four or five, students had to separate those words into at least three different categories, name them, make a scheme using categories names and write a few sentences to explain the scheme. We believed that producing words related to equations and arranging them in categories would raise different ideas, concepts and aspects which students relate to equations.

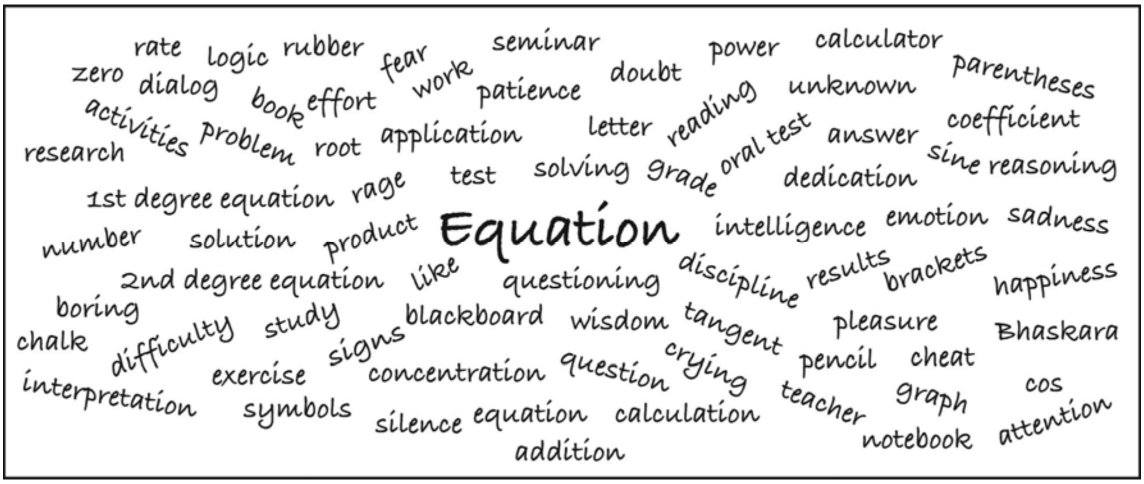

Figure 1. Words from class P16.

The questionnaire we designed had eight questions related to the concept of equation. For this paper, we are going to discuss data collected from the first four of those questions (see Figure 2), designed so that students could write about their conceptions of equations, their solutions and examples. The other questions in the questionnaire were designed in order to raise possible ways in which students solve equations, which mathematical concepts they use in their solving methods and also whether they could model a given problem using a quadratic equation. Those questions and the other instruments of this research are analysed in Lima and Tall (2006a).

1. **What is an equation?**
2. **What is an equation for?**
3. **Give an example of an equation.**
4. **What does the solution of an equation mean?**

Figure 2. The questionnaire.

With those questions and the conceptual maps designed by students, we believe it is possible to uncover aspects of these students' understandings of what an equation might be. We are also comparing data from both instruments to observe whether they uncover similar aspects.

THEORETICAL FRAMEWORK

From the understanding that there are at least three different kinds of mathematical concepts – the ones that come from the study of objects, the ones from the study of actions and the ones from the study of properties (Gray & Tall, 2001; Tall, 2004a, 2004b) – we claim that there are also three different types of cognitive development related to those mathematical concepts which inhabit different worlds of mathematics.

The first world emerges from the perceptions individuals have from the physical objects and their properties. By observing and manipulating objects, it is possible to grasp mathematical concepts from them. With experience, the individual no longer needs the object itself, but s/he can develop mental experiments and extract meaning from them. This is the *conceptual-embodied world* of perceptions, in which individuals observe, describe and manipulate objects in order to give meaning to them. It is important to notice that physical objects and mental representations generated by them are not the only way that mathematical concepts can be embodied: Graphs and visual images may also be seen as ways of embodying concepts, provided that there is an element of body movement involved.

As individuals grow in sophistication, they feel the need to represent the actions they experience in the embodied world. For this, they use symbols that inhabit the *proceptual-symbolic world*. In this world, symbols compress both procedures and concepts associated to the actions. This duality of symbols as processes and concepts is called *procept* by Gray and Tall (1994) and they hypothesise that the flexibility of understanding symbols either as a process or a concept helps individuals to give meaning to mathematical symbols and to extend the meaning given to mathematical concepts in embodied situations, such as the balance or the geometric models.

The third world of mathematics is the *formal-axiomatic world*, composed by the axioms, definitions, properties and theorems which are deduced from them that build the body of knowledge of mathematics. This world is not a focus in the teaching before undergraduate levels, because mathematics is not axiomatically constructed in lower school levels, at least in the Brazilian context. Therefore, this world is not in focus in this paper, as the subjects of our study are of high school level.

In this research study, we search for aspects from both embodied and symbolic worlds that may be presented in students' work and also try to get insights of what kind of meaning they give to equations.

RESULTS

In this session we present the words that came about during the brainstorming. We expected that those words will show what notions students relate to equations. Then, we look at the categories and names given to them and also the schemes that were created from them. Finally, we present the answers to the questionnaire and the ways in which they are related to the findings from the conceptual maps.

Conceptual maps

The conceptual map session took place in a 100 minutes mathematics class and was guided by the teacher. The researcher was present as an observer, but also could be of help to the teacher if requested.

The brainstorming sessions from the three classes are very similar. All of them contain words like “number”, “signs”, “symbols” “calculation”, “addition”, “subtraction”, “multiplication”, “division”, “unknown”, “rules” and “solution”. These words may mean that these students are likely to understand an equation as a calculation with numbers and unknowns in order to get the solution. Words for the operations and the word “rule” may represent the actions performed on symbols that form the equation, as well as the need they might have to perform those actions. This seems to be a general idea of equation present in all classes, but other common words (or absence of them) may also give us more insight from their understanding of equations.

For instance, the teachers with whom we had worked claimed that they use the balance model as an approach to teach equations. However, there is no evidence from the brainstorming sessions in any class of the scale, or balance or even weights among the words presented. So, there is neither evidence that these students relate equations with embodied situations nor any other aspect of the embodied world, such as a problem that could be modelled and solved by an equation. The only words that could be related to embodied aspects are “Biology”, “Physics” and “Chemistry”, in the sense that they could mean a subject where an equation could be used as a tool to solve a problem.

The idea of an equation as a calculation in which certain rules are applied to get a solution place the notion of equation that these students have in the symbolic world. Nevertheless, the lack of words related to equality or equivalence may mean that the symbols that were emphasised here do not carry a conceptual meaning. This leaves them with only procedural ways of thinking.

There were no words that we could relate to the formal world.

The categories which the words were divided into are similar in each class. In the class C15, the main categories are related to *Symbols*, *Formulas*, *Subjects*, *School* and *Objects*. Not all of the small groups of students used all those categories, but all of them used only the categories in this list. *Symbols* is a category where students placed the words like “parenthesis”, “brackets”, “signs” and also “equals sign” and “x and y” (meaning the unknown), while *Formulas* contain words like “quadratic formula”, “Pythagorean theorem” and “Tales’ theorem”. *Subject* and *School* are much related, as *Subjects* refer to the ones they have at school. This was the only class that gave emphasis to different subjects. *Objects* is a category where words for objects that we find in a classroom were placed, such as “pencil” or “rubber”. Analysing this kind of categorization, we conclude that this class may understand equation as a content area that should be worked at school, not only in mathematics, but other subjects as well. It has some symbols and formulas which should be used to solve them. To us, this is a procedural way of seeing an equation, focusing on the symbols from the symbolic world.

The main categorization made by the small groups from class C16 divided the words in *Very related*, *Related* and *Not Related* to equations. This leaves all words that could be referred to Mathematics into the first category. In this way, it is not really possible to do a very detailed analysis of categories in C16. However, it was very interesting to note that the word “letter” is in the *Not Related* category. We claim that the group of students that made this categorization may see an equation as a regular calculation only, and not as the process of finding the value for an unknown.

The whole brainstorming and conceptual map session at class P16 was mainly related to the worries of the student with learning equations. Their categories, as well as their words, mostly make use of phrases that show their feelings, worries and difficulties. For instance, some categories are *Feelings*, *Culture* and *Decision*. But they also use categories named *School*, *Solution*, *Mathematics*, *Learning* and *Methodology*. Into the *Mathematics* category there are words like “rules”, “formulas” and “symbols”, which suggest that they have the same understanding of equation as C15.

We found it useful to look into each category as an attempt to see how students perceived the words in them. For instance, the word “equal”, which appeared only in the brainstorming of the class C15, was used in five of the eight small groups that were formed in this class. Four of them placed this word in categories called *Signs* or *Symbols* and one group placed it in the one called *Operations*. This may be evidence that students are likely to see the equals sign as an operational symbol (Kieran, 1981), which entices them to perform an action or a calculation, instead of representing equivalence. Another example is the word “unknown”, which sometimes was placed in rather peculiar categories, like *Not Useful*. Students who did that may not know either the meaning of the word “unknown” or its role in an equation. Finally, words for subjects, like “Biology”, “Physics” and “Chemistry”, which appeared in the brainstorming of class C15, were placed in categories named like *School* and *Subjects*, showing that they might be only part of the school life, and not as an embodied situation, as we have imagined.

We can see that these categories are related mainly to aspects from the symbolic world, but in a way that only procedural aspects seem to be relevant. Apparently, these students do not relate operations, symbols, and signs with conceptual meanings that the symbols stand for, but only the actions represented by operations and calculations.

One example of conceptual map designed by students can be seen in Figure 3. It is from a group in class P16, and their brainstorming also contains words related to their feelings about previous experiences with equations, like “anger”, “sadness”, “panic”, “doubt”, “fear”, which were placed in the *Feelings* category of this group. The *Actions* category contain words that show the need for abilities, like “dialogue”, “interpretation”, “attention” and others. The *Working Instruments* category contains objects like “blackboard”, “calculator”, or even words related to the classroom, like “tests” and “exercises”. Lastly, the *Equation Elements* category contains all words related to mathematics that appeared on the brainstorming.

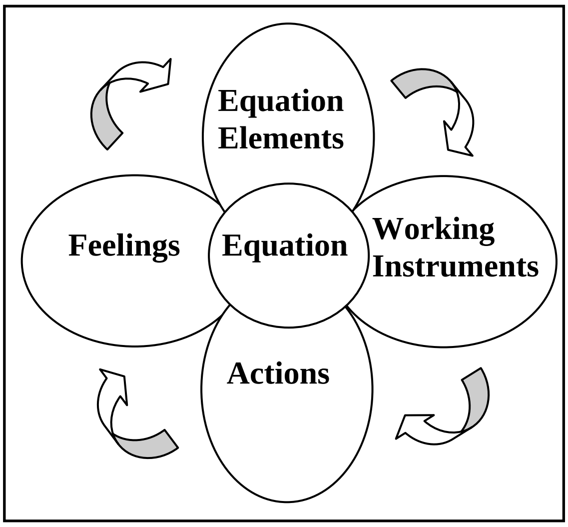

Figure 3. Conceptual map from one group of P16.

Our interpretation of this map leads us to conclude that students who designed it relate mathematical elements mostly with school, which is represented in the category *Working instruments*. To succeed in the learning of equations, they need to make efforts but also to overcome the fears and difficulties they face in this learning.

The understanding we can make from conceptual maps is that those students probably give meaning to equations by relating them to the procedures they need to perform to solve them. Apparently, they are calculations, but not necessarily containing an unknown. There are certain rules or formulas, as well as certain symbols, to be used, although the equals sign is not amongst them.

This view of equation, in our analysis, is placed in the symbolic world. The symbols and actions to manipulate them are in focus. The procedures may be the main important characteristic. The absence of words that refer to conceptual meaning to equation, such as equality or equivalence, guides us to claim that those students may not have proceptual thinking when dealing with equations. They do not seem to emphasise the concepts behind an equation and its solution.

The Questionnaire

The questionnaire was applied on a different day from the conceptual map session. It took place in a 50 minutes mathematics class, under the supervision of the teacher. Each question was given to the students separately to stop them from changing their answers by the influence of other questions.

The most common answers to Question 1 (*What is an equation?*) were *"It is a calculation in which there are additions, multiplications and many symbols like +, ÷"* or *"Equation is a set of calculations"* and *"It is a mix of numbers and letters aiming to find the value of the unknown"*. From this, we understand that these students probably understand equations either as a common calculation, like addition or subtraction, or as a calculation in which it is necessary to have an unknown and to find its value. This is consistent to the notion of equations as calculations that can be understood from the analysis of the brainstorming sessions. Other answers to this question also show difficulties students are faced with while solving an equation and the fear of not being successful (*"I cannot understand them nor solve them"*), characteristics of equations, such as *"they have two members"* or *"they can be linear or quadratic"*, and the search for an unknown number. There is no mention of aspects from embodied world in this question. The main characteristic of an equation that is emphasised is its solving and the need to find a value for the unknown. Procedural thinking may also be present here, but we do not have sufficient evidence to assert that these students make use of proceptual thinking.

The general agreement among students in answers to Question 2 (*What is an equation for?*) is that it is useful *"To find the solution"*, meaning the value for the unknown. Students in P16 also refer to an equation as a mathematical calculation that does not have a practical use. For instance, one student wrote *"I will rarely or never use equations to solve daily life problems"* and other wrote *"in life and everyday, it is not useful for many things, but maybe it is useful for people who like mathematics"*. The usefulness of an equation seems to be, therefore, related to the unknown and to the main action students perform on an equation. A few answers also relate the unknown to mathematics content, as in answers such as *"to find, for instance, the value of an angle or the side of a triangle"*.

45 from 77 students gave an equation with one unknown as example in Question 3 (*Give an example of equation*). Four of them are quadratic, while all the others are linear. Three students presented, together with a linear equation, a calculation like $12 \times 7 = 84$ or $2 + 2 = 4$, written vertically. Numerical and algebraic expressions like "$30 - 200 + 15 \cdot 5 - (-5 + 1) =$" and "$2 + [5x + 13x\{16 - (55x - 10x \div 2)118\} \div 75] =$" were also present in students work. In addition, 47 of the 77 students who answered the questionnaire made an attempt to solve the example they gave, and 27 succeeded. It seems that students are more able to apply rules and procedures to solve equations than to perceive concepts that underlie those solving methods.

Lastly, for Question 4 (*What does the solution of an equation mean?*) the usual answer is *"value for the letter"* (meaning the unknown), or *"the end of the calculation"*. Again, students in P16 showed their feelings, saying that it is *"the end of a great effort"*. The focus in the answers to this question was on the final product of an action. As their understanding and use of equations is mainly addressed to mathematical issues, the solution was attached to the value of the unknown, without any meaning from the situation that might have generated the equation.

Looking through the answers to all four questions, we can see that aspects from the embodied world were not mentioned. These students did not refer to equations as tools to solve real life situations nor as the balance between two plates of a scale. The answers often mentioned symbols and the actions that may be performed to solve equations. These are aspects present in the symbolic world. However, students did not seem to refer to symbols by giving meaning to them. They just mentioned a few actions that might be related to procedures, which may mean that these students do not have proceptual thinking.

Comparing results from both instruments, we can see similar aspects. In both, students present the understanding of an equation as a calculation. Whether this calculation includes an unknown or not, the main idea is to perform operations such as addition and multiplication. This shows the focus on the procedures rather than on concepts, revealing a possibility that those students do not see an equation as a procept nor have proceptual thinking.

DISCUSSION

The conceptual map and questionnaire give us very similar characteristics of what these students understand by equations. Apparently, they focus more on procedural meaning than any other ways of giving meaning to equations. The fact that more than half of the students actually solved the examples they gave is evidence of their understanding of equations as a process, while there is no evidence that they see it as a concept. The lack of symbolic meaning for equations may be evidence of the absence of conceptual understanding of equations. Both these characteristics prevented them from having proceptual understanding and from seeing symbols in a flexible way. Further data (Lima & Tall, 2006a, 2006b) show that this led them to a meaningless use of rules to solve equations.

Although the teachers claim that they usually use the balance model approach to teach equations, none of the students made any reference to aspects related to this approach. Vlassis (2002) claims that giving meaning to equations from such approach, that we consider embodied, made it possible for students to give meaning to the equals sign as an equality symbol and also to the algebraic principle of performing the same operation in both sides of the equation to maintain the equilibrium. We hypothesise that meaning to symbols come from embodied situations and are extended from them. This would help students to reach a proceptual understanding of equations by transferring embodied to symbolic meaning. Unfortunately, if the teachers really have made use of an approach based on the embodied world, it was not apparently sustainable for those students, so they ended up without either embodied and symbolic meaning.

In addition, the absence of meaning for the equals sign may be a reason for students to understand equations as a regular calculation with no difference from multiplication or others. The equals sign appears to just show them where to place the solution.

The emphasis on procedures that is shown in both the conceptual maps and the questionnaire suggest a teaching approach based on procedures and actions to be performed to solve equations, and not on the meaning for symbols. This resulted in an understanding of equations by these students that lacks flexibility to deal with symbols as processes and concepts. Analysis of further data (Lima & Tall, 2006a; 2006b) show that these students use procedures based on procedural embodiments, of picking a number and putting it on the other side, with additionally changing signs. Such procedures are no longer related to algebraic principles, so they can be misused by students, who end up making mistakes as reported in literature (for instance, Freitas, 2002; Cortés & Pfaff, 2000).

For further research, we suggest a teaching approach based on aspects of embodied and proceptual worlds. We believe that, by starting with embodied characteristics, it is possible for students to recognize the equals sign as an equality symbol that balances both sides of an equation. Giving meaning in this world, it would be more reasonable for them to understand symbols as procedures and concepts that give flexibility to the mind and would enable them to grasp the use of rules to solve equations not as a meaningless procedure that is necessary to follow, but as a type of mathematical reasoning that guides them to successful solutions. In addition, links between embodied and proceptual worlds should be emphasised by the use of equations to represent real life situations.

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Qualitative Differences in the Teaching and Learning of the Constant Function

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ABSTRACT: *This paper examines two experienced Turkish teachers' teaching of the constant function and their students' resulting understanding of the notion. Using a theoretical standpoint that emerges from an analysis of APOS theory, the paper illustrates that the teachers differ remarkably in their approaches to the essence of the concept. Though their personal subject knowledge and understanding of the potential difficulties and misconceptions associated with acquisition of aspects of the function concept is similar, and although they use epistemologically the same tasks, their classroom presentation focuses upon qualitatively different aspects of the concept. This in turn has a considerable influence on their students' knowledge construction.*

Key words: *Teaching, Learning, Constant Function, Action-oriented teaching, Process oriented teaching, Algebraic representation, Graphical representation.*

INTRODUCTORY FRAMEWORK

The interest in examining the impact of teaching practices on student learning derives from the belief that teachers play an active and direct role in their students' knowledge construction (see, for example Askew, Brown, Rhodes, William, & Johnson, 1996; Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996).

Studies that deal with the relationship between teaching and learning can be seen in two categories: 'simple process-product research' and 'qualitative process-product research'. Brophy and Good (1986) document the development of the former and suggest it focuses mainly upon directly observable variables in teacher's instruction and relates these to the students' achievement as measured through standard tests. Qualitative process-product research employs in-depth qualitative inquiry to gain an understanding of the psychological and pedagogical aspects of teaching and learning and the interaction between the two (see, for example, Pirie & Kieren, 1992; Cobb, McClain, & Whitenack, 1997).

This paper continues the interest in the latter tradition by examining two Turkish teachers' instruction of the constant function and relates it to their students' learning of the concept.

DIFFICULTIES AND MISCONCEPTIONS ASSOCIATED WITH THE FUNCTION CONCEPT

The Turkish mathematics curriculum introduces the constant function through a definition the specific form of which addresses an 'all-to-one' transformation and, in a mathematical sense, is intended to bring simplicity for learners:

> A function is called a constant function if it matches every element in the domain to one and the same element in the co-domain.
>
> (Çetiner, Yıldız, & Kavcar, 2000, p. 86)

However, previous studies have reported that many students develop serious difficulties and misconceptions with the notion of function. Vinner (1983) and Sfard (1992) have suggested that students have a strong tendency to think of a function as an arithmetic rule or algebraic formula. Such a perception may be explained by a limited understanding of the arbitrariness condition, which rules out attributing a mechanical rule to the function concept. Monitoring the univalence condition in an algebraic situation has also indicated difficulties. For example, in reporting that 62% of university students considered that an equation of a circle ($x^2+ y^2=1$) represents a function, Tall and Bakar (1992) suggest that this phenomenon may be explained by the fact that the terms 'implicit function' or 'many-valued function' are misused in Britain. Whatever the reason, it is clear that these students were unable to see that this expression gives out more than one image for $x\in(-1, 1)$ or does not produce unique images for $x>1$ and $x<-1$.

With particular reference to the constant function, Markovits, Eylon, and Bruckheimer (1986) indicated that very few high school students demonstrated a full understanding of an 'all-to-one' transformation in a situation where $g(x)=4$. Additionally, approximately half of the participants in Tall and Bakar's (1992) study rejected the possibility that a line parallel to the x-axis could represent a function. The students largely claimed that *y* would then be independent of the value of *x*.

It is a common misconception among students that a graph of a function is a smooth and continuous line or curve (Vinner, 1983; Dubinsky & Harel, 1992; Tall & Bakar, 1992) whilst others have suggested that the graph of a function must be a straight line (Leinhardt, Zaslavsky, & Stein, 1990; Janvier, 1998). Markovits et al (1986) reported students' difficulties with the idea that pre-image and image pairs are located respectively on the *x* and *y*-axis and they indicated that more than half of the students in their study claimed that the images on a graph correspond to given points on the *x*-axes or vice versa. Goldenberg (1988) indicated that some students also suggest that the graph of function matches the element where the graph intersects the *x*-axis to that where the graph intersects the *y*-axis.

The studies that have examined the source of student difficulties and misconceptions associated with the function concept have taken a variety of forms. The notions of 'concept definition and concept image' (Tall & Vinner, 1981), 'operational and structural conceptions of function' (Sfard, 1991), 'multiple representation of function' (see, for example, Goldenberg, 1988; Thompson, 1994), 'action-process-object conceptions of function' (Dubinsky, 1991) and 'vertical and horizontal growth of the function concept' (Schwingendorf, Hawks, & Beineke, 1992) each appear to make a contribution towards the recognition of misconceptions and difficulties and of their source. Directly or indirectly, each also provides us with some insight into the teaching and learning of the concept.

Vinner (1983), for example, suggests that since some students use concept images and not the concept definition to think about the function concept, pedagogically appropriate teaching strategies should allow learners to use examples, manipulations and other experiential opportunities before they are given the definition of function. However, it is a conjecture within this paper that teaching approaches which prioritise concept images may restrict students' understanding of the concept to the kinds of examples that are selected and used to create particular images of the constant function. A consequence of this may be rote learning and the accumulation of images without underlying meaning. From a teaching perspective, concept definition and concept image should not be seen as alternatives.

Sfard (1991) also criticized the notion of introducing the concept of function through a definition. Sfard's evidence suggested to her that students possess a cognitive tendency towards an operational conception of function. She adds that a structural conception of function, through which a function is construed as a unified entity in which the process and the properties of the function have been combined, is too difficult for many to acquire. Consequently, she suggests a teaching approach that follows an epistemological and historical evolution of the function concept, conjecturing that this would promote a growing sophistication from an operational to a structural form of understanding.

Advocates of the multi-representation approach to the teaching and learning of the function concept believe that students would develop better understanding if they experienced the concept across a range of representations – graphical, algebraic and tabular forms that include sets of ordered pairs. The underlying philosophy is that different, well chosen representations could each convey part of the meaning of a mathematical concept, but establishing connections between the representations could provide a more coherent and unified message (Goldenberg, 1988).

DEVELOPING A THEORETICAL FRAMEWORK

At the heart of the discussion within this paper is the theoretical framework evolved from 'action-process-object' conceptions of function. A model established from notions of process and object in the context of schema (Dubinsky, 1991) and designed to give some illumination to the notion of advanced mathematical thinking and the way in which students may develop the ability to engage with it was modified to provide the construction of what is now more regularly termed APOS theory (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996). Though now recognized as a four stage model hypothesizing the development of mathematical concepts from an engagement with actions (A), through to the formulation of processes (P) and their encapsulation into objects (O) that may be integrated into the notion of a schema (S), the current paper only focuses on the first two components of this theory, that is actions and processes, and attempts to use these to describe not only qualities of learning but qualities of teaching.

Cottrill et al (1996) indicate that an action conception of a mathematical idea refers to repeatable mental or physical manipulations that transform objects (e.g., numbers, sets) into new ones. An understanding which reflects such a conception would manifest itself as an ability to complete a transformation by performing all appropriate operational steps in a sequence. Dubinsky and Harel (1992) indicated that such a conception involves the ability to substitute a number into an expression and calculate its image (or vice-versa). However, an understanding restricted to actions implies that learners would compose two algebraic functions by replacing each occurrence of the variable in one expression by the other expression and simplifying (Breidenbach et al., 1992).

Breidenbach et al. (1992) restrict their discussion to functions expressed as algebraic expressions. They provide no comment as to whether or not the action conception reflects itself in other representations and, if so, how. They contend that if understanding the function concept is to go beyond mere manipulations with algebraic expressions and set-diagrams it should include the process conception. The ability to interiorize actions and reflect upon them is seen to be critical to the development of this form of understanding. The possession of a process conception of function entails the ability to talk about a function in terms of an input and an output without necessarily performing all the operations of a process in a step-by-step manner. A process can be manipulated in various ways; it can be reversed or combined with other processes (Cottrill et al., 1996).

Dubinsky and Harel (1992) consider that a process conception of function is required to overcome the confusion of 'one-to-one' and 'uniqueness to the right' conditions, but they identified three characteristics that, from a multi-representational context, would be obstacles to its development. One is the continuity restriction, which concerns a misconception that a graph of function should be a continuous line or curve – the ability to interpret a function process in a graph made of discrete points is indicator of a strong process conception. The second obstacle is the lack of an ability to construct a process when none is explicit in the situation. A set of ordered pairs involves no explicit recipe but allows the identification of a process conception if the first and second components are seen as elements of the domain and co-domain respectively and if there is an awareness of the implicit process (function) matching the first components to the second ones. The third identified obstacle is the students' need to have an explicit algebraic formula to decide whether or not a situation represents a function. It is our conjecture that a strong process conception is required to see an 'all-to-one' transformation in the algebraic and graphical forms of the constant function.

Dubinksy and his colleagues assert that the absence of an algebraic description makes it difficult to observe an action conception of function in graphical situations and in ordered pairs. Dubinsky and Harel (1992) reported that some students recognised a graph of a function only from memory, possibly by consulting their mental images. It may be possible to do this without necessarily making sense of a graph as the representation of a function which is a transformation from the x to y-axis, but such behaviour may not involve even an action conception. However, the authors interpret such behaviour as a constraint in the development of a process conception.

Such observations suggest that we could take the view that APOS theory may not be applicable to different representations of the same mathematical idea, in this case functions. However, within this study we are concerned with students' algebraic and graphical conceptions of the constant function. Based upon an interpretation of the literature, we suggest that an action conception of function within the graphical situation is determined whenever a student recognizes that a function transforms elements from the domain (x-axis) to the co-domain (y-axis). In contrast, a well established process conception of function requires the ability to think about the function in the context of the concept definition such as teasing out a function process in a situation made up of discrete points or re-defining a domain set on which a graph in two or three sections represents a function. Understanding that falls between these conceptions is considered to represent progress towards a process conception.

Even though it is not a subject of discussion within this paper it is worth considering how an object notion of function would fit into the previous discussion. Constant reflection upon a process may lead to its eventual encapsulation as an object (Cottrill et al., 1996). The possession of an object conception entails the ability to use a function in further processes and with this understanding a function may be used in the process of derivative and integral. From the graphical perspective, an object conception of function enables one to manipulate a graph of function (e.g., shifting the graph of $f(x)=x^2$ two units along the y-axis in the positive direction to obtain the graph of $g(x)= x^2+2$) without dealing with the graph point by point.

The framework of Schwingendorf et al. (1992) considers the vertical (depth of understanding) and the horizontal (breadth of concept images) growth of the function concept and combines, in essence, the theories of multiple representations and cognitive growth via process-object conceptions. Though they suggest that computer supported instruction aids the development of a process conception of function, and is therefore seen to be a distinct advantage over more traditional forms of teaching, DeMarios and Tall (1999) suggest that a constructivist approach to teaching improved students' flexibility in thinking of the function concept in both directions.

The advantages to be gained from the use of computer technology are not yet available within Turkey, so it is the quality of teaching and learning within a more traditional atmosphere that is the focus within the current paper.

The APOS theory of learning has both its advocates and opponents. Eisenberg (1991) finds it apposite in explaining how a function process is encapsulated into a mathematical object. He believes that the technique of encapsulation is a perfect match for the epistemological development of the function concept. On the other hand, Tall (1999) criticises the universal applicability of APOS theory particularly in its ability to account for geometric development. However, pursuing the discussion above, arising from the notions of action and process conceptions of function this paper conjectures that action-oriented teaching is an approach that can constrain students' understanding of function to the action conception whilst process-oriented teaching encourages students to develop a process conception of function. These aspects of teaching orientations will be considered through the data presentation.

RESEARCH METHOD

Bayazit (2005) explored the relationship between teaching and learning the concept of function. A qualitative case study (in the sense of Merriam, 1988) was used to interpret teaching orientation and its possible impact on students' learning. Two teachers, one each from two different Anatolian High schools participated in the study: Ahmet had 25 years teaching experience and Burak had 24 years. Both teachers were deemed by their colleagues and their head teachers to be highly qualified. The students of both schools were in the 9th grade (age 15). A purposeful sampling strategy (Merriam, 1988) was used to involve teachers who used different approaches to teach functions, to control the students' initial levels of understanding, their socio-economic backgrounds, and other school-related factors such as instructional facilities.

Anatolian High schools are highly regarded in Turkey and the two schools identified for the study were compatible in terms of achievement, teacher expertise, school culture and instructional facilities. Students within these schools are selected on the basis of their achievement in national standardized tests in science and social studies. Mindful of Tolgyesi's (1985) conclusion that parental educational achievement and occupation can have a considerable effect on student achievement in mathematics and science, an analysis of the socio-economic status of the students who participated in the study was carried out. This analysis indicated that there was no significant difference between the students from the two schools, the socio-economic status of the parents being equally distributed to reflect occupations within the Turkish context that were upper level middle class or middle class.

Each teacher was observed teaching all aspects of function but within this paper we concentrate on the constant function. Lessons were tape-recorded and annotated field notes were taken to record the teachers' pedagogical indications and visual attributes not detected by the audiotape. 55 students formed the core sample for the study (28 from Ahmet's class: Class A; and 27 from Burak's Class: Class B). Data associated with the developing understanding of all students was gathered through pre- and post tests that included an opportunity for students to provide reasons for their answers. Clarification interviews with three students from each class were carried out after each test. These students were selected on the basis of their achievement and the quality of their explanations within the pre-test. Since there was no discernable difference in the achievement of students on the basis of grouping them low, average and high achievers, students were selected on differences associated with their informal conceptions of function prior to the formal course, and after consultation with the teachers.

Descriptive statistics were used to analyse the students' test results whilst discourse and content analysis (Philips & Hardy, 2002) were used to consider the transcripts of lessons and student interviews. The objective of the discourse analysis was not to interpret a specific instructional act in its own context but to construe that act within its surrounding conditions. Since the study involved multiple cases, it was necessary to use the strategy of cross-case analysis (Miles & Huberman, 1994) to establish links between variables. To explain the relationships between the teachers' instructional practices and the students' development of function a comparison between the sets of data was made in two ways. Firstly, after an analysis of the instances where teachers differed from each other, correspondingly different occurrences within the students' data were considered. Secondly, we determined the instances in which the classes of students displayed noticeable differences in understanding the constant function, and then looked for the corresponding variables in the teachers' instruction. The same strategy was also utilised to identify the relationships between the teachers' opinions about the students' misconceptions and difficulties with functions and their theoretical approaches to such obstacles.

RESULTS

The results are presented in two ways. First we consider the teachers' instructions of the constant function, and secondly we examine the students' understanding of this concept.

An analysis of preliminary interviews with the two teachers (**A**hmet: Class **A** and **B**urak: Class **B**) established they had similar mathematical backgrounds and teaching experience, possessed a strikingly similar ability to diagnose their students' difficulties and misconception and a similar awareness of the way in which they thought their students would think about the function concept. However, they differed considerably in suggesting ways in which they would overcome any difficulties and misconceptions. Whilst Ahmet recommended a pedagogical approach distinguished by the use of multiple representations and the encouragement of visual thinking, Burak indicated that he would prioritise verbal descriptions and offer everyday analogies to contextualise concepts. He thought that "*using two of more representations together would confuse the students*". Thus, though the two teachers possessed similar pedagogical content knowledge, their beliefs about teaching suggested a divergence in their pedagogical treatment of the function concept.

These differences may be seen in the way the two teachers differed in their introduction of the notion of constant function.

Ahmet presented the concept through an operational verbal definition:

> Consider a function f, $f: A \to B$. If the function f matches every element of the domain, A, to one and the same element in the co-domain, B, we call it constant function.

and he illustrated the meaning behind this definition with multiple representations that included set-diagrams, sets of ordered pairs, graphs, and algebraic expressions. He used set-diagrams as a scaffolding to facilitate the students' understanding of the 'all-to-one' transformation in algebraic and graphical situations.

> This definition indicates that all of the elements in the domain must have one and the same image in the co-domain. … Do not misinterpret it; it does not mean that the co-domain must have a single element. … It would involve several elements, but only one of those…has pre-images…

In contrast, Burak prepared his students for the idea of constant function through an analogy to which he referred quite frequently:

> … Let's examine the term 'constant' in daily meaning. Suppose that you are discussing an issue with a person, OK. Yet, no matter whatever you say that person does not change his mind; he does not accept the idea you are proposing. … He keeps believing that what he says…is a hundred percent true. … We describe such a people as a fixed-minded person, do not we? We could think of the constant function like that.

The sense in which the analogy represented a constant function was not clarified. For example, Burak did not describe the person's mind as a dynamic process (function) receiving different messages (inputs), processing (reasoning) them and then reaching a single conclusion (output – constant).

As part of his lesson Ahmet proceeded to present the following to his students:

> What are the values of 'a' and 'b' for which $f: R \to R$, $f(x)=(a-2)x^2+(b+1)x+5$ represents a constant function?

After bringing the definition of constant function to the students' attention Ahmet continued (Episode A):

> Ahmet: This expression involves something that does not allow the transformation of all the real numbers to one and the same element [1][1]. What should we do so that this function produces the same element irrespective of whatever we put into the x? [2].
>
> Student: The value of 'a' is 2 and the value of 'b' is -1.

[1] The numbers indicate key statement in the teacher's explanation that is referred to in subsequent paragraphs which consider the teacher's instruction from a learning point of view.

Ahmet: How did you find out?

Student: The expression must involve just 5 so that it matches all the values of x to 5. Therefore, I equalised the coefficients of the other terms to 0.

Ahmet: If the rule of a function involves an independent variable like x, that function produces different outputs for the inputs given for that variable [3]. We should fix the value of y, the image [4]. We ensure it as we remove the terms containing xs from the expression [5]. So, we have to equalise the coefficients of x^2 and x to 0.

[Ahmet manipulates the expression and obtains the function $f(x)=5$].

No matter what we put into the x, say -5, 0, 4…, all goes to 5 [under this function]…

It is clear that Ahmet used the definition of constant function as a cognitive tool to establish the solution to the problem. Note also that he uses a guided discovery teaching method (statements [1] and [2] above) to encourage his students to figure out what they had to do (eliminating terms with x's from the expression) and he elaborates why they had to remove the variable x (statements [3], [4] and [5] above).

In contrast, though Burak emphasised that the term with x must be removed from a similar expression he did not clarify 'why it should be removed?'. His students were asked to:

> Work out the precise form of the constant function $f(x)=(4n-2)x+(2n+3)$; and sketch the graph of it.

Handling the first part of the problem Burak explained (Episode B):

Burak: We described it like a fixed minded person… Whatever we say; he never changes his mind. …no matter whatever we put into the x we come up with the same image. … Let's remember the algebraic form of the constant function; it will help us so much… … In general we represented it as $f(x)=a$, $a \in R$. So, could we say that…a constant function involves just a number; this number would be an integer, a natural number… … In this expression [$f(x)=(4n-2)x+(2n+3)$] there are two terms; one is the constant term, $2n+3$, and the other is a term involving x, $(4n-2)x$. …So, first of all we should get rid of the term containing x; because if this is a constant function…it must not involve x. How can we do that…?

Student: We would equalise the coefficient of x to 0.

Burak: Yes! Exactly! This is what we must do here. We should equalise the coefficient of x, $4n-2$, to 0.

[Burak manipulated the expression to obtain the function $f(x)=4$].

Here we see Burak reiterates his analogy but again does not link it to the targeted concept. The notion of constant function appears to receive little emphasis in the response to the question. More importantly, it is not used as a cognitive tool while solving the problem but a strategy is identified to obtain the solution based upon the elimination of the terms involving x from the expression. Bringing the description,

($f(x)=a,\ a\in R$), to the students' attention a truth is stressed: a constant function does not involve an x, and a goal established: *"...we should get rid of the terms contain x ..."*. No clarification is given as to why the terms including x must be removed from the expression. While solving the second part of the question Burak made use of the students' prior knowledge of lines parallel to the coordinate axes but his goal was again to promote the acquisition of procedural knowledge.

Through these two examples, which were typical of their teaching, we can see that the two teachers substantially differed in their approaches to the essence of the constant function. Ahmet usually employed a process-oriented teaching approach and consistently engaged his students with the idea of constant function by using several relevant strategies, the most prominent of which utilised the definition of the constant function as a cognitive tool to provide a basis for process-oriented language. In contrast, Burak used mostly action-oriented teaching and, engaging his students with the visual properties of algebraic and graphical forms of the constant function, placed little emphasis on the connections between their underlying meaning but emphasised the acquisition of factual knowledge.

The key features of each teacher's approach are summarized in Table 1.

Table 1
Key Features of the Teachers' Instructions of the Constant Function

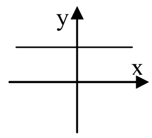
Ahmet	Burak
• Introduces the concept through an operational definition. • Establishes connections between representations and between ideas. • Uses clear concept-driven language to elucidate the idea of constant function. • By solving particular problems in different ways, creates new opportunities to maintain students' engagement with the concept. • Uses set-diagrams and ordered pairs like scaffolding to support students' access to a function process in the algebraic and graphical context. • Encourages students' visualisation of the graph of constant functions.	• Introduces the concept through a structural (an algebraic) description. • Does not establish connections between representations and between ideas. • Uses concise but vague language to explain the concept but clear and explicit language to establish a procedure. • Offers everyday analogies, but does not link them to the concept. • Makes use of the students' previous knowledge to teach the procedure not the concept.
• Implements a guided discovery teaching method.	• Implements a focused-questioning teaching approach.

LEARNING OUTCOMES

Pre-tests given to the students of each class indicated that there was almost no difference in their conceptual knowledge (for example, an understanding of dependence between two varying quantities, the ability to interpret an implicit relation within a set of ordered pairs) and in their procedural skill in manipulating algebraic expressions.

However after the instructional treatment the groups differed considerably in their understanding of constant function. To illustrate this we consider their responses to two situations presented to them.

> **Situation 1:** Does this graph represent a function from R to R?
> Give your answer with the underlying reasons.



25% of Class B and 18% of class A either did not respond to the question or suggested that the graph did not represent a function. Two thirds of Class A established the function process from the *x* to the *y*-axis and identified the 'all-to-one' matching over the graph whilst less than half of the students in Class B did so. Most of the remaining students from Class A simply identified the situation as 'function' because of the vertical line test but for every one of these in class A there were two in Class B whilst yet others in Class B gave the rule provided by Burak during his teaching:

> Yes, we could say, in general, that every line parallel to the *x*-axis represents a constant function. **I suggest you to write down this note on your notebook**.

This reason was reiterated by Belgin and Serap, two students from Class B, who were interviewed about their response to situation 1:

> … I mean, we solved similar problems in the lessons; there are similar examples in my notebook…umm…we learned that every parallel line to the *x*-axis is a graph of function. (Belgin, Class B)

> …as far as I remember every line parallel to the *x*-axis represents a function…
>
> (Serap, Class B)

Belgin's argument suggests that she was acting with her concept image – she had a notion that a graph of function transforms elements from the *x* to the *y*-axis – but she could not indicate how or why the transformation occurred. During the interview she revealed a misconception arising from her interpretation of the situation:

> …this element [where the graph intersects the *y*-axis] has not been assigned to any element in the domain… (Belgin, Class B)

Such comments were in marked contrast to those given during the interviews by the three students of Class A of which Okan's is typical.

> It matches every value of x on the x-axis to this point, say 8 [the intersection point on the y-axis]. I mean, this function matches infinite numbers in R to 8 therefore it is a graph of constant function. (Okan, Class A)

The second situation reported within this paper is the students' understanding of the constant function in the algebraic situation.

Situation 2: Given the functions $f: R \rightarrow R$ $f(x)=5$ and $g: R \rightarrow R$ $g(x)=3$, what is the value of $(fog)(7)$? Give the reason to your answers.

86% of Class A students produced a correct answer to this question whilst only 48% of Class B students did so. The Class A students emphasized the notion that a constant function transforms every input to one and the same output. Half of Class B either produced no answer or moved away from the problem once they had satisfied the composition protocol such that $(fog)(7)=f(g(7))$.

Class differences in this task cannot be simply explained by difference in students' understanding of the composite function because in the post-test 89% of Class A and 93% of Class B worked out the image of 2 when the functions $f(x)=2x+1$ and $g(x)=x^2-1$ were given.

Interviews with the six students provided detail that complemented the results from the questionnaire. The three students from Class A (Okan, Demet, and Erol) indicated a strong process conception of constant function whereas only one student in Class B (Aylin) did so. These four all commented on the 'all-to-one' transformation from x to the y-axis in situation 1. Their responses to situation 2 also articulated the idea that a constant function transforms every input to one and the same output. Aylin's response is typical:

> …I have to find, first, the value of $g(7)$… $g(x)$ matches all the elements to 3. Why? Because it is a constant function; therefore it takes 7 to 3. …$f(x)$ matches all the values of x to 5; again the reason is the same; it is a constant function…so $f(x)$ matches 3 to 5… (Aylin, Class B)

Serap, Class B, appeared to be in transition from an action to a process conception of function in both the graphical and the algebraic contexts. She was able to interpret the implicit processes in the algebraic situations and manipulate the inputs (7 and 3) through the process of $g(x)$ and $f(x)$ respectively but her conception missed the essence of the constant function as an 'all-to-one' transformation. We saw previously that she suggested that every line parallel to the x-axis represented a function. When asked to explain the underlying meaning of this she indicated:

> Let me think…if I say this graph matches the elements on the x-axis to this point [where the graph intersects the y-axis], may I make a mistake…[silence]…I am not sure; but this is the only reason I would give… (Serap, Class B)

Though Serap accurately describes 'all-to-one' transformation there is uncertainty in what she says. It is conjectured that this shows that she had not fully attained a process conception of the constant function.

Belgin, the final student from Class B was considered to be at the action level in both the graphical and the algebraic situations. She did not attempt the latter because from her perspective she needed algebraic formulas to carry out the composition.

Table 2 summarises the interviewees' development of the constant function and indicates class differences identified through the post-test questionnaire.

Table 2
Summary of the Interviewees' Development of the Constant Function in the Graphical and Algebraic Situations

	Class A			Class B		
Representation	Okan	Demet	Erol	Aylin	Serap	Belgin
Graphical Situation (I)	P	P	P	P	A→P	A
Algebraic Expressions (2)	P	P	P	P	A→P	A

Abbreviations: **A:** An action conception of function. **P:** A process conception of function.
A→P: Transition towards a process conception of function

Table 2 suggests that the three students from Class A and one from Class B attained a process conception of constant function. The other two from Class B either indicated progress towards a process conception of constant function or remained at the action level.

DISCUSSION AND CONCLUSION

Simply defined, teaching can refer to instructional acts taken to help students construct knowledge but it is a complex cognitive skill delivered in an ill-structured dynamic environment (Leinhardt, 1993). Learning is a cumulative process that an individual develops through interacting with external or internal stimuli. The mediating process between the two is open to the influence of many factors that may include the individuals' cognitive ability, attitude to mathematics and external factors such as parental involvements in students' education and the type of society in which the students live. The difficulty in controlling all of these influences does not permit an explanation of the relationships between teaching and learning in the sense of a cause-effect relationship. However, our evidence suggests that teaching practices that differ in their approaches to the essence of a concept are likely to produce qualitatively different learning outcomes. Ahmet's process-oriented teaching approach appears to have encouraged his students to develop a process conception of constant function whilst Burak's action-oriented practices largely constrained his students' understanding to an action conception of constant function.

The development in each group of students' understanding cannot be explained by any one particular aspect of the teachers' instruction; instead it can be best construed as the full impact of teaching inputs that make up action-oriented and process-oriented teaching approaches (outlined in Table 1). Additionally, what the students had learned in other lessons on functions might have positively or negatively affected their acquisition of the constant function.

Having said this, to illustrate the distinction in teachers' approaches to the essence of the concept and its consequent learning outcomes, we consider again the teaching discourses associated with the algebraic problem (Situation 1). As has been seen in Episode A, the focus of Ahmet's instruction is on the concept of constant function. The definition of constant function is used as a cognitive tool throughout to derive a solution to the problem. Unlike Burak, he does not set up an easily accessible goal (equalise the coefficients of x^2 and x to zero and calculate the values of 'a' and 'b'). Rather, he acts as a facilitator and prompts the students' thinking through indirect but concept-driven explanations: *"...there is something that does not allow the transformation of all the real numbers to one and the same..."*. The effectiveness of this approach can be seen in the students' data. 86% of his class composed two constant functions at the point 7 with a clear articulation that a constant function does an 'all-to-one' transformation – a feature confirmed through the interviews with the three Class A students.

In contrast, Burak, when solving a similar problem (Episode B), engaged his students with the visual properties of the algebraic form of the constant function. He brought the algebraic description ($f(x)=a,\ a\in R$) to the students' attention and stressed time and again factual knowledge: a constant function does not involve x. Taking this idea as a referent, Burak sets up the goal that the term with x must be removed from the expression, but he does not illustrate nor encourage his students to find an answer to the question 'why should it be removed?'. We can see the negative impact of this action-oriented teaching approach on his students' learning. In response to the questionnaire, less than half of his class indicated a full understanding of the constant function in an algebraic form, a feature replicated during the interviews when only one of his students did so.

In conclusion, the evidence suggests that teachers have considerable role to play in students' knowledge construction. They perform this role by creating opportunities in which individuals construct their own knowledge. Provision of analogies and the use of students' prior knowledge cannot facilitate meaningful learning unless they are (re)organised and presented in a way that communicates the essence and the components of the targeted concept to the learners. The evidence suggests that to help students construct epistemologically correct and conceptually rich knowledge of constant function teachers should prioritise the concept itself. They should allow the students to experience the concept across representations, utilise pedagogically powerful representations to encourage the students' access to the idea in the implicit ones, establish connections between the ideas and between the representations, and provide concept-driven clear and explicit language to illustrate the notion of constant function.

An additional interesting feature that emerges from this study suggests that teachers may have a strong understanding of the ways students think about function and the possible difficulties and misconceptions that the students would encounter, yet this does not imply that they will be attentive to these issues within the classroom. Burak had a good understanding of all of these features but (as he indicated during his interview) external factors, which in this case was the Turkish examination system, could misguidedly influence teachers to side-line or even disregard them in order to ensure that students had appropriate practice in examination type questions.

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The Number Line Representation within the National Numeracy Strategy: Children's estimation skills

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ABSTRACT: *This paper considers children's accuracy in estimating the position of numbers on a 0 to 100 number line, a task inspired by numerous similar examples from within the English National Numeracy Strategy. Children with median ages ranging from 6.5 to 10.5 were invited to position paired numbers on a 0-100 number line segment. Differences between their estimates and the actual position of the numbers indicates that they exaggerate magnitudes in the lower half of the line but underestimate them in the upper half. The findings call into question explanations of behaviour orientated towards linear or logarithmic models and suggest the alternative view that to some extent accuracy is dependant upon perceptual and imagined "correctors".*

Key words: *Errors, Estimating Magnitude, 0 to 100 number line segment, National Numeracy Strategy, Number Line.*

INTRODUCTION

Within England and Wales the process of improving standards in mathematics is specified within two documents. The first, the National Curriculum for Mathematics (DfEE, 1999a), identified the contents of the mathematics curriculum that should be taught during Key Stage 1 (KS1, age 5 to 7) and Key Stage 2 (KS2, age 8 to 11). The second, the National Numeracy Strategy (NNS) (DfEE, 1999b), identified, almost through a process of genetic decomposition, how and what should be taught to instil a sense of being numerate. The NNS is fully aligned with the Programmes of Study outlined within the National Curriculum and therefore provides a detailed basis for implementing the statutory requirements. Its developmental approach for the acquisition of numerical skills and concepts sees the transition from the use of perceptual items to "more sophisticated mental counting strategies" finally sublimated by the acquisition of basic number facts and the development of knowledge of the number system. To achieve these ends the NNS suggests that a daily mathematics lesson is appropriate for almost all pupils with a high proportion of the time devoted to whole class teaching through which mental and oral work should feature strongly.

The question arises as to whether or not the initiatives within the UK have provided opportunities for children, particularly for those considered to be at the lower end of the spectrum of achievement, to develop a sophisticated means of thinking. This paper gives a response to this question by considering one small aspect of the initiative, the accuracy with which children with a median age range of 6.5 to 10.5 estimate the position of numbers on a 0 to 100 number line segment.

THE NUMBER LINE: REPRESENTATION AND TOOL

The larger study of which this paper is part (Doritou, 2006), was informed by the legal requirements of the English National Curriculum and was associated with curricula guidance presented within the National Numeracy Strategy. The study focused on a particular representation, the number line, “a key classroom resource” (DfEE, 1998, p. 23) that is frequently illustrated within the NNS to support the development of numeracy concepts and skills. However, a perusal of the documentation suggests that the number line is a resource more frequently associated with procedural applications rather than the conceptual development that may arise from its inherent conceptual structure. For instance, the number line is seen as a mechanism for showing how forwards and backwards counting works but there is an implicit ambiguity between its use and the use of the number track. These two representations are different in nature — one is an analogy of the continuity associated with number, the other emphasises the discrete nature of whole number. Differences between the two are manifest within the NNS by perceptual differences associated with either labelling spaces or labelling points. There is no reference to what the points actually mean and this can undermine the initial development of what may become a sophisticated mean of representing the number system.

The association between number (real number) and line has been evident since Babylonian times (Wilder, 1968). The Greeks intuitively conceived real numbers as corresponding to linear magnitudes and their idea of "magnitude", which is substituting magnitude for number, implied that one might think of "numbers as measured off on a line" (Bourbaki, 1984/1994, p. 121). The number line is, therefore, an abstraction of a representation strongly associated with the notion of a measure instrument since continuity underscores it — starting from the Euclidean line, a "sense of continuity" can be created for and by the individual and the result be used as a number line to represent natural numbers.

In the context of school mathematics and somewhat in contrast to the above statement, the Association of Teachers of Mathematics (ATM, 1967) referred to the number line as if it was a tactile object, and suggested either "cutting", "wrapping" or "stitching" it round the circumference of a circle. The number line was presented as a result of a series of mappings, frequently associated with the notion of multiple but particular attention was given to the visual effect created by the varied-sized jumps that created a range of "curve stitching" presentations.

In 1979 the Department of Education and Science (DES) in England, published a Handbook of Suggestions in Mathematics for the ages 5 to 11 and stressed that the number line was extremely important during the early stages of mathematics to support the development of addition and subtraction. Structured number lines and empty number lines were presented within the suggestions as equidistant points for ordering the natural numbers. Later conceptions of the empty number line emphasized the maintenance of number order but not the scale (Dettmer-Kratzin, 1997).

Williams and Shuard (1976) extensively discussed the nature and use of the number line as a representation of the number system within primary mathematics classrooms. They pointed out that it could support conceptual understanding of number development and specified that its underlying continuity highlighted its underlying characteristic that each point on the line corresponds to a unique number. Between the natural numbers there is an infinite number of points that correspond to other numbers such as fractions and decimals. They emphasised the importance of orderly and equally spaced points and the property that the number line could be extended in both directions to infinity.

More recently, Herbst (1997) indicated that the number line might be seen as a metaphor of the number system. He defines it as the consecutive translation of a specified segment U as a unit, from zero. U itself can be partitioned in an infinite number of ways (i.e. fractions of U). In its completeness the number line is a very sophisticated metaphor, but it is conjectured that recognition of this sophistication can grow through experience and use.

In their review of mathematical representations, seen as the mediators of mathematical concepts and processes between teachers and pupils, Lesh, Post and Behr (1987) categorise the number line as a manipulable model that has built in relationships and operations that can fit many learning situations. The consequence is that there have been a range of suggested modifications and uses of the number line as a representation to support the development of mathematical skill and understanding. It is considered as a diagnostic tool for both the pupil and the teacher (Rousham, 1997) and it can be a helping tool for teaching whole number operations (see for example Behr & Post, 1992). The fact that pupils can physically act on it allows them to be cognitively involved in their own actions and make decisions about where to make a mark and write down the numbers on the line (Beishuizen, 1997; Klein, Beishuizen & Treffers, 1998).[1] Thus the "dense" nature of the number line not only permits it to be seen as a representation of the number system but also enables it to be used as a "solving and justifying" tool (Herbst, 1997). It may help in obtaining the solution to a problem on the

[1] The reader should note that it is not the purpose of this paper to consider modifications to the notion of number line that include the 'empty number line' (see Klein *et al*., 1998), nor to discuss the merits or otherwise of associated models such as the number track, a notion that may cause considerable confusion. The distinction has been clarified by Skemp (1989).

line, as well as provide justification of the way of thinking and to the way an answer is obtained.

Throughout the 1999 version of the NNS, reference to using the number line to support the development of an understanding of the number system is continual. It is associated with the development of counting skill to order numbers to 10 (Reception, median age (MA) 4.5), to 20 (Year 1, MA 5.5), to 100 (Y2, MA 6.5) and it is used to develop understanding of fractions (Y2 onwards), decimals and negative numbers (Y4 onwards). Additionally, we may see it used from Y2 to develop skill in addition and subtraction. It is worth indicating that the recently published version of the NNS (DfES, 2006a) has a tendency to require that objectives associated with number knowledge, the introduction of operational concepts and the extension of the number system to include fractions, are introduced approximately one year earlier than the 1999 document. It is suggested that this would ensure an increasing level of high achievement, particular in those areas where progress is not being sustained. Such a change may be seen to contribute towards the “drive to raise standards and personalize learning so that all of our children can achieve their full potential.” (DfES, 2006a, Introductory Statement).

IDENTIFYING NUMBER POSITION ON THE NUMBER LINE

Throughout the early stages of the 1999 NNS document there were recurring references suggesting that children within Years 2 to 4 (median ages 6.5 to 8.5) should be able to have experiences that would enable them to interpolate the position of a particular number on a partially numbered or an empty number line, record their estimates and find the difference between the estimate and the actual number. For Year 2 children the presented examples include number lines with the ends marked 0 and 10 and for Years 3 and 4 number lines are marked from 0 to 100. It is implicit that children within the Years 5 and 6 possess the skill and understanding to do this. Continued reference to these forms of activity is made throughout the document identifying the links between the 1999 the 2006 recommendations (DfES, 2006b).

Several studies have investigated number line representation by individuals with particular reference to estimating the position of a particular number. Dehaene (1997) considered the internal representation of numbers by adults and suggested that they express a logarithmic pattern, smaller numbers are to the left and larger numbers, tending to be compressed into a smaller space, are to the right. He argued that this is an inherent and intuitive mental idea of the number line that we all possess and that this idea:

> "… is automatically activated whenever we see a number and which specifies that 82 is smaller than 100 without requiring any conscious effort. This "number sense" is embodied in a mental number line oriented from left to right."
>
> (Dehaene, 1997, p.151)

He further suggested that on this mental number line not all numbers are represented with the same accuracy.

> Subjectively speaking, the distance between 8 and 9 is not identical to that between 1 and 2. The 'mental ruler' with which we measure numbers is not graduated with regularly spaced marks. It tends to compress larger numbers into a smaller space. … As soon as a continuum needs to be divided into discrete categories, intuition dictates the selection of a compressed scale, most often logarithmic, which tightly matches our internal representation of numbers.
>
> (p. 76, 77)

He argued that the error for smaller numbers is smaller because humans express smaller numbers more often than larger numbers throughout their lifetime. Smaller numbers are taught first and they are repeated consistently.

Siegler and Opfer (2003) and Siegler and Booth (2004) argue that estimations associated with representation of numerical quantities on a number line can be either logarithmic or linear or an amalgam of the two. Linear representations suggest a constant growth in the magnitude of the errors. A logarithmic scale would suggest that as we move away from zero, errors associated with estimations become larger. Siegler and Booth (2005) concluded that kindergarteners' estimations expressed a consistently logarithmic pattern, 1st graders' an amalgam of logarithmic and linear patterns and 2nd graders' purely linear. Siegler and Opfer (2003) concluded that children’s (2nd, 4th and 6th grades) estimations of numbers in the range 0 to 100 expressed linear mapping between the numbers and their magnitudes that improves in consistency with age. Children’s estimation of numbers from 0 to 1000 expressed a logarithmic pattern which:

> "… exaggerates the distance between the magnitudes of numbers at the low end of the range and minimizes the distance between magnitudes of numbers in the middle and upper ends of the range." (Siegler & Booth, 2004; p. 429)

Siegler and Booth (2004) and Booth and Siegler (2006) suggest that linear representations are associated with high achievement related to age and experience, as well as an approach to estimation that involves dividing the line at specific points and the use of these points as references to locate other numbers (Siegler & Opfer, 2003). An accumulative approach towards estimating the position of numbers results in the logarithmic pattern. Given the recommendations of the NNS which suggest that younger children should be given opportunities to estimate positions on a 0 to 100 line, our intention was to consider the estimations of children from Y2 and above within one English primary school on a similar line.

METHOD

As we have seen, the NNS (DfEE, 1999a; DfES, 2006b) suggests that children should be able to associate their inherent knowledge of the number system from 0 to 100 with magnitude to estimate the position of numbers on a 0 to 100 number line. This paper reports on the quantitative outcomes derived from the presentation of a series of 0 to 100 number lines to children with the invitation that they place particular numbers on individual number lines. The individual items were compatible with items identified from within the 1999 version of the NNS (Section 5; pp. 8, 9, 17) but were also selected to consider differences in the children's ability to estimate numbers that were equidistant from the end points 0 and 100. This was inspired by a comment within QCA (1999a):

> An important aspect of having an appreciation of number is to know when a number is close to 10 or a multiple of 10: to recognise, for example, that 47 is 3 away from 50, or that 96 is 4 away from 100. (p. 28)

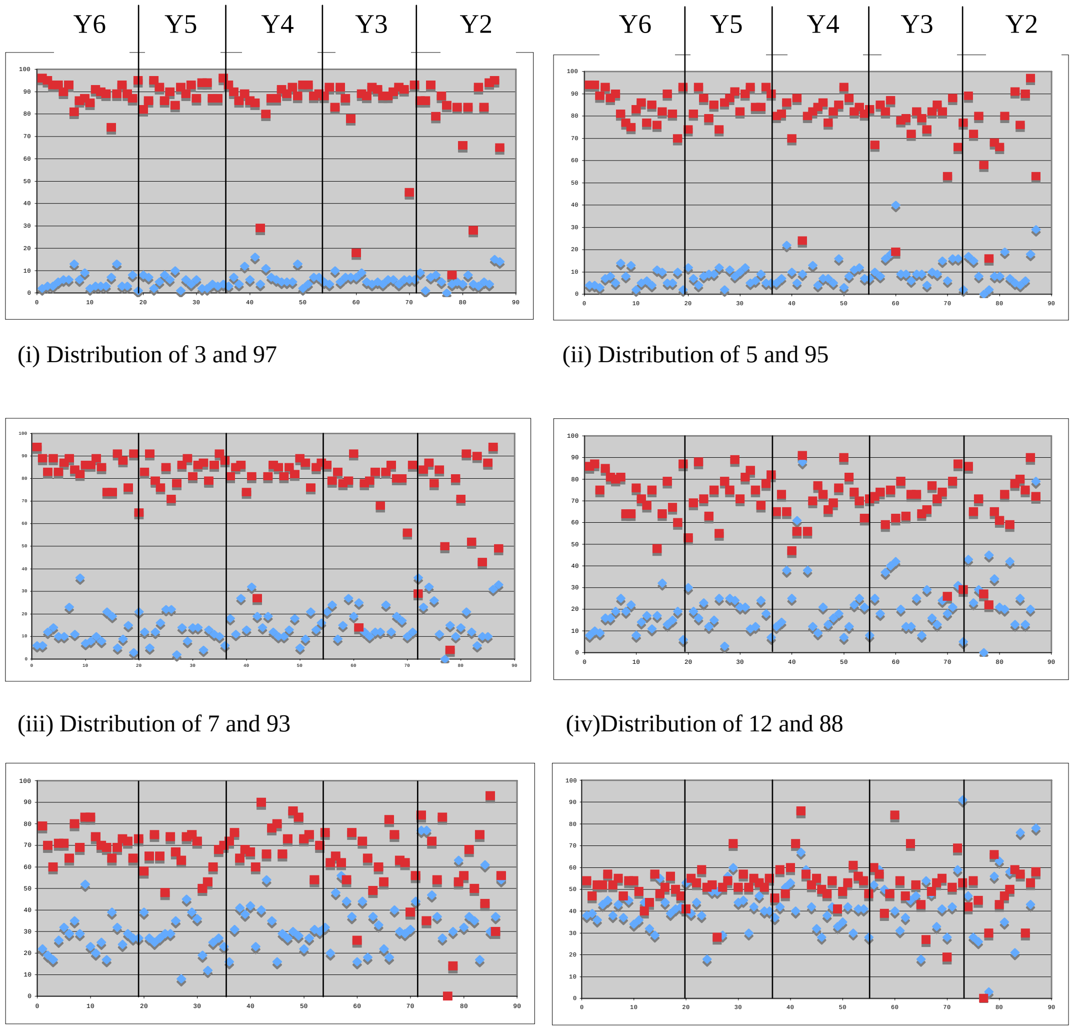
The numbers, in the order that the children responded to them were: 93, 45, 12, 5, 75, 3, 7, 25, 97, 93, 88 and 55.

The sample responding to the pinpointing questions consisted of 91 children (18 from Year 2, 18 from Year 3, 19 from Year 4, 16 from Year 5 and 19 from Year 6). The questionnaire was designed so that the children were presented separate sheets with a series of numbers and required to place each number on a 0 to 100 number line segment placed horizontally on each sheet.

The data, drawn from children from a school within the English Midlands had, by accident rather than design, remarkable similarities to that of Siegler — the school was within a relatively deprived area with children from diverse cultural backgrounds. The school has extensively used the NNS since 1999.

RESULTS

Figure 1 provides an illustrative example of the way of the distributions of the estimations associated with each pair of numbers. The figure is constructed by providing all of the estimates from all of the children in the sample and the distributions associated with each year group may be identified within the relative columns of each graph. In each instance the horizontal scale is marked to 91 (the total number of children) whilst the vertical, to maintain consistency, is on a 0 to 100 scale subdivided into units of ten.



(i) Distribution of 3 and 97

(ii) Distribution of 5 and 95

(iii) Distribution of 7 and 93

(iv)Distribution of 12 and 88

(v) Distribution of 25 and 75

(v) Distribution of 45 and 55

Figure 1. Comparative distribution of estimations associated with number pairs by children, within Y2 to Y6.

From Figure 1 it can be seen that:

- Children tend to underestimate the position of numbers near 100 and overestimate numbers near 0. The distribution of the children's estimates of magnitude appears far more widespread for the numbers 93 and 88 than they are for the numbers 7 and 12. The numbers 93 and 88 are almost always under-estimated, but there was a tendency for the children to over-estimate 7 (88% of the full sample) and 12 (80% of the full sample).
- The overall accuracy of the estimations by the children decreases as the magnitude of the number to be estimated moves away from 0 but begins to improve as the numbers move closer to 100. For example when estimating 97

and 95, a considerable number of children, particularly from within Years 2, 3 and 4 displayed a remarkable degree of inaccuracy. For example, 50% of Year 6 children and 75% of Year 2 children give an estimation of 97 that is below 90 whilst 70% of the former and 80% of the latter give estimations of 95 that are below 90. Of these, 40% of Y6 children gave estimations below 80, whilst 90% of the Y2 children did so. In comparison, only 10% of the Y6 children and 27% of the children from Y2 gave an estimate of 5 that was larger than 10.

- The distributions associated with 25 and 75 appear almost random, although each estimate is generally confined to an appropriate half of the 0 to 100 line, that is the former is estimated to be below 50 whilst the latter is estimated to be above 50. Again, the main distinction between the two numbers is generally a tendency to overestimate 25 and underestimate 75.
- The distributions associated with 45 and 55 are interesting. Though there is the possibility to provide either an underestimate or an overestimate for these numbers, the distribution of the estimates would seem to have more in common with the distributions for 97 and 3 than with the distributions associated with other number estimates. Pinpointing these numbers does not have the limitations that are associated with pinpointing 7, 5, 93 or 95 — there are no upper and lower limits — and yet there is a sense that the children were being guided to hone in on the numbers. Over 60% of the children pinpointed 55 to within ±5 units of accuracy. 40% did so to 45.

Absolute accuracy, defined as an exact estimation, was only achieved in 4% of the 1080 estimates that were considered. Almost 50% of these were identified in the instances where 3 and 5 were considered. Absolute accuracy was identified equally for both of these numbers. Accurate estimates for 12 and 55, each contributed 15% to this 4%. No absolute accuracy was identified from estimates of the numbers 93, 95 or 97. A more generous interpretation of accuracy, identified by using a range of ±2 units from the actual number, is achieved by almost 20% of the responses of the full sample. Three numbers account for 50% of this proportion: 3(22%), 5(16%) and 55(13%). The remaining 50% is accounted for by the other nine numbers of which the lowest proportions are 93(2.5%), 97(3%) and 88(4%).

Not unexpectedly, the greatest accuracy was noted in the responses of children from within Y6 where 28% of the responses satisfied this degree of accuracy. The lowest proportion was noted from the responses of children within Y3 (10%) and Y2 (13%). While there is a sense that accuracy improves as children get older, there is little evidence that there is an overall consistency in accuracy of estimation and that general mathematical achievement contributes to accuracy.

The characteristics identified from Figure 1 are shown in sharper relief, if each year group is highlighted and if we draw on distributions that are associated with differences between estimations and actual numbers.

Figure 2 provides an illustrative example of one pair, 88 and 12, of each child’s estimate and the distribution of these estimates within each year group.

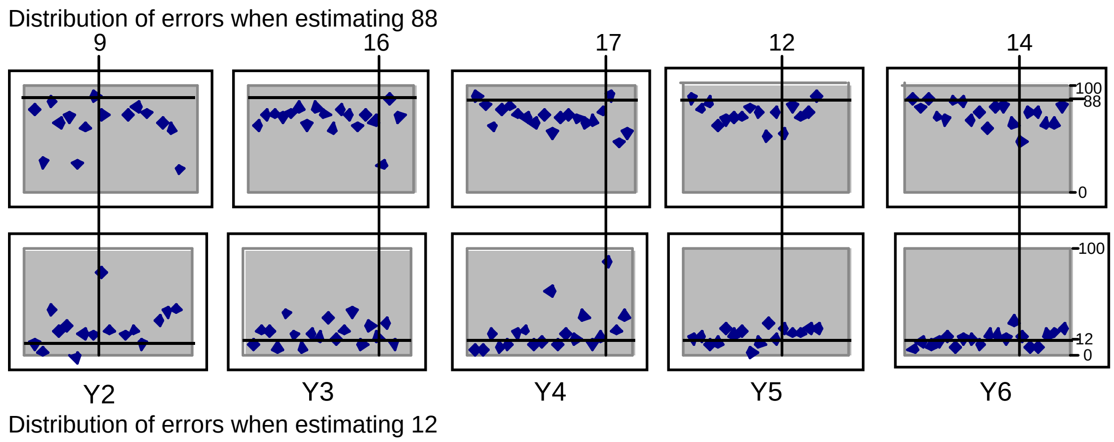


Figure 2. Full sample estimations of 12 and 88 identified by year group.

Each child’s estimate of the numbers 88 and 12 (marked on the figure) is identified on a vertical scale marked 0 to 100. The distributions are arranged so that children within each year group are given in an order that reflect their overall achievement within mathematics as measured by inter school tests and teacher predictions of achievement within Standard Attainment Tasks. High achievers are to the left and low achievers are to the right.

Even though the numbers to be estimated are equidistant from the end points of the line several features emerge from Figure 2:

1. A general trend amongst the majority of children to underestimate the position of 88 and over-estimate the position of 12.
2. The overall accuracy of estimations tends to be associated with age. The estimations of the older children, particularly their estimation of the position of 12, are generally far more accurate than those of any of the younger children but though we note a gradual convergence in the children’s estimates of 88, even amongst the older children there is a tendency to extensively underestimate.
3. A child’s ability to provide a reasonable estimate of one number does not imply that the child will also provide a reasonable estimate of its paired counterpart. Note particularly the relatively good estimates of 12 given by children 16(Y3), 12(Y5) and 14(Y6) (see vertical lines in Figure 2) and yet each child gives one of the worst estimates of 88 in their particular class. Conversely, consider the good estimates of 88 by children 9(Y2) and 17(Y4) but their relatively poor estimates of 12.

4. Since each diagram is formed to reflect an order of achievement there is little evidence to suggest that mathematically higher achieving children are better at estimating than their lower achieving peers.

Looking at the sum of the absolute errors provides the basis for considering the distribution of the errors associated with each of the numbers pinpointed.

Figure 3 illustrates the size and direction of the difference between the average of the absolute difference between each child's estimation of the position of a number allocated, corrected to establish whether each average is an underestimate or an over-estimate, and the actual point. The curves are diagrammatic to illustrate the differences between Y2 (continuous line) and Y5 (dashed line). For example, the mean of the absolute differences between Year 2 children's estimates of the position of 12 on the number line and its actual position was 17 whilst that for Y5 children was 7. The general observations noted from within Figures 1&2 are supported through a consideration of Figure 3.

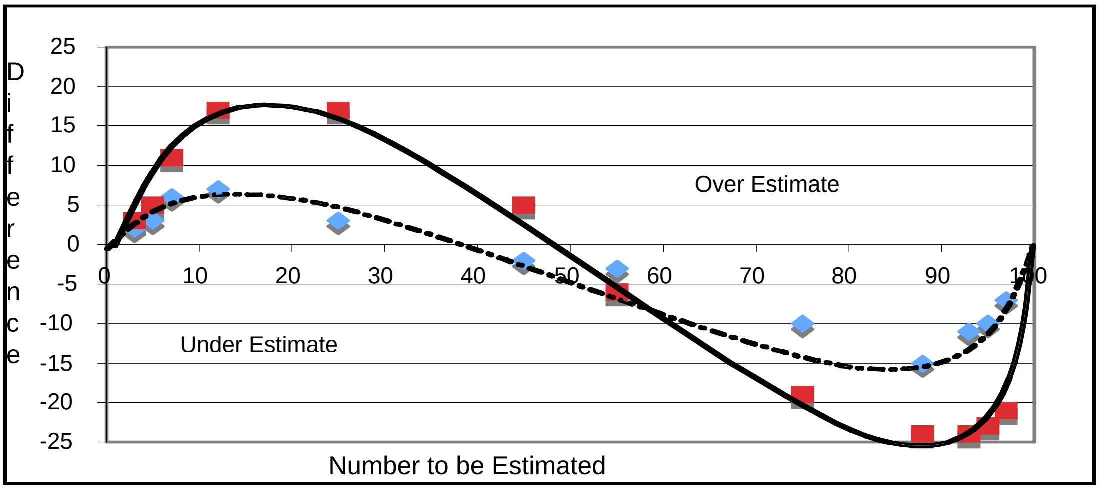
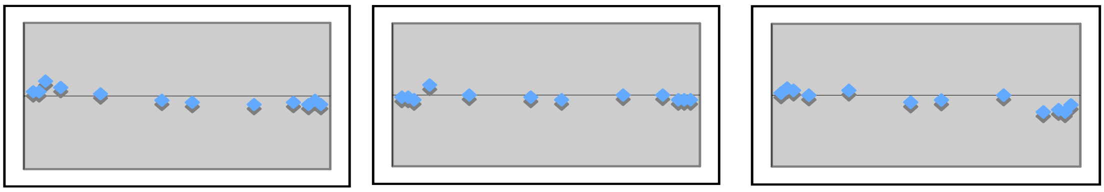


Figure 3. Difference between actual numbers on the number line and the mean of the absolute differences between that number and the estimates given by Years 2 and 5.

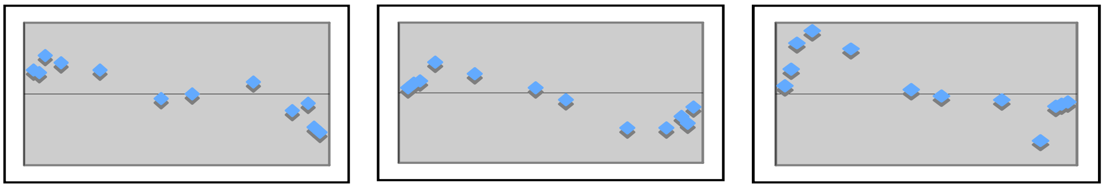
- Apart from one instance, estimating the position of 55, the magnitude of the differences identified from the results of the older children are always less than those of the younger children (Y2).
- Whilst Figure 2 indicates that the magnitude of 12 is over-estimated and that of 88 is underestimated, Figure 3 suggest the tendency to over-estimate magnitudes below 50 and underestimate magnitudes above 50.
- 45 is the only number for which the absolute differences are in the opposite direction. Y2 over-estimate and Y5 underestimate.

The distribution of the estimations of the sample of children within this study do not appear to match the general distributions suggested by Siegler and Booth even though there are strong similarities in the type of sample. One reason for this may be the restricted nature of the number line used in the current study. The children were unable to provide an estimation below 0 or above 100 and therefore these two points act as limits upon which estimations, particularly those of numbers towards the extremes, would seem to converge. However, the evidence does appear to suggest that there can be a wide range of error associated with the estimations of numbers between these limits and it also suggests that the estimations of individual children may display different features.



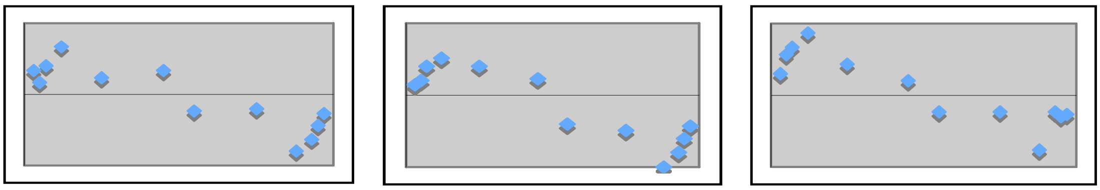
CHILD 6.4 CHILD 5.3 CHILD 4.16

(i) General sound estimation. All errors between 0 and 5 units



CHILD 6.7 CHILD 3.10 CHILD 2.4

(ii) Reasonable estimation. Several errors between 5 and 10 units



CHILD 6.15 CHILD 5.1 CHILD 4.4

(iii) Poor estimation. Several errors between 10 and 20 units

Figure 4. Illustrative individual estimates of magnitudes.

Figure 4 is established by considering the direction of errors associated with individual estimations. Each child pinpointed the numbers to be estimated with either absolute accuracy (and as indicated earlier only in 4% of the total number of estimations was this achieved) or the child's estimation was associated with an error, for example in estimating the magnitude of 25 a child may have actually pinpointed the position of 27, thus giving an overestimate of 2 unit errors. By considering the sum of the absolute value of these units of errors by each child, it was possible to identify the distribution of absolute error ranges categories across the sample of children. Figure 4 presents a selection of children from three identified error ranges. The errors associated with each child's estimation of the 12 magnitudes to be identified were considered and then each range of errors of each individual categorized as a degree of accuracy in their estimation; generally sound estimation (no error larger than five error units), reasonable estimation (several errors between five and ten error units) poor estimation (several errors between ten and twenty error units). Within each diagram the horizontal bar indicates zero error whilst errors placed above the bar indicate an over-estimate of the number, errors below the bar indicate an underestimate.

Though the three groups of children reflect different general levels of accuracy in their estimations, the trends in their responses appear to have similar characteristics and as their accuracy in pinpointing becomes less accurate these characteristics become sharper. Figure 4 confirms at the individual level what has been earlier suggested at the general level, that is underestimation of numbers within the first half of the 0 to 100 number line segment — and here the notion of half should be taken a little liberally since several of the children's estimates of 45 may not satisfy this conclusion — and overestimation of numbers within the second half. The children identified within Figures 4 (ii) and (iii) appear to treat 45 and 55 differently to the way in which they treat the numbers that fall before and after these numbers. On the one hand, the numbers between zero and 45 are over-estimated, with varying degrees of error; whilst on the other the numbers between 55 and 100 are under-estimated.

The general features within all of the presentations within Figure 4 are characterised by a "sinusoidal" shaped curve. The use of the term "sinusoidal" is not intended to have any mathematical significance beyond the descriptive. It is simply a mechanism to illustrate features that emerge from the distributions of the estimations within the three groups. Figure 5 draws upon these features to present a comparative illustration of the trends in the distributions of the mean differences of all of the year groups across the range of estimated numbers. To achieve simplicity it does not include pinpointed differences.
The indications from Figure 5 are that:

- Amongst all year groups there is a general tendency to over-estimate the positions of numbers in the first half of the number line segment 0 to 100 and underestimate the position of numbers in the second half.
- The maximum extent of underestimation can be greater than that of overestimation.
- The trends noted amongst children within Years 3, 4 and 5 are similar. Years 2 and 6 demonstrate departures from these in that differences identified within Year 2 are greater than those of all other years whilst those identified for Year 6 are less.
- The remarkable accuracy with which all year groups, apart from Year 6, identify the position of 45. It is equally remarkable that they do not then reflect similar accuracy with 55 (since 45 and 55 are equidistant from the middle of the line segment).

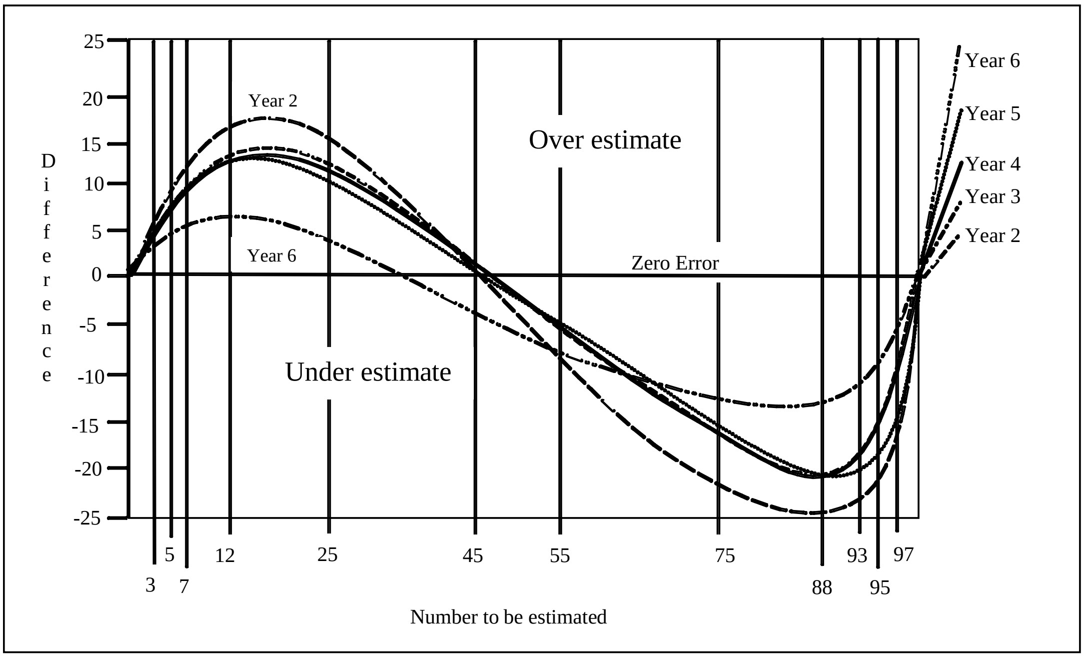


Figure 5. Summary of general trends in children's efforts to estimate position of numbers on a number line.

DISCUSSION

Given that two points marked on the number line segment were 0 and 100 it is perhaps not surprising that as children are required to pinpoint numbers close to the points, the errors that arise are generally small and unidirectional. Therefore, the accuracy of an estimation would appear to increase the nearer the required estimation is to an end point. Here there is a decreasing opportunity to estimate to an alternative side of the target — to make a negative error when pinpointing near 0, and to make a positive error when pinpointing near 100.

However, when the differences between the mean of the estimates of numbers within the extreme decades are considered, views on end point accuracy seem less tenable. The mean deviation of the magnitude for 3, 5 and 7 is an overestimate of 5 while that for the numbers 93, 95 and 97 is an underestimate almost three times as large, -14. Although the children cannot place a number beyond 100, a sense that they display accuracy because the number is near 100 is questionable. Though they appear to illustrate number sense associated with a left to right orientation of a number line, thus enabling them to specify that 93 is less than 100 or that 7 is greater than 1 (Deheane, 1997), the relative equality of the magnitudes of these numbers in the context of their defining frames seems to be missing despite their experiences following the NNS.

What is striking from Figure 5 is not so much that children of different ages appear to be more accurate in estimating magnitudes on the 0 to 100 number line segment line, but that the distribution of the magnitudes should have such a remarkable similarity.

The general evidence would seem to support the conclusions of Siegler and Booth (2004) in that the children exaggerate magnitudes in the lower half of the line but underestimate them in the upper half, and that estimations improve in accuracy as children get older and more experienced. However, the nature of the distribution of the differences does not support either a linear or a logarithmic theory. Nor does it allow us to subscribe to the view that with the range of children we considered there is a change in the pattern of the distribution. Although there is some evidence of growing accuracy this appeared to stagnate between Years 3 and 5.

Over 45% of the children below Y6 within this study chose to estimate the position of a number by counting and marking ones starting from zero. A further 45% used counting without marking. Only 20% of Year 6 counted from zero marking counted ones. Counting activities both in teacher demonstration and in children’s actions with the number line were observed throughout the spectrum of observations and interviews within the classes from within which the children were drawn. Such actions emphasised jumps and consequently it appears that the children's embodiments of estimating number position on the 0 to 100 number line segment were actions associated with counting. The year group errors associated with the numbers 25 (range from an overestimate of 18 for Y2 and 6 for Y6) and 75 (range from an underestimate of 22 for Y2 and 6 for Y6), suggest that it is not apparent that there is tendency for the children to divide the line at specific points and use these points as references to locate other numbers as suggested by Siegler and Opfer (2003). Although there was very limited evidence of children applying a strategic approach to their estimations (for example halving and quartering of the number line), the degree of accuracy associated with 55 and 45, and in particular the latter, seems to suggest an additional explanation.

The visible points 0 and 100 provide cues for positioning the numbers relatively close to them and therefore it could be argued that they serve as “correctors” where adjustments could be made. However, we also suggest that a middle number, possibly 50, may also serve the same purpose but its effect is limited (as also appears to be the case of the visible correctors). It may be that a clearer view of the children’s behaviour can be related to Fischer’s (1994) suggestion that the bias is associated with perception, both real and imagined and that in some instances perception overcomes procedural approaches. On the other hand, it may be conjectured that for middle numbers, children have developed a sense that these numbers are in the middle of the total distance throughout repeated practice in lessons. Finding the middle of lengths, meter sticks or number lines was an activity frequently observed within each class during the wider part of the study.

CONCLUSION

The findings of the study have implications for the objectives (outcomes) and the teaching approaches suggested by documents associated with the National Numeracy Strategy. The curriculum documentation used as a background for this study did not suggest that the number line could be seen as an abstract conception of the number system. It was (and still is within the 2006 documentation) presented as a concrete model that supports actions and, as such, it appears to make a limited contribution towards the development of conceptual understanding of the number system.

Within the school observed, the pedagogic input from the teachers was grounded within the requirements and recommendations of their national documents and the interpretation they appeared to make of these was associated with the development of procedural skills. Little emphasis was placed upon developing greater sophistication in understanding conceptual features. A consequence of this, it is suggested, is that the children's embodiment of the number line and its associated structure was formed from their relatively early experiences and changed very little with subsequent experience.

Doritou (2006) suggested that the development of a meaningful and sophisticated understanding of the number line appeared to elude these children. She obtained little evidence to suggest that they had an integrated view of the different features of the number system that could be represented on the number line. The interpretations they made of the number line, the actions associated with it and the communication of mathematical meaning did not give them a global picture of a number system but instead a picture made up of several discrete packages, whole numbers, fractions and decimals. As each package developed, their understanding of the number line remained embodied in a whole number world that, it is conjectured, did not reflect the assumptions of the teachers or the expectations of the NNS. The teachers' use of the number line to emphasise counting procedures suggested an embodiment that was clearly apparent in the children's attempts to estimate numerical position. Evidence of the use of a strategic approach that may have grown out of a perception of partitioning based upon reconsidering the 0 to 100 segment as a new whole subdivided into halves and quarters was virtually absent. The results would suggest that the experience advocated by the NNS and interpreted by the teachers did not appear to translate into a sense of number pinpointing that based upon the application of general number knowledge to a sophisticated representation.

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A Life-Time's Journey from Definition and Deduction to Ambiguity and Insight

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ABSTRACT: *In this paper I speak of a personal journey in mathematics and mathematical thinking that began in a mathematical world of precision and certainty and found a world of mathematical thinking full of ambiguity and insight. It is a journey on which I have had many fellow travellers from whom I have learnt most of what I know, particularly my colleague Eddie Gray and our research students, and other very special co-authors. They have accompanied me on various quests in a life-time's journey by way of 'concept image', 'measuring number', 'local straightness', 'generic organiser', 'cognitive root', 'procept', 'cognitive unit', 'advanced mathematical thinking', to 'three worlds of mathematics' where the route to the world of formal mathematics is by way of two other mental worlds— conceptual embodiment (thought experiments based on perceptions) and proceptual symbolism (actions, such as counting, symbolised as concepts, such as number). On the journey I also discovered new insights, at least for myself: that the proliferation of mathematics education research is constructing a growing environment for individuals to build theories and careers, leading to a welcome level of productivity, but that this productivity has led to a wealth of complication that needs compression into an insightful simplicity. We know more than we knew thirty years ago, but our knowledge has not produced universal success in teaching mathematics. Looking back, I see my journey as a quest to seek the underlying simplicity that enables us to think in a powerful mathematical way in our increasingly complicated lives.*

Key words: *Concept image, measuring number, local straightness, generic organiser, cognitive root, procept, cognitive unit, advanced mathematical thinking, three worlds of mathematics.*

STARTING FROM WHERE WE ARE

We all begin our journey from where we are at the start. My own journey in mathematics education began from my position as a mathematics lecturer in a university mathematics department which coloured the ways in which I viewed mathematics and mathematical thinking. I sought clear definitions, clear deductions and the construction of a coherent theory of mathematics education.

In the early 1970s, the main cognitive frameworks available were stimulus-response behaviourism (hardly appropriate for the subtleties of mathematical thinking) and Piaget's epistemological approach to child development. The simple but profound book on *Psychology of Learning Mathematics* by Richard Skemp (1971) came as a breath of fresh air. I was so enamoured of his work that I invited him to speak to the Mathematics Department at Warwick and the mathematicians were so impressed that, when the Education Professor resigned that year, Richard was invited to apply for the post and became Professor of Education.

When Richard was asked to review Freudenthal's book *Mathematics as an Educational Task*, having already bought his own copy and not wanting another, he passed the invitation to me. To review a work of the great Freudenthal was a huge task for a young mathematics lecturer and I sought advice from a senior colleague, James Eels, who knew him well. He confirmed that I should say *exactly* what I felt and, emboldened by his advice, I wrote a welcoming but critical essay. I received a post-card from Freudenthal after the review appeared: "thank you for the review which I enjoyed, *especially* the critical parts."

My position in the Mathematics Department was as 'a lecturer in mathematics with special interests in education'. Rolph Schwarzenberger, a professor and a friend who gave me guidance, encouraged me to begin research into the learning of undergraduates and I began a study on students' understanding of limit processes (Tall, 1976) where I found that most students believed that '0.98 is just less than one'. I had data, but where was the theory?

I still saw mathematical thinking from the viewpoint of a mathematician, becoming enamoured of catastrophe theory through the research of my departmental chair Christopher Zeeman (1977). The paper I presented at the very first meeting of PME (Tall, 1977) used catastrophe theory to describe how the brain could leap suddenly from one viewpoint that becomes untenable to another that is more stable.

This paper interested Shlomo Vinner and I travelled to Israel in 1979 to work with him. There I met Efraim Fischbein, the first President of PME, who was at that time working on the concept of infinity. As a mathematician, I suggested to him that the conflicts in the data he had collected related in part to two different ways of looking at infinity: cardinal infinity arising as an infinite extension of counting and the infinities and infinitesimals of the calculus of Leibniz that I suggested were infinite extensions of 'measuring' (Tall, 1980a). He told me he wanted to 'see' an infinitesimal and I suggested that rational functions could be ordered by looking at their graphs and defining $f(x) > g(x)$ if the graph of f is above the graph of g for sufficiently large values of x. Functions such as $y = x$ or $y = x^2$ are, in this sense, greater than zero but ultimately below any positive constant graph so they are 'infinitesimals'. He protested saying that he wanted to 'see' an infinitesimal as a tiny value and these rational functions did not look ‘small’ to him. He remained unmoved when I suggested we shift our attention to a vertical line far off to the right to see that constant functions $y = c$ met the line in fixed points, but variable functions met the line in variable points where those that were positive but ultimately lower than any given positive constant were infinitesimal. I wrote about this in a paper (Tall, 1980b) and was met with the wrath of Israel when Tommy Dreyfus alerted me that Fischbein was concerned that he had more strength in psychology than in mathematics and that, in referring to his rejection of mathematical constructions, I had touched a sensitive spot. I wrote an immediate letter of apology to Efraim and he and his student Dina Tirosh became long-term friends and continued to keep in contact over the years. When he passed away, I was invited to give his eulogy at PME.

Back in Warwick, at the age of around forty, I started a second PhD in Mathematics Education with Richard Skemp who was about to become the second President of PME. He was such an inspiration with his clear and simple approach to theory, and gave me great insight into the psychological side of mathematical thinking. I was invited to present the eulogy in his memory at PME too.

When Shlomo Vinner visited me at Warwick at Easter in 1980, I had a great deal of data available and no way of making sense of it because the students’ answers weren’t mathematically coherent. He showed me his paper with Rina Hershkowitz (1980) prepared for PME later that year in which students’ interpretations of geometric concepts were analysed in terms of the new notion of ‘concept image’ and ‘concept definition’. It was exactly what I needed to make sense of my data. I began writing with great speed and energy and wrote the first draft of a paper in a day, finishing it off in subsequent discussion with Shlomo (Tall & Vinner, 1981).

This became my second source of embarrassment with my Israeli friends. Following the custom in my experience as a mathematician, authors’ names were written in alphabetical order and so it happened that ‘Tall and Vinner’ became quoted as the origin of ‘concept image’ and ‘concept definition’, when it really was the invention of Vinner, shared earlier with Rina Hershkowitz. In my defence, I recalled Richard Skemp’s immortalization of ‘instrumental and relational understanding’ using terms formulated by his friend Stieg Mellin-Olsen in a social setting, which Richard converted to a psychological setting. His paper became a classic. In the case of ‘concept image’ and ‘concept definition’, I took Shlomo’s definition of the terms in two separate compartments in the philosophical mind and turned them into mental conceptions in the biological brain. This paper became a classic too. Publication is all. The credit goes to those who publish first, even if they acknowledge earlier sources.

As I worked on my doctoral thesis in the early eighties, computers arrived and I began to bring together my mathematical knowledge with my personal version of cognitive psychology. I knew that students had a serious difficulty with limits, which they saw as potential processes that were never finished rather than fixed concepts. I had used infinitesimal concepts in teaching a course on ‘Development of Mathematical Concepts’ long ago in the early seventies and a young David Pimm—then a mathematics undergraduate—had persuaded me to teach a mathematics course on non-standard analysis. I therefore knew the mathematical theorem that when the graph of a differentiable function is magnified by an infinite quantity, with infinitesimals too small and infinite elements too far away to see, the resulting image is precisely an infinite straight real line (The technical details are in Tall, 1980c). So I was on mathematically solid ground when I introduced students to 'locally straight' functions that looked straight under appropriately high magnification. This led to my programming the software for *Graphic Calculus*, which was the first approach to the calculus using a balanced combination of mathematical correctness and visual conceptual meaning.

As I studied for a PhD in education under Richard Skemp, my earlier PhD in mathematics entitled me to have my own PhD student, John Monaghan. He radically changed my earlier analysis of infinite concepts by showing me that his students thought that $\sqrt{2}$ was an infinite number too. It is infinite *in extent* because its decimal goes on and on forever, 1.414… . He found that students conceived of 'proper' numbers that could be calculated precisely, such as integers, fractions and finite decimals, and 'improper' numbers that could not. (Monaghan, PhD[1]).

The eighties became a period of great excitement as I travelled to show *Graphic Calculus*, and developed more theory relating to the visualisation of concepts. I theorised that a picture can be *specific*, such as the graph of $\sin x$ and its slope function, looking like $\cos x$. It can also be imagined as being typical of more general locally straight functions whose slope function could be seen by looking along the changing slope of the graph. So I formulated the notion of 'generic organiser' as an environment that enables the learner to manipulate examples and (if possible) non-examples of a specific mathematical concept or a related system of concepts (Tall, 1989). This was considered as a complementary construction to the notion of advance organizer (Ausubel *et al.*, 1978), a higher-level structure used to organize future learning from above, whereas a generic organiser builds up generalisations from below. I also proposed the notion of 'cognitive root', which is 'an anchoring concept which the learner finds easy to comprehend, yet forms a basis on which a theory may be built' (Tall, 1989). A good example of a cognitive root is the concept of 'local straightness'.

At this time in the mid-eighties I was becoming considered as an 'expert' in visualization, though I knew little about the wider cognitive aspects of the topic. I worked with the idea of linking visualization and symbolism with my second PhD student Michael Thomas who has continued to work with me ever since. He built up the idea of 'versatile thinking' in which visualization and symbolism were used in complementary ways (Thomas PhD; Tall & Thomas, 1991) and together we began to build a 'principle of selective construction' where the computer could be used to carry part of the burden of internal computation while the learner could focus on the higher level relationships. In doing so, we began to be interested in the relationship between process and concept.

[1] In this paper, all PhD theses are given in a supplementary list at the end in chronological order.

At this point there was a veritable explosion of related ideas. Ed Dubinsky (1991) was developing his APOS theory whereby ACTIONS were routinized into PROCESSES, encapsulated into OBJECTS to become part of a larger SCHEMA of thought, and Anna Sfard was developing a theory with complementary aspects of STRUCTURAL and OPERATIONAL mathematics (Sfard, 1991). I spent time with both, but we never came close enough to write joint papers in process-object theory, although my thinking was greatly influenced by both of them. I misunderstood Anna's notion of 'structural' because she related it to the way Ian Stewart and I had written about complex numbers as ordered pairs of real numbers and I naturally related the term to the structural approach of Bourbaki with axiomatic definitions and formal deduction.

It was some time before the light dawned. Eddie Gray and I had been colleagues for many years at Warwick when he began his own research in the late eighties. His first major work (Gray, 1991) studied the responses of young children to simple arithmetic problems and he agreed to submit his current research for a PhD. One late Thursday afternoon in December 1990 as we looked at his data on the relationship between counting processes and number concepts, it became clear that we did not have a word to cover both possibilities. Dubinsky (1991) had his notion of process that could be encapsulated into object and de-encapsulated into process, Sfard had her notion of process that could be reified into object, but she referred to process and object as 'two sides of the same coin' (Sfard, 1991), and asked, "How can anything be a process and an object at the same time?" Suddenly I suggested we call this 'thing' that can be both process *and* concept, a *procept*. It was a moment of supreme revelation, comparable with the time that Shlomo showed me his paper on concept image. All the disparate pieces fell into place. Some children were thinking flexibly, shifting seamlessly from process to concept, others were fixed in the procedures of counting. Immediately we saw applications in algebra (expressions as process or concept) and analysis (limits as process or concept). This was huge!

At that very moment, Rolph Schwarzenberger, currently the head of our department, walked into my office. Before he could speak, I said, "On your knees, Schwarzenberger! You were not there when Leibniz first said 'functio', or when Cantor first said, 'set'. But you were there when Tall said 'procept'!" He smiled indulgently and said, "I am not sure of the etymology of the word," meaning that the prefix 'pro' was a Greek term with a different usage. I looked back and said, "If you have a delicate new plant, the last thing you do is to prune it, instead you nurture it and pour manure all over it." (Except that I did not say the word 'manure'.)

That evening we were delighted with our work and considered all the possibilities that lay before us. But then, as I sat in my car in the car park in the gathering gloom, I suddenly lost heart and said to Eddie: "I'm not so sure; all we have is a word." He leaned in through the car window and said, "We have more than that, we have *duality*, *ambiguity* and *flexibility*!" This most profound insight became the title of our first paper together: 'Duality, Ambiguity and Flexibility in Thinking' (Gray & Tall, 1991). It was sometime later that Eddie's wife Mareea christened it the DAFT paper.

Eddie used the theory to suggest the idea of the 'proceptual divide' in the spectrum of performance between those children who cling to remembered procedures and those who become increasingly flexible through the use of proceptual thinking. The notion of procept was the product of an equal partnership in which we shared different experiences to produce something new that was genuinely greater than the sum of its parts. It began a productive relationship over a decade and a half, in which we worked with our doctoral students to develop and enrich the theoretical framework.

Meanwhile, a development that had begun back at the beginning of the 1980s and had run in parallel for years began to bear fruit. The papers presented in the first years of PME were mainly about mathematics education at elementary level. I presented a paper in 1981 which sounded the clarion call to build a theory of mathematics education from early beginnings right through to university undergraduate and research level (Tall, 1981). Gontran Ervynck was also working on this idea and coined the phrase 'advanced mathematical thinking', proposing a working group of that title and inviting me to share in its organisation. I had no talent for administration and he did it all.

By 1979 we had agreed to write a book that I edited and it appeared as *Advanced Mathematical Thinking* in 1991. As I wrote the epilogue to that book, I saw the contrast between two distinct strands of thought leading to formal mathematics.

> There are therefore (at least) two different kinds of mathematics. One builds from gestalts, through identification of properties and their coherence, on to definition and deduction at advanced levels of mathematical thinking. The other continually encapsulates processes as concepts, to build up arithmetic, then generalizes these ideas in algebra before formalizing them as definitions and deductive theorems in the advanced mathematics of abstract algebra.
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> (David Tall, 1991, p.254)

At the same time, PhD student Md.Nor Bakar had introduced me to the theory of prototypes (Smith, 1988), which resonated with my thoughts on concept images and generic examples and led to our realisation that students' conceptions of functions were very much influenced—often unconsciously—by their previous experiences (Bakar, PhD; Bakar & Tall, 1991).

Meanwhile, Norman Blackett studied trigonometric ratios as process and concept and researched how students might gain more conceptual insight using interactive visual software to explore the properties of trigonometric ratios in right-angled triangles. As a by-product it gave a significant gender difference, with the girls cooperating while the boys took the activity less seriously so that a previous advantage enjoyed by the boys was turned around (Blackett, PhD; Blackett & Tall, 1991).

In 1993 I was struck down by an illness (sarcoidosis) which left me exhausted and I had fourteen months off work. I continued to see my research students during this time, but everything else stopped. On attempting to return to work, it was clear that I could cope for a day, perhaps two, but then the duties began to fall behind schedule and within a week it had all ground to a halt. So I took early retirement in 1994, while continuing to work at the university on a one-third timetable. The remainder of my working life was the best time of all. I had a stream of willing PhD students, supported by Eddie as second supervisor and we developed a range of new ideas as I shifted from writing a majority of solo papers to joint papers with others.

My three American PhD Students each gave me insight into the nature of American College Mathematics. Phil Demarois studied the relationship between different representations (facets) and different levels (layers) of encapsulation in the function concept (Demarois, PhD; Demarois & Tall, 1996). Mercedes McGowen studied the cognitive collages of knowledge construction produced by college students to show how the students concept maps changed with time, as the more successful built new knowledge on old while the less successful had more fragile structure which failed to build on previous knowledge (McGowen, PhD; McGowen & Tall, 1999). Lillie Crowley revealed how students could be awarded the same grade at one stage, yet diverge in the next stage with the more successful producing coherent knowledge structures while the less successful remained with limited procedures that trapped them in less appropriate ways of working (Crowley, PhD; Crowley & Tall, 1999).

Malaysian PhD student Yudariah Yusof took me along a different track by investigating student attitudes to problem-solving (Yusof, PhD; Yusof & Tall, 1995). Amazingly, she found that the qualities that mathematicians desired of their students were encouraged in problem-solving, but suppressed in regular mathematics lectures.

Maselan bin Ali studied how Malaysian students coped with the symbolic routines of differentiation, showing that the more successful tended to have two or more procedures available for a given problem (classified as multi-procedure or process) while the less successful were more likely to have at most one (classified as procedural) (Ali, PhD; Ali & Tall, 1996).

Robin Foster studied young children's solutions of equations of the form $3+4=\square$, $3+\square=7$ and $\square+4=7$. We hypothesised together that the first could be solved by any method of addition, the second solved by 'counting-on' from 3, but the third would be more difficult for children at the 'count-all' stage because it would require them to guess a starting point from which to count-on, to see if the desired objective was reached (Foster, 1994). He also considered the way in which children used Dienes' Multibase Blocks to solve subtraction problems and found that while the most successful would use the materials to illustrate what they knew, a second group might use the materials to see the underlying mathematics, a third group might function satisfactorily with the materials, but not without, and a fourth group might fail entirely. (Foster, PhD). This has widespread importance in using statistics in any controlled experiment where one group are given a special treatment: the statistics only measure the change in *one* sub-group—the second sub-group who have a genuine improvement in long-term learning—while the first, third and fourth have no permanent long-term effect at all.

At University Level, Marcia Pinto from Brazil considered how students at Warwick coped with a first course of university analysis, distinguishing between those who rely on their concept imagery to give meaning to the definition (natural learners) and those who give meaning to the definition through studying the formal definitions and related proofs to derive properties (formal learners) (Pinto, PhD; Pinto & Tall, 1999).

Erh-Tsung Chin (Abe) from Taiwan studied how students made sense of equivalence relations, revealing that underlying embodiments interfered with their understanding of the three properties of an equivalence relation (Chin, PhD; Chin & Tall, 2002).

Soo Duck Chae from Korea studied the use of computer software in investigating the meaning of the bifurcation of solutions of iteration of the equation $f(x) = \lambda x(1-x)$ as λ increases. Initially the iteration successively replacing x by $f(x)$ tends to a limit for $0 < \lambda \leq 3$ then bifurcates to an orbit of period 2 at a value of $\lambda = \lambda_1$, and to orbits of period 4, 8, 16, … at $\lambda_2, \lambda_3, \lambda_4, \ldots$ to give a sequence that converges to the Feigenbaum constant λ_∞. According to Dubinsky's original APOS theory, it is possible to distinguish three stages of encapsulation: from the process of performing the iteration encapsulated as a final orbit, from the varying orbit to the sequence of bifurcation points, and from the sequence of bifurcation points to the Feigenbaum limit. APOS theory as originally formulated implies that encapsulation need be performed at each stage so that the objects formed at that stage could be used at the next. Yet, Soo Duck found that for $\lambda \leq 3$ many students were still at the process stage getting closer and closer to the limit, while for $\lambda > 3$, they switched focus to the visual picture of the orbit and used this as an object which bifurcated to reveal the sequence (λ_n) whose approximate numerical values looked as if they would converge geometrically (Chae PhD; Chae & Tall, 2001). This confirmed for me that a symbolic APOS theory required complementing with embodied visualisation and human action to explain how mathematical learning occurred. On the other hand, the embodied theory of Lakoff (Lakoff & Nunez, 2000) did not focus on process-object encapsulation at all. In 2001, Eddie and I discussed the relationship between embodied objects and symbolic procepts (Gray & Tall, 2001).

The major step in my journey to link embodiment and the symbolic compression from process to concept occurred when Anna Poynter (previously Anna Watson) revealed the insight of a student Joshua. In talking about the sum of two vectors, he explained that the sum was a single vector 'had the same effect' as the combination of two individual vectors. This key unlocked the door of the relationship between embodiment and process-object encapsulation. Embodied encapsulation involves a 'delicate shift of attention' (in the sense of Mason, 1980) from the *action* being performed to the *effect* of that action (Poynter, PhD; Watson, 2004).

This parallel between embodiment and symbolism led me immediately to the idea of *three distinct worlds of mathematics* (Tall, 2004). We begin in *a world of conceptual embodiment* focusing on (real-world) objects and actions and by thought experiments focusing on generic properties, we construct hierarchies of mental objects through to the platonic world of Euclidean geometry and visual representations of algebra and the calculus. *The world of proceptual symbolism* compresses actions from processes into thinkable concepts (procepts) to lead to a hierarchical development in arithmetic, algebra and symbolic calculus. The two combined lead by natural or formal thinking to *the formal axiomatic world* of concept definition and formal proof.

I had begun thirty years before in the formal world of mathematics and I had backtracked and found a route from the perceptions of a child to the conceptions of mathematicians. And virtually every insight came from someone else! As Richard Skemp taught me, 'pleasure is a signpost, not a destination'. 'The journey is the reward'. The journey still continues, clarifying issues in the theoretical framework, for instance, in the way in which learners build on previous knowledge to produce what I call 'met-befores' in the concept image that can be helpful in some contexts, but act as cognitive obstacles in others. We already have a map of the cognitive growth of procepts through arithmetic, algebra, calculus and on to the formal definitions of analysis and other formal mathematics (Tall, Gray, Ali, Crowley, Demarois, McGowen, Pitta, Pinto, Thomas, & Yusof, 2001).

The journey uses the natural strengths and limitations of the biological brain to compress complicated detail into the simplicity of thinkable concepts that can be handled by the limited focus of attention (Akkoc & Tall, 2004).

The research continued as Amir Asghari produced a highly personal Lakotos-style thesis in which he challenged not only my work in textbooks with Ian Stewart and research with Abe (Erh-Tsung) Chin, but also analysed that of the Greeks, Gauss, Russell to point out subtle flaws in the theories of Dienes and Skemp. Once again, challenge has led to personal enrichment. (Asghari, PhD; Asghari & Tall, 2005).

The journey has been enhanced by the companionship and insight of many other companions who have taught me so much. Other than my full PhD students, Tony Barnard shared with me his concept of 'cognitive unit', Dina Tirosh her research on infinity, John Pegg his research in SOLO and Van Hiele Theory; Ian Stewart collaborated in the writing of three mathematics texts, and a host of others shared ideas including Gary Davis, Demetra Pitta, Shakar Rasslan, Juan Pablo Meija-Ramos, my two Brazilian visiting students, Victor Giraldo and Rosana Nogueira de Lima, and Adrian Simpson, the organiser of this volume and the related celebrations. But it is to my research students I give the greatest thanks, particularly Eddie Gray, John Monaghan and Michael Thomas, all of whom have distinguished careers supervising doctoral students in a direct line from Richard Skemp through myself and them to succeeding generations.

Christopher Zeeman once said to me that the main test for PhD students to be awarded the degree is that they have taught their supervisor something important. It is an interesting idea.

An individual cannot be the source of limitless power of thought. In the real world there is no such thing as perpetual motion. To develop one needs new sources of energy and that energy comes from other people freely giving of their ideas. In the recent review of the first thirty years of PME (Guttiérrez & Boero, 2006), the ideas discussed in this paper have the largest number of references for a single person in the whole volume. I take quiet pride in this statistic for someone who retired over a decade ago on ill-health. Of course, it is not the work of a single person, but the contributions of a whole family working for a common purpose. As can be seen from this review, almost everything I have done has been gifted to me willingly by others. My work has only been made possible by the support of all my collaborators, especially the PhD students at Warwick University who all earned the award of a doctorate for teaching their supervisor something important.

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PhD THESES

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